MATH 436: EXAM 1

Tuesday, September 22, 2009.

Do all five problems. Write clearly and legibly in grammatically correct English prose.

(1) Let $\alpha \neq 0$ be a parameter. Consider the space curve $\gamma(t)$:

$$\gamma(t) := \begin{bmatrix} e^t \cos(\alpha t) \\ e^t \sin(\alpha t) \\ e^t \end{bmatrix}$$

  (a) Prove that γ is regular.
  (b) Prove that γ lies on a cone.
  (c) Parametrize γ by arclength.
  (d) What is $\lim_{t\to\infty} k(t)$?

(2) Determine all unit-speed curves $\gamma(s)$ starting at $\gamma(0) = (0,0,0)$ with $k(s) = 1$ and $\tau(s) = 0$.

(3) (The Darboux vector) Let $(\mathsf{T}(s), \mathsf{N}(s), \mathsf{B}(s))$ be the Frenet-Serret frame of a unit-speed curve $\gamma(s)$. Using the structure equations

$$\frac{d}{ds}\mathsf{T}(s) = k(s)\mathsf{N}(s)$$
$$\frac{d}{ds}\mathsf{N}(s) = -k(s)\mathsf{T}(s) + \tau(s)\mathsf{B}(s)$$
$$\frac{d}{ds}\mathsf{B}(s) = -\tau(s)\mathsf{N}(s)$$

find a $\mathsf{D}(s)$ such that

$$\frac{d}{ds}\mathsf{T}(s) = \mathsf{D}(s) \times \mathsf{T}(s)$$
$$\frac{d}{ds}\mathsf{N}(s) = \mathsf{D}(s) \times \mathsf{N}(s)$$
$$\frac{d}{ds}\mathsf{B}(s) = \mathsf{D}(s) \times \mathsf{B}(s)$$

Date: October 8, 2009.

(4) Let α, β be parameters, and consider the *twisted cubic curve* defined by

$$\gamma(t) := \begin{bmatrix} t \\ \alpha t^2 \\ \beta t^3 \end{bmatrix}.$$

Compute the curvature $k(0)$, torsion $\tau(0)$, unit tangent vector $\mathsf{T}(0)$, unit normal vector $\mathsf{N}(0)$, unit binormal vector $\mathsf{T}(0)$ for the parameter value $t = 0$. (Since this curve is not unit-speed, you may use the curvature formula (Proposition 2.1)

$$k(t) = \frac{\|\gamma''(t) \times \gamma(t)\|}{\|\gamma'(t)\|^3}$$

and the torsion formula (Proposition 2.3)

$$\tau(t) = \frac{\gamma'(t) \times \gamma''(t) \cdot \gamma'''(t)}{\|\gamma'(t) \times \gamma''(t)\|^2}$$

valid whenever $\gamma''(t) \neq 0$.)

(5) Recall that a *rigid motion* of $\mathbb{E}^n$ is an isometry which *preserves orientation.* An *isometry* is an affine map of the form

$$p \xmapsto{g} Ap + b$$

where A is an orthogonal matrix and b is a vector (defining a translation). Suppose that g is an isometry of $\mathbb{E}^3$ which leaves invariant a plane P. Prove or disprove:

(a) The restriction of g to P is an isometry.
(b) The restriction of g to P is a rigid motion.

(Hint: can you reduce to the case $P = \mathbb{E}^2 \times \{0\}$?)