Dynamics on character varieties

Bill Goldman
University of Maryland

5 April 2024
Maryland Dynamics Conference
celebrating Giovanni Forni’s 60th birthday
Dynamics on character varieties and geometric structures

Abstract

Classifying geometric structures on manifolds naturally leads to actions of mapping class groups on character varieties. For example, complete affine structures on closed surfaces are classified by $\text{GL}(2, \mathbb{Z})$-orbits on $\mathbb{R}^2$. Particularly basic are the automorphisms of the variant of the Markoff surface $x^2 + y^2 + z^2 - xyz = 20$ where the dynamics bifurcates between ergodic (level $< 20$) and not ergodic (level $> 20$).
References

Classifying geometric structures

Lie and Klein (1872): A geometry consists of the properties of a space $X$ invariant under transitive action of a Lie group $G$.

Ehresmann (1936): Manifolds locally modeled on $(G, X)$.

Examples include:

- Euclidean structures (flat Riemannian metrics);
- Hyperbolic structures (metrics of curvature $-1$);
- Affine structures (flat affine connections of zero torsion).

Can introduce isolated singularities with specified cone angles — for example, translation surfaces are very special Euclidean structures.
Classifying geometric structures

- Lie and Klein (1872): A *geometry* consists of the properties of a space $X$ invariant under transitive action of a Lie group $G$. 

- Examples include:
  - Euclidean structures (flat Riemannian metrics);
  - Hyperbolic structures (metrics of curvature $-1$);
  - Affine structures (flat affine connections of zero torsion).

- Can introduce isolated singularities with specified cone angles — for example, translation surfaces are very special Euclidean structures.
Classifying geometric structures

- Lie and Klein (1872): A *geometry* consists of the properties of a space $X$ invariant under transitive action of a Lie group $G$.
- Ehresmann (1936): Manifolds locally modeled on $(G, X)$. 

Examples include:
- Euclidean structures (flat Riemannian metrics);
- Hyperbolic structures (metrics of curvature $-1$);
- Affine structures (flat affine connections of zero torsion).

Can introduce isolated singularities with specified cone angles — for example, translation surfaces are very special singular Euclidean structures.
Classifying geometric structures

- Lie and Klein (1872): A geometry consists of the properties of a space $X$ invariant under transitive action of a Lie group $G$.
- Ehresmann (1936): Manifolds locally modeled on $(G, X)$.
- Examples include:
  - Euclidean structures (flat Riemannian metrics);
  - Hyperbolic structures (metrics of curvature $-1$);
  - Affine structures (flat affine connections of zero torsion).
  - Can introduce isolated singularities with specified cone angles—for example, translation surfaces are very special singular Euclidean structures.
Classifying geometric structures

- Lie and Klein (1872): A *geometry* consists of the properties of a space $X$ invariant under transitive action of a Lie group $G$.
- Ehresmann (1936): Manifolds locally modeled on $(G, X)$.
- Examples include:
  - Euclidean structures (flat Riemannian metrics);
Classifying geometric structures

- Lie and Klein (1872): A *geometry* consists of the properties of a space $X$ invariant under transitive action of a Lie group $G$.
- Ehresmann (1936): Manifolds locally modeled on $(G, X)$.
- Examples include:
  - Euclidean structures (flat Riemannian metrics);
  - Hyperbolic structures (metrics of curvature $-1$);
Classifying geometric structures

- Lie and Klein (1872): A geometry consists of the properties of a space $X$ invariant under transitive action of a Lie group $G$.
- Ehresmann (1936): Manifolds locally modeled on $(G, X)$.
- Examples include:
  - Euclidean structures (flat Riemannian metrics);
  - Hyperbolic structures (metrics of curvature $-1$);
  - Affine structures (flat affine connections of zero torsion).
Classifying geometric structures

- Lie and Klein (1872): A *geometry* consists of the properties of a space $X$ invariant under transitive action of a Lie group $G$.
- Ehresmann (1936): Manifolds locally modeled on $(G, X)$.
- Examples include:
  - Euclidean structures (flat Riemannian metrics);
  - Hyperbolic structures (metrics of curvature $-1$);
  - Affine structures (flat affine connections of zero torsion).
- Can introduce isolated singularities with specified cone angles — for example, translation surfaces are *very special* singular Euclidean structures.
Markings and the Mapping Class Group

Classifying such \((G, X)\)-structures on a fixed topology \(\Sigma\) leads to action of the mapping class group \(\text{Mod}(\Sigma) := \pi_0(\text{Diff}(\Sigma)) \to \text{Out}(\pi)\) on deformation space \(\text{Def}(G, X)(\Sigma)\) of marked \((G, X)\)-structures.

\(\text{Def}(G, X)(\Sigma)\) itself locally modeled on \(\text{Rep}(\pi, G)\).

The \(\text{Mod}(\Sigma)\)-action on \(\text{Def}(G, X)(\Sigma)\) corresponds to \(\text{Out}(\pi)\)-action on \(\text{Rep}(\pi, G)\).
Markings and the Mapping Class Group

Classifying such \((G, X)\)-structures on a fixed topology \(\Sigma\) leads to action of the mapping class group

\[
\text{Mod}(\Sigma) := \pi_0(\text{Diff}(\Sigma)) \rightarrow \text{Out}(\pi)
\]

on deformation space \(\text{Def}_{(G,X)}(\Sigma)\) of marked \((G, X)\)-structures.
Markings and the Mapping Class Group

Classifying such \((G, X)\)-structures on a fixed topology \(\Sigma\) leads to action of the mapping class group

\[
\text{Mod}(\Sigma) := \pi_0(\text{Diff}(\Sigma)) \to \text{Out}(\pi)
\]

on deformation space \(\text{Def}_{(G, X)}(\Sigma)\) of marked \((G, X)\)-structures.

\(\text{Def}_{(G, X)}(\Sigma)\) itself locally modeled on \(\text{Rep}(\pi, G)\)
Markings and the Mapping Class Group

Classifying such \((G, X)\)-structures on a fixed topology \(\Sigma\) leads to action of the *mapping class group* \(\text{Mod}(\Sigma) := \pi_0(\text{Diff}(\Sigma)) \to \text{Out}(\pi)\) on deformation space \(\text{Def}_{(G, X)}(\Sigma)\) of marked \((G, X)\)-structures.

- \(\text{Def}_{(G, X)}(\Sigma)\) itself locally modeled on \(\text{Rep}(\pi, G)\)
- \(\text{Mod}(\Sigma)\)-action on \(\text{Def}_{(G, X)}(\Sigma)\) corresponds to \(\text{Out}(\pi)\)-action on \(\text{Rep}(\pi, G)\).
Euclidean geometry occurs where $X = \mathbb{R}^n$ and $G = \text{Isom}(X)$. Let $\Sigma$ be the $n$-torus.
Classification of Flat Tori

*Euclidean geometry* occurs where $X = \mathbb{R}^n$ and $G = \text{Isom}(X)$. Let $\Sigma$ be the $n$-torus.

- A *Euclidean structure* on $\Sigma$ (a flat Riemannian metric) identifies $\Sigma$ with a flat torus $\mathbb{R}^n/\Lambda$ where $\Lambda < \mathbb{R}^n$ is a lattice.
Classification of Flat Tori

Euclidean geometry occurs where $X = \mathbb{R}^n$ and $G = \text{Isom}(X)$. Let $\Sigma$ be the $n$-torus.

- A Euclidean structure on $\Sigma$ (a flat Riemannian metric) identifies $\Sigma$ with a flat torus $\mathbb{R}^n/\Lambda$ where $\Lambda \subset \mathbb{R}^n$ is a lattice.
- A marking of $\Sigma$ is just a basis of $\Lambda$. 
Classification of Flat Tori

*Euclidean geometry* occurs where $X = \mathbb{R}^n$ and $G = \text{Isom}(X)$. Let $\Sigma$ be the $n$-torus.

