MATH 730: EXAM 1 (GENERAL TOPOLOGY)

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(1) Prove or disprove the following elementary statements:
   (a) If X is compact and Y is Hausdorff, then every continuous map $X \longrightarrow Y$ is closed.
   (b) If X is compact and Y is Hausdorff, then every continuous map $Y \longrightarrow X$ is closed.
   (c) Every compact subset of a topological space is closed.

(2) (August 1995) Consider the following subsets of $\mathbb{R}$:

$$Z := (3,4) \cup \{5\} \cup (6,7) \cup \{8\} \cup \cdots \cup (3n, 3n+1) \cup \{3n+2\} \cup \dots$$

$$X := (0,1) \cup \{2\} \cup Z$$

$$Y := (0,1] \cup Z.$$

Prove or disprove:
   (a) $\exists$ a continuous bijection $X \longrightarrow Y$;
   (b) $\exists$ a continuous bijection $Y \longrightarrow X$;
   (c) X and Y are homeomorphic.

(3) Recall that an n-dimensional *manifold* is a Hausdorff topological space for which every point has an open neighborhood homeomorphic to $\mathbb{R}^n$, or equivalently, an open ball in $\mathbb{R}^n$. Prove or disprove the following statements:
   (a) Every open subset of $\mathbb{R}^n$ is an n-dimensional manifold.
   (b) The *n-sphere*

$$S^n := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + x_1^2 + \cdots + (x_n)^2 = 1\}.$$

   is an n-dimensional manifold.
   (c) The one-point compactification of an n-dimensional manifold is never an n-dimensional manifold.
   (d) The one-point compactification of an n-dimensional manifold is always an n-dimensional manifold.

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