MATH 437: FIRST HOMEWORK

Due Thursday, February 6, 2014.

The problem numbers and page numbers for these seven problems refer to Bachman, *A Geometric Approach to Differential Forms,* 2nd edition.

(1) (Problem 2.6, p.9) Let

$$D(x,y) := \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix}.$$

If for some point (x_0, y_0), you know $D(x_0, y_0) > 0$, then show that the signs of

$$\frac{\partial^2 f}{\partial x^2}(x_0, y_0), \quad \frac{\partial^2 f}{\partial y^2}(x_0, y_0)$$

agree. Find examples of such functions where $(x_0, y_0) = (0, 0)$.

(2) (Problem 2.11, p.13) The tangent lines to the graph $z = f(x, y)$ of a function $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}$ at a point $\big(x_0, y_0, f(x_0, y_0)\big)$ correspond, by *horizontal projection*

$$\mathbb{R}^3 \xrightarrow{\Pi} \mathbb{R}^2$$

$$(x, y, z) \longmapsto (x, y)$$

to lines $\ell \subset \mathbb{R}^2$ through (x_0, y_0). Define the *slope* of such a line to be the slope of that line in the vertical plane $\Pi^{-1}(\ell)$ which projects to ℓ, where the z-coordinate is considered "vertical." Let $f(x, y)$ be the function

$$f(x, y) := \frac{1}{2}x^2 + 3xy$$

Find the largest slope of any tangent line to the graph $z = f(x, y)$ of $f(x, y)$ at the point $(1, 1)$.

Date: January 30, 2014.

(3) (Problem 2.18, p.16) Draw the curves given by the following parametrizations:
   (a) (t, t^2) where $0 \leq t \leq 1$;
   (b) (t^2, t^3) where $0 \leq t \leq 1$;
   (c) $\big(2\cos(t), 3\sin(t)\big)$ where $0 \leq t \leq 2\pi$;
   (d) $\big(\cos(2t), \sin(3t)\big)$ where $0 \leq t \leq 2\pi$;
   (e) $\big(t\cos(t), t\sin(t)\big)$ where $0 \leq t \leq 2\pi$.

(4) (Problem 2.20, p.17) Parametrize, in two different ways, the line segment connecting the points $(1, 1)$ and $(2, 5)$. Find a parametrization which has unit speed (that is, parametrize the curve by arc length).

(5) (Problem 2.35, p.21) Consider the parametrization

$$\phi(r, \theta) = \big(r\cos(\theta), r\sin(\theta), \sqrt{r}\big)$$

where $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$.
   (a) Sketch the surface parametrized by ϕ.
   (b) Find two tangent vectors to this surface at the point $(0, 1, 1)$.

(6) (Problem 2.36, p. 22) Let B be the ball of radius 1 in $\mathbb{R}^3$, that is, the set of points $(x, y, z) \in \mathbb{R}^3$ satisfying $x^2 + y^2 + z^2 \leq 1$).
   (a) Use spherical coordinates to find a parametrization for B.
   (b) Parametrize the intersection of B with the first octant, that is, the region where all x, y, z are all positive.
   (c) Parametrize the intersection of B with the octant where x, y, z are all negative.

(7) (Problem 2.41, p. 23) Consider the following parametrization:

$$\psi(r, \theta) = \big(2r\cos(\theta), r\sin(\theta) + 1\big),$$

where $0 \leq r \leq 1$ and $0 \leq \theta \leq \pi$.
   (a) Sketch the region of $\mathbb{R}^2$ parametrized by ψ.
   (b) Find the vectors $\partial\psi/\partial r$ and $\partial\psi/\partial\theta$.
   (c) Find the area of the parallelogram spanned by these two vectors.