- A *Euclidean structure* on $\Sigma$ (a flat Riemannian metric) identifies $\Sigma$ with a flat torus $\mathbb{R}^n/\Lambda$ where $\Lambda < \mathbb{R}^n$ is a lattice.
- A *marking* of $\Sigma$ is just a basis of $\Lambda$.
- The space of marked lattices (bases of $\mathbb{R}^n$) is just $\text{GL}(n, \mathbb{R})$. 


Classification of Flat Tori

Euclidean geometry occurs where \( X = \mathbb{R}^n \) and \( G = \text{Isom}(X) \). Let \( \Sigma \) be the \( n \)-torus.

- A Euclidean structure on \( \Sigma \) (a flat Riemannian metric) identifies \( \Sigma \) with a flat torus \( \mathbb{R}^n / \Lambda \) where \( \Lambda < \mathbb{R}^n \) is a lattice.
- A marking of \( \Sigma \) is just a basis of \( \Lambda \).
- The space of marked lattices (bases of \( \mathbb{R}^n \)) is just \( \text{GL}(n, \mathbb{R}) \).
- Thus the deformation space \( \text{Def}_{(G,X)}(\Sigma) \) of isometry classes of marked flat tori is just the space \( \text{GL}(n, \mathbb{R}) / \text{O}(n) \).
Classification of Flat Tori

*Euclidean geometry* occurs where $X = \mathbb{R}^n$ and $G = \text{Isom}(X)$. Let $\Sigma$ be the $n$-torus.

- A *Euclidean structure* on $\Sigma$ (a flat Riemannian metric) identifies $\Sigma$ with a flat torus $\mathbb{R}^n/\Lambda$ where $\Lambda < \mathbb{R}^n$ is a lattice.
- A *marking* of $\Sigma$ is just a basis of $\Lambda$.
- The space of marked lattices (bases of $\mathbb{R}^n$) is just $\text{GL}(n, \mathbb{R})$.
- Thus the deformation space $\text{Def}_{(G,X)}(\Sigma)$ of isometry classes of marked flat tori is just the space $\text{GL}(n, \mathbb{R})/\text{O}(n)$.
- The mapping class group $\text{Mod}(\Sigma) = \text{GL}(n, \mathbb{Z})$. 

\[\blacktriangleright\]
Classification of Flat Tori

Euclidean geometry occurs where $X = \mathbb{R}^n$ and $G = \text{Isom}(X)$. Let $\Sigma$ be the $n$-torus.

- A Euclidean structure on $\Sigma$ (a flat Riemannian metric) identifies $\Sigma$ with a flat torus $\mathbb{R}^n/\Lambda$ where $\Lambda < \mathbb{R}^n$ is a lattice.
- A marking of $\Sigma$ is just a basis of $\Lambda$.
- The space of marked lattices (bases of $\mathbb{R}^n$) is just $\text{GL}(n, \mathbb{R})$.
- Thus the deformation space $\text{Def}_{(G,X)}(\Sigma)$ of isometry classes of marked flat tori is just the space $\text{GL}(n, \mathbb{R})/\text{O}(n)$.
- The mapping class group $\text{Mod}(\Sigma) = \text{GL}(n, \mathbb{Z})$.
- The moduli space of flat $n$-tori is the biquotient

$$\text{GL}(n, \mathbb{Z}) \backslash \text{GL}(n, \mathbb{R})/\text{O}(n)$$
A more familiar analog:
Marked Riemann surfaces and Teichmüller space
A more familiar analog: Marked Riemann surfaces and Teichmüller space

- The Riemann moduli space $\mathcal{M}(\Sigma)$ parametrizes Riemann surfaces $M$ of a fixed topology $\Sigma$. 
A more familiar analog:
Marked Riemann surfaces and Teichmüller space

- The *Riemann moduli space* $\mathcal{M}(\Sigma)$ parametrizes Riemann surfaces $M$ of a fixed topology $\Sigma$.
  - Quasiprojective variety over $\mathbb{C}$; singular exactly at Riemann surfaces with *nontrivial* automorphisms.
A more familiar analog:
Marked Riemann surfaces and Teichmüller space

- The *Riemann moduli space* $\mathcal{M}(\Sigma)$ parametrizes Riemann surfaces $M$ of a fixed topology $\Sigma$.
  - Quasiprojective variety over $\mathbb{C}$; singular exactly at Riemann surfaces with *nontrivial* automorphisms.
- More tractable object:
  *Teichmüller space* $\mathcal{T}(\Sigma)$ of marked Riemann surfaces $(M, f)$: metric space/complex manifold $\simeq \mathbb{R}^{6g-6}$. 
A more familiar analog:
Marked Riemann surfaces and Teichmüller space

- The *Riemann moduli space* $\mathcal{M}(\Sigma)$ parametrizes Riemann surfaces $M$ of a fixed topology $\Sigma$.
  - Quasiprojective variety over $\mathbb{C}$; singular exactly at Riemann surfaces with *nontrivial* automorphisms.

- More tractable object:
  *Teichmüller space* $\mathcal{T}(\Sigma)$ of marked Riemann surfaces $(M, f)$: metric space/complex manifold $\approx \mathbb{R}^{6g-6}$.
  - *Marking*: Diffeomorphism $\Sigma \xrightarrow{f} M$; Riemann surface $M$ varies, but the topology $\Sigma$ fixed.
A more familiar analog:
Marked Riemann surfaces and Teichmüller space

- The *Riemann moduli space* $\mathcal{M}(\Sigma)$ parametrizes Riemann surfaces $M$ of a fixed topology $\Sigma$.
  - Quasiprojective variety over $\mathbb{C}$; singular exactly at Riemann surfaces with *nontrivial* automorphisms.

- *More tractable object:*

  Teichmüller space $\mathcal{T}(\Sigma)$ of marked Riemann surfaces $(M, f)$: metric space/complex manifold $\approx \mathbb{R}^{6g-6}$.
  - *Marking:* Diffeomorphism $\Sigma \xrightarrow{f} M$; Riemann surface $M$ varies, but the topology $\Sigma$ fixed.
  - Equivalence classes $\leftrightarrow \text{Mod}(\Sigma)$-orbits on $\mathcal{T}(\Sigma)$. 
A more familiar analog:
Marked Riemann surfaces and Teichmüller space

- The \textit{Riemann moduli space} $\mathcal{M}(\Sigma)$ parametrizes Riemann surfaces $M$ of a fixed topology $\Sigma$.
  - Quasiprojective variety over $\mathbb{C}$; singular exactly at Riemann surfaces with \textit{nontrivial} automorphisms.

- \textit{More tractable object}:
  \textit{Teichmüller space} $\mathcal{T}(\Sigma)$ of marked Riemann surfaces $(M, f)$:
  metric space/complex manifold $\approx \mathbb{R}^{6g-6}$.
  - \textit{Marking}: Diffeomorphism $\Sigma \xrightarrow{f} M$;
    Riemann surface $M$ varies, but the topology $\Sigma$ fixed.
  - Equivalence classes $\leftrightarrow$ \text{Mod}(\Sigma)-orbits on $\mathcal{T}(\Sigma)$.

- $\mathcal{M}(\Sigma) = \mathcal{T}(\Sigma)/\text{Mod}(\Sigma)$. 
Coordinate atlases and development

Geometry: Homogeneous space $X = G/H$.

Topology: Manifold $\Sigma$ with universal covering $\tilde{\Sigma} \to \Sigma$ and fundamental group $\pi$.

Marking: Diffeomorphism $\Sigma \xrightarrow{f} M$; the geometry on $M$ varies, but the topology of $\Sigma$ remains fixed.

Patches $U \subset M$: Coordinate atlas of charts $U \to X$ defining local coordinates on $U$ modeled on $X$.

On overlapping patches, coordinate changes extend (locally uniquely) to transformations of $X$ from $G$.

