Let $V$ be a vector space, and let $\Lambda^\bullet(V^*)$ be its exterior algebra. Recall that $\Lambda^0(V^*) = \mathbb{R}$ and $\Lambda^1(V^*) = V^*$. For any vector $v \in V$, define a linear map $\Lambda^1(V^*) \xrightarrow{\iota_v} \Lambda^0(V^*)$ by:

$$V^* = \Lambda^1(V^*) \xrightarrow{\iota_v} \mathbb{R} = \Lambda^0(V^*)$$

$\psi \mapsto \iota_v(v)$

- Show that $\iota_v$ extends to a derivation of $\Lambda^\bullet(V^*)$ of degree $-1$, that is a family of linear maps

$$\Lambda^k(V^*) \rightarrow \Lambda^{k-1}(V^*)$$

satisfying, for $\alpha \in \Lambda^k(V^*)$, $\beta \in \Lambda^l(V^*)$,

$$\iota_v(\alpha \wedge \beta) = \iota_v(\alpha) \wedge \beta + (-1)^k \alpha \wedge \iota_v(\beta)$$

- Show that the set $\text{Der}^{-1}(\Lambda^\bullet(V^*))$ of all such derivations of degree $-1$ forms a vector space and that

$$V \rightarrow \text{Der}^{-1}(\Lambda^\bullet(V^*))$$

$v \mapsto \iota_v$

is an isomorphism of vector spaces.

- For any graded algebra $A$, define a notion of derivation of degree $k$. Suppose that $D_1, D_2$ are derivations of degree $k_1, k_2$ respectively. Show that their commutator

$$[D_1, D_2] := D_1D_2 - (-1)^{k_1k_2}D_2D_1$$

is a derivation of $A$ of some degree $k$, and compute $k$.

- Now let $M$ be a manifold (for example, an open subset of $\mathbb{R}^m$) and $\Omega^\bullet(M)$ the graded algebra of exterior differential forms on $M$. Let $\xi \in \text{Vect}(M)$ be a vector field on $M$. Define an operation of interior multiplication $\iota_\xi$ on $\Omega^\bullet(M)$ which is a derivation of degree $-1$. Compute $[\iota_{\xi_1}, \iota_{\xi_2}]$ for $\xi_1, \xi_2 \in \text{Vect}(M)$. 

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