A (linear) plane in $\mathbb{R}^3$ is defined by nonzero row vector $\rho = [a \ b \ c]$. It consists of points (represented by column vectors)

$$
\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}
$$

satisfying

$$0 = \rho \cdot \vec{v} = ax + by + cz.
$$

An affine plane in $\mathbb{R}^3$ is defined by nonzero row vector $\rho = [a \ b \ c]$ by the formula

$$d = \rho \cdot \vec{v} = ax + by + cz.
$$

where $d \in \mathbb{R}$. We can “enhance” $\rho$ to

$$
\tilde{\rho} := [a \ b \ c \ -d]
$$

and $\vec{v}$ to

$$
\tilde{\vec{v}} := \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
$$

so that the affine plane is given by

$$0 = \tilde{\rho} \cdot \tilde{\vec{v}} = ax + by + cz - d.
$$

Another way to define a plane is parametrically. That is, given numbers $s, t \in \mathbb{R}$, the general point on the plane is given by an expression

$$f(s,t) = \vec{v}_0 + sv_1 + tv_2
$$

where $\vec{v}_0, \vec{v}_1, \vec{v}_2 \in \mathbb{R}^3$ are vectors.

1. Suppose that $\rho_1$ and $\rho_2$ define the same linear plane. How are these row vectors related?
**Solution:** \( \rho_1 \) and \( \rho_2 \) are nonzero scalar multiples of each other. I gave full points for just writing this down, but what follows is an explanation of why they must be scalar multiple.

It is obvious that multiplying \( ax + by + cz = 0 \) by a non-zero scalar yields an equivalent equation. Less obvious is the fact that if \( a'x + b'y + c'z = 0 \) defines the same set in \( \mathbb{R}^3 \), then there is a nonzero scalar \( k \) such that \( ka' = a, kb' = b, kc' = c \). Although we can see this intuitively if we realize that \( \begin{bmatrix} a \\ b \\ c \end{bmatrix} \) is the normal (column) vector to our plane, and all the normal vectors to a plane are in the same linear subspace. But that’s an argument using column vectors and we’d like to have an argument using row vectors.

I want to make a clarifying note that I think will alleviate some confusion. We are viewing row vectors as linear map from \( \mathbb{R}^3 \) into \( \mathbb{R} \). We already know from linear algebra that the linear maps \( \mathbb{R}^m \to \mathbb{R}^n \) are \( n \times m \) matrices. Thus a linear map \( \mathbb{R}^3 \to \mathbb{R} \) is a \( 1 \times 3 \) matrix. It looks exactly like a row vector, but in many cases in helps to think of it as a matrix. We showed in homework one that the space of linear maps \( \mathbb{R}^m \to \mathbb{R}^n \) always form a vector space. And so the space \( L(\mathbb{R}^3, \mathbb{R}) \) form a vector space that looks exactly like \( \mathbb{R}^3 \) except all the vectors are lying on their sides. Obviously we can associate row vectors to column vectors via the transpose operation. But it’s good to keep in mind the distinction between our space (column vectors) and the linear maps on our space (row vectors).

Now \( \rho_1 \) and \( \rho_2 \) are linear maps \( \mathbb{R}^3 \to \mathbb{R} \), which means they have a 2-dimensional null space (the null space is also called the “kernel”). The plane associated with a row vector \( \rho_i \) is just its null space. So the hypothesis that \( \rho_1 \) and \( \rho_2 \) define the same plane is a way of saying that they have the same null space. The Rank-nullity theorem then implies that the row spaces of \( \rho_1 \) and \( \rho_2 \) must be the same one-dimensional space. But the row space is just the span of row vectors in \( \rho_i \), and \( \rho_i \) is a single row vector, so this directly implies \( \rho_1 = k\rho_2 \) for \( k \in \mathbb{R} \setminus \{0\} \).

2. Same question for affine planes.

**Solution:** Again \( \tilde{\rho}_1 \) and \( \tilde{\rho}_2 \) are non-zero scalar multiples of each other.

Again it’s clear that \( \tilde{\rho}_1 \) and \( k\tilde{\rho}_1 , \; k \neq 0 \) define the same affine plane. Less clear is that this accounts for all row vectors that define the same plane. In this case, because projectivization is involved, it may just be less confusing to do the problem using normal column vectors rather than linear maps (row vectors) using the implicit isomorphism between them.