Local charts define development immersion $\tilde{\Sigma} \hookrightarrow X$, equivariantly respecting holonomy homomorphism $\pi \to G$.

Development globalizes coordinate charts.

Holonomy globalizes coordinate changes.
Coordinate atlases and development

- **Geometry**: Homogeneous space $X = G/H$. 
- **Topology**: Manifold $\Sigma$ with universal covering $\tilde{\Sigma} \to \Sigma$ and fundamental group $\pi$.
- **Marking**: Diffeomorphism $\Sigma \xrightarrow{f} M$; the geometry on $M$ varies, but the topology of $\Sigma$ remains fixed.
- **Patches** $U \subset M$: Coordinate atlas of charts $U \to X$ defining local coordinates on $U$ modeled on $X$.
- On overlapping patches, coordinate changes extend (locally uniquely) to transformations of $X$ from $G$.
- Local charts define development immersion $\tilde{\Sigma} \hookrightarrow X$, equivariantly respecting holonomy homomorphism $\pi \to G$.
- Development globalization coordinate charts.
- Holonomy globalization coordinate changes.
Coordinate atlases and development

- **Geometry**: Homogeneous space $X = G/H$.
- **Topology**: Manifold $\Sigma$ with universal covering $\tilde{\Sigma} \to \Sigma$ and fundamental group $\pi$.
Coordinate atlases and development

- **Geometry**: Homogeneous space $X = G/H$.
- **Topology**: Manifold $\Sigma$ with universal covering $\tilde{\Sigma} \rightarrow \Sigma$ and fundamental group $\pi$.
- **Marking**: Diffeomorphism $\Sigma \stackrel{f}{\rightarrow} M$; the geometry on $M$ varies, but the topology of $\Sigma$ remains fixed.
Coordinate atlases and development

- **Geometry:** Homogeneous space $X = G/H$.
- **Topology:** Manifold $\Sigma$ with universal covering $\widetilde{\Sigma} \rightarrow \Sigma$ and fundamental group $\pi$.
- **Marking:** Diffeomorphism $\Sigma \xrightarrow{f} M$; the geometry on $M$ varies, but the topology of $\Sigma$ remains fixed.
  - **Patches** $U \subset M$: Coordinate atlas of charts $U \rightarrow X$ defining local coordinates on $U$ modeled on $X$.

- **Development:** Globalizes coordinate charts.
- **Holonomy:** Globalizes coordinate changes.
Coordinate atlases and development

- **Geometry:** Homogeneous space $X = G/H$.
- **Topology:** Manifold $\Sigma$ with universal covering $\tilde{\Sigma} \to \Sigma$ and fundamental group $\pi$.
- **Marking:** Diffeomorphism $\Sigma \to M$; the geometry on $M$ varies, but the topology of $\Sigma$ remains fixed.
  - Patches $U \subset M$: Coordinate atlas of charts $U \to X$ defining local coordinates on $U$ modeled on $X$.
  - On overlapping patches, coordinate changes extend (locally uniquely) to transformations of $X$ from $G$. 

Local charts define development immersion $\tilde{\Sigma} \to X$, equivariantly respecting holonomy homomorphism $\pi \to G$.

Development globalizes coordinate charts.

Holonomy globalizes coordinate changes.
Coordinate atlases and development

- **Geometry**: Homogeneous space $X = G/H$.
- **Topology**: Manifold $\Sigma$ with universal covering $\tilde{\Sigma} \to \Sigma$ and fundamental group $\pi$.
- **Marking**: Diffeomorphism $\Sigma \xrightarrow{f} M$; the geometry on $M$ varies, but the topology of $\Sigma$ remains fixed.
  - Patches $U \subset M$: Coordinate atlas of charts $U \to X$ defining local coordinates on $U$ modeled on $X$.
  - On overlapping patches, coordinate changes extend (locally uniquely) to transformations of $X$ from $G$.
  - Local charts define *development* immersion $\tilde{\Sigma} \hookrightarrow X$, equivariantly respecting *holonomy* homomorphism $\pi \to G$. 
Coordinate atlases and development

- **Geometry**: Homogeneous space $X = G/H$.
- **Topology**: Manifold $\Sigma$ with universal covering $\tilde{\Sigma} \rightarrow \Sigma$ and fundamental group $\pi$.
- **Marking**: Diffeomorphism $\Sigma \xrightarrow{f} M$; the geometry on $M$ varies, but the topology of $\Sigma$ remains fixed.
  - Patches $U \subset M$: Coordinate atlas of charts $U \rightarrow X$ defining local coordinates on $U$ modeled on $X$.
  - On overlapping patches, coordinate changes extend (locally uniquely) to transformations of $X$ from $G$.
  - Local charts define *development* immersion $\tilde{\Sigma} \hookrightarrow X$, equivariantly respecting *holonomy* homomorphism $\pi \rightarrow G$.
    - Development globalizes coordinate charts.
Coordinate atlases and development

- **Geometry:** Homogeneous space $X = G/H$.
- **Topology:** Manifold $\Sigma$ with universal covering $\tilde{\Sigma} \rightarrow \Sigma$ and fundamental group $\pi$.
- **Marking:** Diffeomorphism $\Sigma \xrightarrow{f} M$; the geometry on $M$ varies, but the topology of $\Sigma$ remains fixed.
  - Patches $U \subset M$: Coordinate atlas of charts $U \rightarrow X$ defining local coordinates on $U$ modeled on $X$.
  - On overlapping patches, coordinate changes extend (locally uniquely) to transformations of $X$ from $G$.
  - Local charts define *development* immersion $\tilde{\Sigma} \hookrightarrow X$, equivariantly respecting *holonomy* homomorphism $\pi \rightarrow G$.
    - Development globalizes coordinate charts.
    - Holonomy globalizes coordinate changes.
Ehresmann-Weil-Thurston principle
Ehresmann-Weil-Thurston principle

- Construct *deformation space* of marked \((G, X)\)-structures on \(\Sigma\) up to appropriate equivalence relation.
Ehresmann-Weil-Thurston principle

- Construct deformation space of marked $(G, X)$-structures on $\Sigma$ up to appropriate equivalence relation.
- Holonomy defines a mapping

\[ \text{Def}_{(G,X)}(\Sigma) \xrightarrow{\mathcal{H}} \text{Rep}(\pi, G) \]
Ehresmann-Weil-Thurston principle

- Construct *deformation space* of marked $(G, X)$-structures on $\Sigma$ up to appropriate equivalence relation.
- Holonomy defines a mapping

$$\text{Def}_{(G, X)}(\Sigma) \xrightarrow{\mathcal{H}} \text{Rep}(\pi, G)$$

- Best cases stratify into smooth manifolds and $\mathcal{H}$ “tries to be” local diffeomorphism (Thurston 1979).
Ehresmann-Weil-Thurston principle

- Construct *deformation space* of marked \((G, X)\)-structures on \(\Sigma\) up to appropriate equivalence relation.

- Holonomy defines a mapping

\[
\text{Def}_{(G,X)}(\Sigma) \xrightarrow{\mathcal{H}} \text{Rep}(\pi, G)
\]

- Best cases stratify into smooth manifolds and \(\mathcal{H}\) “tries to be” local diffeomorphism (Thurston 1979).

- Changing marking corresponds to action of *mapping class group* on \(\text{Def}_{(G,X)}(\Sigma)\).
Construct *deformation space* of marked \((G, X)\)-structures on \(\Sigma\) up to appropriate equivalence relation.

Holonomy defines a mapping

\[
\text{Def}_{(G, X)}(\Sigma) \xrightarrow{\mathcal{H}} \text{Rep}(\pi, G)
\]

Best cases stratify into smooth manifolds and \(\mathcal{H}\) “tries to be” local diffeomorphism (Thurston 1979).

Changing marking corresponds to action of *mapping class group* on \(\text{Def}_{(G, X)}(\Sigma)\).

