(1) Let $a, b, c, d \in \mathbb{R}$ be real numbers. Represent a linear transformation $\mathbb{R}^2 \xrightarrow{\phi} \mathbb{R}^2$ by the matrix

$$A := \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$ 

The basis $\mathcal{B} := (1, i)$ of $\mathbb{C}$ (over $\mathbb{R}$) establishes an isomorphism (of real vector spaces)

$$\mathbb{R}^2 \xrightarrow{\Upsilon} \mathbb{C}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto x + iy$$

$$[z]_\mathcal{B} \mapsto z$$

where $z = x + iy$. In this basis $\mathcal{B}$, multiplication by $i$ is represented by the matrix:

$$J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$ 

(a) Show that $A$ corresponds to multiplication by a complex number if and only $A$ commutes with $J$, that is, if $AJ = JA$.

In this case we say that $A$ is complex-linear. What is the condition on the matrix entries $a, b, c, d$?

(b) The opposite condition is that $A$ anticommutes with $J$, that is:

$$A_\perp J = -JA_\perp$$

In that case, we say that $A$ is complex-antilinear. What is the condition on the matrix entries $a, b, c, d$?

(c) Show that if $\psi \in \mathbb{C}$, then the transformation

$$\mathbb{C} \xrightarrow{\psi} \mathbb{C}$$

$$z \mapsto \psi \overline{z}$$

corresponds to a complex-antilinear transformation

$$\mathbb{R}^2 \xrightarrow{\Upsilon_0 \psi_0 \Upsilon^{-1}} \mathbb{R}^2.$$
(d) Express $A$ as the sum $A = A_+ + A_-$, where $A_+$ commutes with $\mathbb{J}$ (and thus corresponds to a complex scalar $\xi$) and $A_-$ anticommutes with $\mathbb{J}$ as above.

(e) Using the previous expressions for $A_+$ and $A_-$, write the transformation $\Upsilon \circ \phi \circ \Upsilon^{-1}$ in the following form:

$$
\begin{align*}
\mathbb{C} &\rightarrow \mathbb{C} \\
z &\mapsto \xi z + \psi \bar{z}
\end{align*}
$$

where $z = x + iy$, $\xi$, $\psi \in \mathbb{C}$ are complex numbers.