Note that the row vector \( \tilde{\rho}_1 = \begin{bmatrix} a & b & c & -d \end{bmatrix} \) defines an affine plane with normal column vector \( \begin{bmatrix} a \\ b \\ c \end{bmatrix} \), and so since \( \tilde{\rho}_2 \) defines the same plane, its
normal vector must have the form \( N = \begin{bmatrix} ka \\ kb \\ kc \end{bmatrix}, \ k \neq 0 \). Any affine plane parallel the linear plane \( N^\perp \) defined by \( N \) has the form \( kax + kby + kc + r = 0 \) for some \( r \in \mathbb{R} \), so \( \tilde{\rho}_2 \) has the form \( \begin{bmatrix} ka & kb & kc & r \end{bmatrix} \). Now we can compare \( k\tilde{\rho}_1 \) and \( \tilde{\rho}_2 \):

\[
kax + kby + kc + r = 0 \quad kax + kby + kc + d = 0
\]

Subtracting \( k(ax + by + cz) \) on both sides we get that \( r = -kd \), so \( \tilde{\rho}_2 = k\tilde{\rho}_1 \) as we suspected.

3. Describe the row vectors corresponding to the coordinate planes (\( xy \), \( yz \), \( zx \)-planes).

**Solution:** The plane corresponding to a row vector is its null space. So the row vectors are \( \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \).

4. What is the condition (in terms of \( \tilde{\rho} \)) that an affine plane is linear?

**Solution:** We can read this condition directly off the definitions. The condition is \( d = 0 \) if \( \tilde{\rho} = \begin{bmatrix} a & b & c & -d \end{bmatrix} \).

5. Here is a specific example. Write down the plane corresponding to \( \tilde{\rho} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \). Find a parametric expression for it.

**Solution:** The plane is \( \mathcal{P} = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + 3z = -4\} \). The corresponding linear plane is \( x + 2y + 3z = 0 \). A parametric equation for \( \mathcal{P} \) is \( p + s v_1 + t v_2, \ s, t \in \mathbb{R} \) where \( p \in \mathcal{P} \), and \( v_1 \) and \( v_2 \) are linearly independent vectors spanning the linear plane. For example \( p = \begin{bmatrix} -4 \\ 0 \\ 0 \end{bmatrix} \),

\[
v_1 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.
\]

6. Consider the three vectors

\[
a := \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad b := \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad c := \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}
\]

as representing the homogeneous coordinates of three points \( a, b, c \in \mathbb{P}^2 \).

**Solution:** I declare them so-considered.
7. Find a row vector (representing a covector $\mathbb{R}^3 \rightarrow \mathbb{R}$) representing the line $\ell := \overrightarrow{ab}$ in homogeneous coordinates.

**Solution:** Let $P : \mathbb{R}^3 \rightarrow \mathbb{P}^2$ be the projection map. $P(a) = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$, $P(b) = \begin{bmatrix} 4/6 \\ 5/6 \\ 5/6 \end{bmatrix}$. Applying point-slope form or slope-intercept form to these two points gives the slope as $1/2$, and the implicit equation $-x/2 + y - 1/2 = 0$. We can read the row vector from the coefficients in this equation as $[-1/2 \ 1 \ -1/2]$ or any non-zero scalar multiple thereof.

Note that it doesn’t work to subtract $a$ from $b$ before you projectivize, as several people attempted. $b - a = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$, which projects to the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ which corresponds to a slope of 1 instead of $1/2$. Thus projectivization is not a linear operator since $P(b - a) \neq P(b) - P(a)$. In fact, $P(b - a)$ and $P(b) - P(a)$ aren’t even scalar multiples of each other. You already knew this in your heart of hearts, though, since it corresponds to how we add fractions: $a/b + c/d \neq (a + c)/(b + d)$.

8. Does the point $c$ lie on $\ell$?

**Solution:** Yes. $\begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = -7/2 + 8 - 9/2 = -16/2 + 8 = -8 + 8 = 0$.

9. Let $V,W$ be a pair of vector spaces and denote their dual vector spaces $V^*,W^*$. Let $v \in V$, and $\phi \in V^*$. Given a linear map $W \xrightarrow{A} V$, write down how the covector $\phi$ changes under the map $A$. Hint: Let $\phi^T, v \in V$ are column vectors. Consider the pairing $(\phi, v)$ is $\phi \cdot v$. $(\phi, Aw)$ and $(\phi)(Aw) = (\phi A) \cdot w$.

**Solution:** Recall that $\phi$ is the matrix of a linear map in addition to be a row vector. $A$ is also the matrix of a linear map. And composition of linear maps is given by multiplication of their matrices, so $\phi \circ A = \phi A$. If we think of $\phi$ as a vector, this means $A$ operates on $\phi$ by right-multiplication, $\phi \mapsto \phi A$.

In terms of the pairing $(\phi, w) = \phi w$, we have $(\phi, Aw) = \phi Aw$ as the multiplication of three matrices \(^1\), which by associativity is equal to $(\phi A)w = (\phi A, w)$.

\(^1\)For a column vector $v$ and matrix $A$ you may not think of $Av$ as matrix multiplication. In actuality, the fact that we can view $v$ as an $n \times 1$ matrix and calculate $Av$ using standard matrix multiplication is one of the main reasons we use column vectors in linear algebra. The difference between column vectors and row vectors is that we tend to think of row vectors as linear maps, but column vectors as points.