Orbits comprise *moduli space* of (unmarked) \((G, X)\)-structures on \(\Sigma\).
Ehresmann-Weil-Thurston principle

- Construct *deformation space* of marked \((G, X)\)-structures on \(\Sigma\) up to appropriate equivalence relation.
- Holonomy defines a mapping
  \[
  \text{Def}_{(G,X)}(\Sigma) \xrightarrow{\mathcal{H}} \text{Rep}(\pi, G)
  \]
- Best cases stratify into smooth manifolds and \(\mathcal{H}\) “tries to be” local diffeomorphism (Thurston 1979).
- Changing marking corresponds to action of *mapping class group* on \(\text{Def}_{(G,X)}(\Sigma)\)
- Orbits comprise *moduli space* of (unmarked) \((G, X)\)-structures on \(\Sigma\).
  - Analogous to *Riemann space* \(\mathcal{M}(\Sigma) \leftrightarrow \mathcal{I}(\Sigma)/\text{Mod}(\Sigma)\).
Example of trivial (proper) dynamics: marked hyperbolic surfaces
Example of trivial (proper) dynamics: marked hyperbolic surfaces

\[ X = \mathbb{H}^2, \quad G = \text{Isom}(\mathbb{H}^2) \cong \text{PGL}(2, \mathbb{R}) : \]

Then \( \text{Def}(G, X)(\Sigma) \) is the Fricke space \( F(\Sigma) \leftarrow \rightarrow T(\Sigma) \).

Embedding \( F(\Sigma) \hookrightarrow \text{Rep}(\pi, G) \) as connected component:

Trivial dynamics:
The action of \( \text{Mod}(\Sigma) \) on \( F(\Sigma) \) is proper.

Quotient identifies with the Riemann moduli space \( M(\Sigma) \).

For \( \Sigma = T_2 \), the deformation space of unit-area Euclidean structures identifies with the upper half-plane \( \mathbb{H}^2 \).

\( \text{Modular group} \, \text{Mod}(\Sigma) \cong \text{GL}(2, \mathbb{Z}) \) acts properly by linear fractional transformations on \( \mathbb{H}^2 \).
Example of trivial (proper) dynamics: marked hyperbolic surfaces

- $X = \mathbb{H}^2$, $G = \text{Isom}(\mathbb{H}^2) \cong \text{PGL}(2, \mathbb{R})$:
  - Then $\text{Def}_{(G, \chi)}(\Sigma)$ is the Frick space $\mathfrak{F}(\Sigma) \leftrightarrow \mathcal{T}(\Sigma)$.

- Embedding $\mathfrak{F}(\Sigma) \hookrightarrow \text{Rep}(\pi, G)$ as connected component:
  - Trivial dynamics: action of $\text{Mod}(\Sigma)$ on $\mathfrak{F}(\Sigma)$ is proper.

- Quotient identifies with the Riemann moduli space $\mathcal{M}(\Sigma)$.

- For $\Sigma = T^2$, the deformation space of unit-area Euclidean structures identifies with the upper half-plane $\mathbb{H}^2$.

- Modular group $\text{Mod}(\Sigma) \cong \text{GL}(2, \mathbb{Z})$ acts properly by linear fractional transformations on $\mathbb{H}^2$. 
Example of trivial (proper) dynamics: marked hyperbolic surfaces

- $X = \mathbb{H}^2$, $G = \text{Isom}(\mathbb{H}^2) \cong \text{PGL}(2, \mathbb{R})$:
  - Then $\text{Def}_{(G, X)}(\Sigma)$ is the Fricke space $\mathcal{F}(\Sigma) \leftrightarrow \mathcal{I}(\Sigma)$.
  - Embedding $\mathcal{F}(\Sigma) \xrightarrow{\mathcal{H}} \text{Rep}(\pi, G)$ as connected component:
Example of trivial (proper) dynamics: marked hyperbolic surfaces

- $X = \mathbb{H}^2$, $G = \text{Isom}(\mathbb{H}^2) \cong \text{PGL}(2, \mathbb{R})$:
  - Then $\text{Def}_{(G,X)}(\Sigma)$ is the Fricke space $\mathcal{F}(\Sigma) \leftrightarrow \mathcal{T}(\Sigma)$.
  - Embedding $\mathcal{F}(\Sigma) \xrightarrow{\mathcal{H}} \text{Rep}(\pi, G)$ as connected component:
  - **Trivial dynamics**: Action of $\text{Mod}(\Sigma)$ on $\mathcal{F}(\Sigma)$ is proper.
Example of trivial (proper) dynamics: marked hyperbolic surfaces

- $X = \mathbb{H}^2$, $G = \text{Isom}(\mathbb{H}^2) \cong \text{PGL}(2, \mathbb{R})$:
  - Then $\text{Def}_{(G,\chi)}(\Sigma)$ is the Fricke space $\mathcal{F}(\Sigma) \hookrightarrow \mathcal{T}(\Sigma)$.
  - Embedding $\mathcal{F}(\Sigma) \overset{\mathcal{H}}{\longrightarrow} \text{Rep}(\pi, G)$ as connected component:

- **Trivial dynamics:** Action of $\text{Mod}(\Sigma)$ on $\mathcal{F}(\Sigma)$ is proper.
  - Quotient identifies with the *Riemann moduli space* $\mathcal{M}(\Sigma)$.
Example of trivial (proper) dynamics: marked hyperbolic surfaces

- \( X = \mathbb{H}^2, \ G = \text{Isom}(\mathbb{H}^2) \cong \text{PGL}(2, \mathbb{R}) \):
  - Then \( \text{Def}_{(G,X)}(\Sigma) \) is the Fricke space \( \mathcal{F}(\Sigma) \leftarrow \mathcal{T}(\Sigma) \).
  - Embedding \( \mathcal{F}(\Sigma) \xrightarrow{\mathcal{H}} \text{Rep}(\pi, G) \) as connected component:
  - **Trivial dynamics**: Action of \( \text{Mod}(\Sigma) \) on \( \mathcal{F}(\Sigma) \) is proper.
  - Quotient identifies with the Riemann moduli space \( \mathcal{M}(\Sigma) \).
- For \( \Sigma = T^2 \), the deformation space of unit-area Euclidean structures identifies with the upper half-plane \( \mathcal{H}^2 \).
Example of trivial (proper) dynamics: marked hyperbolic surfaces

\( X = H^2, \ G = \text{Isom}(H^2) \cong \text{PGL}(2, \mathbb{R}) : \)

\begin{itemize}
  \item Then \( \text{Def}_{(G,\chi)}(\Sigma) \) is the \textit{Fricke space} \( \mathcal{F}(\Sigma) \leftarrow \to \mathcal{T}(\Sigma) \).
  \item Embedding \( \mathcal{F}(\Sigma) \xrightarrow{\mathcal{H}} \text{Rep}(\pi, G) \) as connected component:
  \item \textit{Trivial dynamics}: Action of \( \text{Mod}(\Sigma) \) on \( \mathcal{F}(\Sigma) \) is proper.
    \item Quotient identifies with the \textit{Riemann moduli space} \( \mathcal{M}(\Sigma) \).
  \item For \( \Sigma = T^2 \), the deformation space of unit-area Euclidean structures identifies with the upper half-plane \( H^2 \).
    \item Modular group \( \text{Mod}(\Sigma) \cong \text{GL}(2, \mathbb{Z}) \) acts \textit{properly} by linear fractional transformations on \( H^2 \).
\end{itemize}
Examples of nonproper (interesting) dynamics

Proper (trivial) dynamics: $\text{PGL}(2, \mathbb{Z})$-action on $H^2$

For $\Sigma = T^2$, the deformation space of unit-area Euclidean structures is the upper half-plane $H^2$ with action the modular group $\text{Mod}(\Sigma) \cong \text{GL}(2, \mathbb{Z})$ acting properly by linear fractional transformations.

If $\chi(\Sigma) < 0$, my work with Suhyoung Choi implies $\text{Mod}(\Sigma)$ acts properly on the deformation space $\mathbb{R}P^2(\Sigma)$ of marked real projective structures.

In contrast, complete affine structures with usual linear action of $\text{GL}(2, \mathbb{Z})$. (O. Baues 2000).
Examples of nonproper (interesting) dynamics

Proper (trivial) dynamics: $\text{PGL}(2, \mathbb{Z})$-action on $H^2$

- For $\Sigma = T^2$, the deformation space of unit-area Euclidean structures is the upper half-plane $H^2$ with action the modular group $\text{Mod}(\Sigma) \cong \text{GL}(2, \mathbb{Z})$ acting properly by linear fractional transformations.
Examples of nonproper (interesting) dynamics

Proper (trivial) dynamics: $\text{PGL}(2, \mathbb{Z})$-action on $H^2$

- For $\Sigma = T^2$, the deformation space of unit-area Euclidean structures is the upper half-plane $H^2$ with action the modular group $\text{Mod}(\Sigma) \cong \text{GL}(2, \mathbb{Z})$ acting properly by linear fractional transformations.

- If $\chi(\Sigma) < 0$, my work with Suhyoung Choi implies $\text{Mod}(\Sigma)$ acts properly on the deformation space $\mathbb{RP}^2(S)$ of marked real projective structures.
Examples of nonproper (interesting) dynamics

Proper (trivial) dynamics: $\text{PGL}(2, \mathbb{Z})$-action on $\mathcal{H}^2$

- For $\Sigma = T^2$, the deformation space of unit-area Euclidean structures is the upper half-plane $\mathcal{H}^2$ with action the modular group $\text{Mod}(\Sigma) \cong \text{GL}(2, \mathbb{Z})$ acting properly by linear fractional transformations.
- If $\chi(\Sigma) < 0$, my work with Suhyoung Choi implies $\text{Mod}(\Sigma)$ acts properly on the deformation space $\mathbb{R}P^2(S)$ of marked real projective structures.
- In contrast, complete affine structures on with usual linear action of $\text{GL}(2, \mathbb{Z})$. (O. Baues 2000).
Complete affine surfaces

\[ T^2 \rightarrow \mathbb{R}^2 / \Lambda \] are all affinely isomorphic and correspond to the origin \( 0 \in \mathbb{R}^2 \).

Others obtained from the polynomial diffeomorphism

\[ \mathbb{R}^2 \phi \rightarrow \mathbb{R}^2 (x, y) \mapsto (x + y^2, y) \]

as \( T^2 \sim = \mathbb{R}^2 / \phi \Lambda \phi^{-1} \). (Kuiper 1950)

If translation \( \lambda(x, y) = (x + s, y + t) \) lies in the lattice \( \Lambda \), then \( (x, y) \phi \lambda \phi^{-1} \rightarrow (x + 2ty + (s + t^2), y + t) \) is affine.
Complete affine surfaces

Euclidean structures $T^2 \xrightarrow{f} \mathbb{R}^2 / \Lambda$ are all *affinely isomorphic* and correspond to the origin $0 \in \mathbb{R}^2$. 
Complete affine surfaces

Euclidean structures \( T^2 \xrightarrow{f} \mathbb{R}^2/\Lambda \) are all \textit{affinely isomorphic} and correspond to the origin \( 0 \in \mathbb{R}^2 \).

Others obtained from the \textit{polynomial diffeomorphism}

\[
\mathbb{R}^2 \xrightarrow{\phi} \mathbb{R}^2 \\
(x, y) \mapsto (x + y^2, y)
\]

as \( T^2 \xrightarrow{\cong} \mathbb{R}^2/\phi\Lambda\phi^{-1} \). (Kuiper 1950)
Complete affine surfaces

- Euclidean structures $T^2 \xrightarrow{f} \mathbb{R}^2/\Lambda$ are all *affinely isomorphic* and correspond to the origin $0 \in \mathbb{R}^2$.
- Others obtained from the *polynomial diffeomorphism*

$$\mathbb{R}^2 \xrightarrow{\phi} \mathbb{R}^2$$

$$(x, y) \longmapsto (x + y^2, y)$$

as $T^2 \xrightarrow{\cong} \mathbb{R}^2/\phi\Lambda\phi^{-1}$. (Kuiper 1950)

- If translation $\lambda(x, y) = (x + s, y + t)$ lies in the lattice $\Lambda$, then

$$\left(x, y\right) \xrightarrow{\phi\lambda\phi^{-1}} \left(x + 2ty + (s + t^2), y + t\right)$$

is affine.
The twisted cubic cone

Baues showed that these correspond to invariant affine structures on the torus as a Lie group.

\[
\begin{align*}
X : Y : Z : W 
\in \mathbb{R}^4 \mid XZ - Y^2 = YW - Z^2 = 0
\end{align*}
\]

Deligne (2021: This deformation space is naturally a twisted cubic cone \( \{ [X:Y:Z:W] \in \mathbb{R}^4 \mid XZ - Y^2 = YW - Z^2 = 0 \} \), the image of the \( \text{GL}(2,\mathbb{Z}) \)-equivariant Veronese embedding \( \mathbb{R}^2 \to \mathbb{R}^4 \) \( [x:y] \mapsto [x^3 : x^2 y : xy^2 : y^3] \).
The twisted cubic cone

Baues showed that these correspond to invariant affine structures on the torus as a Lie group.

Deligne (2021: This deformation space is naturally a twisted cubic cone

\[
\left\{ [X : Y : Z : W] \in \mathbb{R}^4 \mid XZ - Y^2 = YW - Z^2 = 0 \right\},
\]

the image of the \( \text{GL}(2, \mathbb{Z}) \)-equivariant Veronese embedding

\[
\begin{align*}
\mathbb{R}^2 & \longrightarrow \mathbb{R}^4 \\
\begin{bmatrix} x \\ y \end{bmatrix} & \mapsto \begin{bmatrix} x^3 \\ x^2y \\ xy^2 \\ y^3 \end{bmatrix}
\end{align*}
\]
Chaotic dynamics

The linear action of $\text{Mod}(\mathbb{T}^2) \approx \text{GL}(2, \mathbb{Z})$ on $\mathbb{R}^2$ is chaotic — no reasonable quotient.

Euclidean area on $\mathbb{R}^2$ is invariant.

(Moore 1966) Action is ergodic:

Every invariant function is a.e. constant.

Almost every orbit is dense.

... although discrete orbits exist, e.g. $\frac{1}{n} \mathbb{Z}^2$...

Therefore, the classification of geometric structures should be more insightfully regarded as a dynamical system, since the moduli space — its quotient — is often intractable.
Chaotic dynamics

The linear action of $\text{Mod}(T^2) \cong \text{GL}(2, \mathbb{Z})$ on $\mathbb{R}^2$ is chaotic — no reasonable quotient.

Therefore, the classification of geometric structures should be more insightfully regarded as a dynamical system, since the moduli space — its quotient — is often intractable.
Chaotic dynamics

- The linear action of $\text{Mod}(T^2) \cong \text{GL}(2, \mathbb{Z})$ on $\mathbb{R}^2$ is chaotic — no reasonable quotient.
  - Euclidean area on $\mathbb{R}^2$ is invariant.
Chaotic dynamics

- The linear action of $\text{Mod}(T^2) \cong \text{GL}(2, \mathbb{Z})$ on $\mathbb{R}^2$ is chaotic — no reasonable quotient.
  - Euclidean area on $\mathbb{R}^2$ is invariant.
  - (Moore 1966) Action is ergodic:
Chaotic dynamics

- The linear action of \( \text{Mod}(T^2) \cong \text{GL}(2, \mathbb{Z}) \) on \( \mathbb{R}^2 \) is chaotic — no reasonable quotient.
  - Euclidean area on \( \mathbb{R}^2 \) is invariant.
  - (Moore 1966) Action is \textit{ergodic}:
    - Every invariant function is a.e. constant.
Chaotic dynamics

- The linear action of $\text{Mod}(T^2) \cong \text{GL}(2, \mathbb{Z})$ on $\mathbb{R}^2$ is chaotic — no reasonable quotient.
  - Euclidean area on $\mathbb{R}^2$ is invariant.
  - (Moore 1966) Action is *ergodic*:
    - Every invariant function is a.e. constant.
    - Almost every orbit is dense.
Chaotic dynamics

- The linear action of $\text{Mod}(T^2) \cong \text{GL}(2, \mathbb{Z})$ on $\mathbb{R}^2$ is chaotic — no reasonable quotient.
  - Euclidean area on $\mathbb{R}^2$ is invariant.
  - (Moore 1966) Action is ergodic:
    - Every invariant function is a.e. constant.
    - Almost every orbit is dense.
    - ... although discrete orbits exist, e.g. $\frac{1}{n}\mathbb{Z}^2$ ...
Chaotic dynamics

- The linear action of $\text{Mod}(T^2) \cong \text{GL}(2, \mathbb{Z})$ on $\mathbb{R}^2$ is chaotic — no reasonable quotient.
  - Euclidean area on $\mathbb{R}^2$ is invariant.
  - (Moore 1966) Action is ergodic:
    - Every invariant function is a.e. constant.
    - Almost every orbit is dense.
    - ... although discrete orbits exist, e.g. $\frac{1}{n}\mathbb{Z}^2$ ...

- Therefore, the classification of geometric structures should be more insightfully regarded as a dynamical system, since the moduli space — its quotient — is often intractable.
Elements $\gamma \in \pi_1(\Sigma)$ define character functions on $\text{Rep}$:

$$\text{Rep}(\pi, G) \xrightarrow{\gamma} \mathbb{R} \left[ \rho \right] \rightarrow \mathbb{R} \left( \text{Tr} \rho(\gamma) \right)$$

with Hamiltonian vector fields $\text{Ham}(\gamma \rightarrow \mathbb{R})$.

For the Fricke-Teichmüller component when $G = \text{PSL}(2, \mathbb{R})$, $\gamma$ corresponding to a simple loop, $\text{Ham}(\gamma \rightarrow \mathbb{R})$ generates the Fenchel-Nielsen twist flows (Wolpert 1982).

For $G = \text{SL}(2)$, character functions $\gamma \rightarrow \mathbb{R}$ of simple $\gamma$ generate coordinate ring of $\text{Rep}(\pi, G)$. 

Character functions and Hamiltonian twist flows
Elements $\gamma \in \pi_1(\Sigma)$ define *character functions* on $\text{Rep}$:

$$\text{Rep}(\pi, G) \xrightarrow{f_\gamma} \mathbb{R}$$

$$[\rho] \mapsto \Re(\text{Tr}\rho(\gamma))$$

with Hamiltonian vector fields $\text{Ham}(f_\gamma)$. 
Elements $\gamma \in \pi_1(\Sigma)$ define character functions on $\text{Rep}$:

$$\text{Rep}(\pi, G) \xrightarrow{f_\gamma} \mathbb{R}$$

$$[\rho] \mapsto \Re(\text{Tr}\rho(\gamma))$$

with Hamiltonian vector fields $\text{Ham}(f_\gamma)$.

For the Fricke-Teichmüller component when $G = \text{PSL}(2, \mathbb{R})$, $\gamma$ corresponding to a simple loop, $\text{Ham}(f_\gamma)$ generates the Fenchel-Nielsen twist flows, (Wolpert 1982).
Character functions and Hamiltonian twist flows

Elements $\gamma \in \pi_1(\Sigma)$ define character functions on $\text{Rep}$:

$$\text{Rep}(\pi, G) \xrightarrow{f_\gamma} \mathbb{R}$$

$$[\rho] \mapsto \mathbb{R}(\text{Tr}\rho(\gamma))$$

with Hamiltonian vector fields $\text{Ham}(f_\gamma)$.

For the Fricke-Teichmüller component when $G = \text{PSL}(2, \mathbb{R})$, $\gamma$ corresponding to a simple loop, $\text{Ham}(f_\gamma)$ generates the Fenchel-Nielsen twist flows, (Wolpert 1982).

For $G = \text{SL}(2)$, character functions $f_\gamma$ of simple $\gamma$ generate coordinate ring of $\text{Rep}(\pi, G)$. 
Let $G = SU(2)$. Dehn twist $Tw_\gamma$ generates a lattice inside the $R$-action corresponding to $\text{Ham}(f_\gamma)$-orbits.

$\rho(\gamma) \in G_{\text{elliptic}} \Rightarrow$ Integral curves of $\text{Ham}(f_\gamma)$ are circles $C_\gamma \rho$.

For almost every value of $f_\gamma$, the Dehn twist $Tw_\gamma$ defines an ergodic translation of $C_\gamma \rho$.

**Ergodic decomposition**: Every $Tw_\gamma$-invariant function is a.e. $\text{Ham}(f_\gamma)$-invariant.

If $f_\gamma$ generate the coordinate ring of $\text{Rep}(\pi, G)$, their differentials $df_\gamma$ span every cotangent space.

$\text{Ham}(f_\gamma)$ span every tangent space.

Flows of $\text{Ham}(f_\gamma)$ generate a transitive action on each connected component of where the vector fields span.

Mod(\Sigma)-action ergodic on regions where simple loops have elliptic holonomy.
Hamiltonian flows and Dehn twists

Let $G = SU(2)$. Dehn twist $Tw_\gamma$ generates lattice inside $\mathbb{R}$-action corresponding to $\text{Ham}(f_\gamma)$-orbits.
Let $G = SU(2)$. Dehn twist $Tw_\gamma$ generates lattice inside $\mathbb{R}$-action corresponding to $\text{Ham}(f_\gamma)$-orbits.

$\gamma \in G$ elliptic $\implies$ Integral curves of $\text{Ham}(f_\gamma)$ are circles $C_{\rho^\gamma}$. 
Let $G = SU(2)$. Dehn twist $Tw_\gamma$ generates lattice inside $\mathbb{R}$-action corresponding to $\text{Ham}(f_\gamma)$-orbits.

1. $\rho(\gamma) \in G$ elliptic $\implies$ Integral curves of $\text{Ham}(f_\gamma)$ are circles $C_\rho^\gamma$.
2. For almost every value of $f_\gamma$, the Dehn twist $Tw_\gamma$ defines ergodic translation of $C_\rho^\gamma$. 

**Ergodic decomposition**: Every $Tw_\gamma$-invariant function is a.e. $\text{Ham}(f_\gamma)$-invariant.

**If** $f_\gamma$ generate the coordinate ring of $\text{Rep}(\pi, G)$, their differentials $df_\gamma$ span every cotangent space.

$\text{Ham}(f_\gamma)$ span every tangent space.

Flows of $\text{Ham}(f_\gamma)$ generate transitive action on each connected component of where the vector fields span.

**Mod(\Sigma)-action ergodic on regions where simple loops have elliptic holonomy.**
Hamiltonian flows and Dehn twists

- Let $G = \text{SU}(2)$. Dehn twist $\text{Tw}_\gamma$ generates lattice inside $\mathbb{R}$-action corresponding to $\text{Ham}(f_\gamma)$-orbits.
  - $\rho(\gamma) \in G$ elliptic $\implies$ Integral curves of $\text{Ham}(f_\gamma)$ are circles $C_\rho^\gamma$.
  - For almost every value of $f_\gamma$, the Dehn twist $\text{Tw}_\gamma$ defines ergodic translation of $C_\rho^\gamma$.

- *Ergodic decomposition*: Every $\text{Tw}_\gamma$-invariant function is a.e. $\text{Ham}(f_\gamma)$-invariant.
Let $G = \text{SU}(2)$. Dehn twist $\text{Tw}_\gamma$ generates lattice inside $\mathbb{R}$-action corresponding to $\text{Ham}(f_\gamma)$-orbits.

$\rho(\gamma) \in G$ elliptic $\implies$ Integral curves of $\text{Ham}(f_\gamma)$ are circles $C_\rho^\gamma$.

For almost every value of $f_\gamma$, the Dehn twist $\text{Tw}_\gamma$ defines ergodic translation of $C_\rho^\gamma$.

**Ergodic decomposition:** Every $\text{Tw}_\gamma$-invariant function is a.e. $\text{Ham}(f_\gamma)$-invariant.

If $f_\gamma$ generate the coordinate ring of $\text{Rep}(\pi, G)$, their differentials $df_\gamma$ span every cotangent space.
Hamiltonian flows and Dehn twists

- Let $G = SU(2)$. Dehn twist $\text{Tw}_\gamma$ generates lattice inside $\mathbb{R}$-action corresponding to $\text{Ham}(f_\gamma)$-orbits.
  - $\rho(\gamma) \in G$ elliptic $\implies$ Integral curves of $\text{Ham}(f_\gamma)$ are circles $C_{\rho}^\gamma$.
  - For almost every value of $f_\gamma$, the Dehn twist $\text{Tw}_\gamma$ defines ergodic translation of $C_{\rho}^\gamma$.

- **Ergodic decomposition:** Every $\text{Tw}_\gamma$-invariant function is a.e. $\text{Ham}(f_\gamma)$-invariant.
  - If $f_\gamma$ generate the coordinate ring of $\text{Rep}(\pi, G)$, their differentials $df_\gamma$ span every cotangent space.
  - $\text{Ham}(f_\gamma)$ span every tangent space.
Let $G = SU(2)$. Dehn twist $Tw_\gamma$ generates lattice inside $\mathbb{R}$-action corresponding to $\text{Ham}(f_\gamma)$-orbits.

- $\rho(\gamma) \in G$ elliptic $\implies$ Integral curves of $\text{Ham}(f_\gamma)$ are circles $C_{\rho\gamma}$.
- For almost every value of $f_\gamma$, the Dehn twist $Tw_\gamma$ defines ergodic translation of $C_{\rho\gamma}$.

**Ergodic decomposition:** Every $Tw_\gamma$-invariant function is a.e. $\text{Ham}(f_\gamma)$-invariant.

- If $f_\gamma$ generate the coordinate ring of $\text{Rep}(\pi, G)$, their differentials $df_\gamma$ span every cotangent space.
- $\text{Ham}(f_\gamma)$ span every tangent space.
- Flows of $\text{Ham}(f_\gamma)$ generate transitive action on each connected component of where the vector fields span.
Let $G = SU(2)$. Dehn twist $Tw_\gamma$ generates lattice inside $\mathbb{R}$-action corresponding to $\text{Ham}(f_\gamma)$-orbits.

- $\rho(\gamma) \in G$ elliptic $\implies$ Integral curves of $\text{Ham}(f_\gamma)$ are circles $C_\rho^\gamma$.
- For almost every value of $f_\gamma$, the Dehn twist $Tw_\gamma$ defines ergodic translation of $C_\rho^\gamma$.

**Ergodic decomposition:** Every $Tw_\gamma$-invariant function is a.e. $\text{Ham}(f_\gamma)$-invariant.

- If $f_\gamma$ generate the coordinate ring of $\text{Rep}(\pi, G)$, their differentials $df_\gamma$ span every cotangent space.
- $\text{Ham}(f_\gamma)$ span every tangent space.
- Flows of $\text{Ham}(f_\gamma)$ generate transitive action on each connected component of where the vector fields span.

- $\text{Mod}(\Sigma)$-action ergodic on regions where simple loops have elliptic holonomy.
Vogt-Fricke theorem and $F_2$

Let $F_2 = \langle X, Y \rangle$ be free of rank two. Then $\text{Hom}(F_2, \text{SL}(2)) \cong \text{SL}(2) \times \text{SL}(2)$ and $\text{Rep}(F_2, \text{SL}(2))$ is its quotient under $\text{Inn}(\text{SL}(2))$.

The $\text{Inn}(\text{SL}(2))$-invariant mapping $\text{Hom}(F_2, \text{SL}(2)) \rightarrow \mathbb{C}^3$ $\rho \mapsto \begin{bmatrix} \xi := \text{Tr}(\rho(X)) \\ \eta := \text{Tr}(\rho(Y)) \\ \zeta := \text{Tr}(\rho(XY)) \end{bmatrix}$ defines an isomorphism $\text{Rep}(F_2, \text{SL}(2)) \cong \mathbb{C}^3$. 
Vogt-Fricke theorem and $F_2$

Let $F_2 = \langle X, Y \rangle$ be free of rank two. Then

$$\text{Hom}(F_2, \text{SL}(2)) \cong \text{SL}(2) \times \text{SL}(2)$$

and $\text{Rep}(F_2, \text{SL}(2))$ is its quotient under $\text{Inn}(\text{SL}(2))$. 
Vogt-Fricke theorem and $F_2$

Let $F_2 = \langle X, Y \rangle$ be free of rank two. Then

$$\text{Hom}(F_2, \text{SL}(2)) \cong \text{SL}(2) \times \text{SL}(2)$$

and $\text{Rep}(F_2, \text{SL}(2))$ is its quotient under $\text{Inn}(\text{SL}(2))$.

The $\text{Inn}(\text{SL}(2))$-invariant mapping

$$\text{Hom}(F_2, \text{SL}(2)) \longrightarrow \mathbb{C}^3$$

$$\rho \longmapsto \begin{bmatrix} \xi := \text{Tr}(\rho(X)) \\ \eta := \text{Tr}(\rho(Y)) \\ \zeta := \text{Tr}(\rho(XY)) \end{bmatrix}$$

defines an isomorphism

$$\text{Rep}(F_2, \text{SL}(2)) \cong \mathbb{C}^3.$$
Boundary trace for the one-holed torus $\Sigma_{1,1}$
Boundary trace for the one-holed torus $\Sigma_{1,1}$

- $\text{Out}(F_2)$-invariant commutator trace function:

$$\text{Rep}(F_2, \text{SL}(2)) \cong \mathbb{C}^3 \xrightarrow{\kappa} \mathbb{C}$$

$$(\xi, \eta, \zeta) \mapsto \xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta - 2 = \text{Tr}[\rho(X), \rho(Y)]$$

- Nielsen: Every automorphism of $F_2$ maps $[X, Y]$ to a conjugate of itself or its inverse.

- Every homotopy-equivalence $\Sigma_{1,1} \xrightarrow{\sim} \Sigma_{1,1}$ is homotopic to a homeomorphism of $\Sigma_{1,1}$.

- $\text{Mod}(\Sigma) \cong \text{Out}(\pi)$, just like closed surfaces.

- $\text{Out}(F_2) \cong \text{GL}(2, \mathbb{Z}) = \text{Mod}(\Sigma_{1,1})$

- This isomorphism with $\mathbb{C}^3$ depends on a superbasis of $F_2$:

$$\text{Isomorphism } F_2 \cong \langle X, Y, Z | XYZ = 1 \rangle$$

- Superbases are vertices in the Markoff-Bowditch tree associated to the character variety of $F_2$. 
Boundary trace for the one-holed torus $\Sigma_{1,1}$

- Out($F_2$)-invariant commutator trace function:

$$\text{Rep}(F_2, SL(2)) \cong \mathbb{C}^3 \xrightarrow{\kappa} \mathbb{C}$$

$$(\xi, \eta, \zeta) \mapsto \xi^2 + \eta^2 + \zeta^2 - \xi \eta \zeta - 2 = \text{Tr}[\rho(X), \rho(Y)]$$

- (Nielsen): Every automorphism of $F_2$ maps $[X, Y]$ to a conjugate of itself or its inverse.
Boundary trace for the one-holed torus $\Sigma_{1,1}$

- $\text{Out}(F_2)$-invariant commutator trace function:
  
  \[
  \begin{align*}
  \text{Rep}(F_2, \text{SL}(2)) \cong \mathbb{C}^3 & \xrightarrow{\kappa} \mathbb{C} \\
  (\xi, \eta, \zeta) & \mapsto \xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta - 2 \\
  & = \text{Tr}[\rho(X), \rho(Y)]
  \end{align*}
  \]

- (Nielsen): Every automorphism of $F_2$ maps $[X, Y]$ to a conjugate of itself or its inverse.
  - Every homotopy-equivalence $\Sigma_{1,1} \rightsquigarrow \Sigma_{1,1}$ is homotopic to homeomorphism of $\Sigma_{1,1}$. 

- $\text{Mod}(\Sigma_{1,1}) \cong \text{Out}(\pi_1(\Sigma_{1,1}))$, just like closed surfaces.
- $\text{Out}(F_2) \cong \text{GL}(2, \mathbb{Z}) = \text{Mod}(\Sigma_{1,1})$.

- This isomorphism with $\mathbb{C}^3$ depends on a superbasis of $F_2$: an isomorphism $F_2 \cong \langle X, Y, Z \mid XYZ = 1 \rangle$.

- Superbases are vertices in the Markoff-Bowditch tree associated to the character variety of $F_2$. 

---


---
Boundary trace for the one-holed torus $\Sigma_{1,1}$

- $\text{Out}(F_2)$-invariant commutator trace function:

$$
\text{Rep}(F_2, \text{SL}(2)) \cong \mathbb{C}^3 \xrightarrow{\kappa} \mathbb{C}
$$

$$(\xi, \eta, \zeta) \mapsto \xi^2 + \eta^2 + \zeta^2 - \xi \eta \zeta - 2 = \text{Tr}[\rho(X), \rho(Y)]$$

- (Nielsen): Every automorphism of $F_2$ maps $[X, Y]$ to a conjugate of itself or its inverse.

- Every homotopy-equivalence $\Sigma_{1,1} \leadsto \Sigma_{1,1}$ is homotopic to homeomorphism of $\Sigma_{1,1}$.

- $\text{Mod}(\Sigma) \cong \text{Out}(\pi)$, just like closed surfaces.
Boundary trace for the one-holed torus $\Sigma_{1,1}$

- Out($F_2$)-invariant commutator trace function:
  \[
  \text{Rep}(F_2, \text{SL}(2)) \cong \mathbb{C}^3 \overset{\kappa}{\longrightarrow} \mathbb{C} \\
  (\xi, \eta, \zeta) \longmapsto \xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta - 2 = \text{Tr}[\rho(X), \rho(Y)]
  \]

- (Nielsen): Every automorphism of $F_2$ maps $[X, Y]$ to a conjugate of itself or its inverse.
  - Every homotopy-equivalence $\Sigma_{1,1} \sim \Sigma_{1,1}$ is homotopic to homeomorphism of $\Sigma_{1,1}$.
    - $\text{Mod}(\Sigma) \cong \text{Out}(\pi)$, just like closed surfaces.
  - Out($F_2$) $\cong \text{GL}(2, \mathbb{Z}) = \text{Mod}(\Sigma_{1,1})$
Boundary trace for the one-holed torus $\Sigma_{1,1}$

- Out($F_2$)-invariant commutator trace function:

$$\text{Rep}(F_2, \text{SL}(2)) \cong \mathbb{C}^3 \xrightarrow{\kappa} \mathbb{C}$$

$$(\xi, \eta, \zeta) \longmapsto \xi^2 + \eta^2 + \zeta^2 - \xi \eta \zeta - 2 = \text{Tr}[\rho(X), \rho(Y)]$$

- (Nielsen): Every automorphism of $F_2$ maps $[X, Y]$ to a conjugate of itself or its inverse.

- Every homotopy-equivalence $\Sigma_{1,1} \rightsquigarrow \Sigma_{1,1}$ is homotopic to homeomorphism of $\Sigma_{1,1}$.

- Mod($\Sigma$) $\cong$ Out($\pi$), just like closed surfaces.

- Out($F_2$) $\cong$ GL($2, \mathbb{Z}$) = Mod($\Sigma_{1,1}$)

- This isomorphism with $\mathbb{C}^3$ depends on a superbasis of $F_2$: an isomorphism $F_2 \cong \langle X, Y, Z \mid XYZ = 1 \rangle$. Superbases are vertices in the Markoff-Bowditch tree associated to the character variety of $F_2$. 

Boundary trace for the one-holed torus $\Sigma_{1,1}$

- Out($F_2$)-invariant commutator trace function:

$$\text{Rep}(F_2, \text{SL}(2)) \cong \mathbb{C}^3 \xrightarrow{\kappa} \mathbb{C}$$

$$(\xi, \eta, \zeta) \mapsto \xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta - 2 = \text{Tr}[\rho(X), \rho(Y)]$$

- (Nielsen): Every automorphism of $F_2$ maps $[X, Y]$ to a conjugate of itself or its inverse.
  - Every homotopy-equivalence $\Sigma_{1,1} \sim \Sigma_{1,1}$ is homotopic to homeomorphism of $\Sigma_{1,1}$.
    - $\text{Mod}(\Sigma) \cong \text{Out}(\pi)$, just like closed surfaces.
  - Out($F_2$) $\cong \text{GL}(2, \mathbb{Z}) = \text{Mod}(\Sigma_{1,1})$

- This isomorphism with $\mathbb{C}^3$ depends on a superbasis of $F_2$: an isomorphism $F_2 \cong \langle X, Y, Z \mid XYZ = 1 \rangle$.
  - Superbases are vertices in the Markoff-Bowditch tree associated to the character variety of $F_2$. 
Invariant Poisson structure

The boundary trace \( \kappa(\xi,\eta,\zeta) := \xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta - 2 \)
determines the Poisson structure on \( C^3 \) defined by bivector
\[
d\kappa \cdot \partial_\xi \wedge \partial_\eta \wedge \partial_\zeta = (2\xi - \eta\zeta) \partial_\eta \wedge \partial_\zeta + (2\eta - \zeta\xi) \partial_\zeta \wedge \partial_\xi + (2\zeta - \xi\eta) \partial_\xi \wedge \partial_\eta.
\]

Symplectic structure on level sets \( \kappa^{-1}(k) \) include:
- Weil-Petersson symplectic structure on Fricke spaces \( k \leq -2 \);
- Narasimhan-Atiyah-Bott structure for \( G = SU(2) \).

\( \kappa^{-1}(k) \) are the relative character varieties.
Invariant Poisson structure

The boundary trace

\[ \kappa(\xi, \eta, \zeta) := \xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta - 2 \]

determines the Poisson structure on \( \mathbb{C}^3 \) defined by bivector

\[ d\kappa \cdot \partial_\xi \wedge \partial_\eta \wedge \partial_\zeta \]
\[ = (2\xi - \eta\zeta) \partial_\eta \wedge \partial_\zeta \]
\[ + (2\eta - \zeta\xi) \partial_\zeta \wedge \partial_\xi \]
\[ + (2\zeta - \xi\eta) \partial_\xi \wedge \partial_\eta. \]
Invariant Poisson structure

The boundary trace

$$\kappa(\xi, \eta, \zeta) := \xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta - 2$$

determines the Poisson structure on $\mathbb{C}^3$ defined by bivector

$$\begin{align*}
d\kappa \cdot \partial_\xi &\wedge \partial_\eta \wedge \partial_\zeta \\
&= (2\xi - \eta\zeta) \partial_\eta \wedge \partial_\zeta \\
&\quad + (2\eta - \zeta\xi) \partial_\zeta \wedge \partial_\xi \\
&\quad + (2\zeta - \xi\eta) \partial_\xi \wedge \partial_\eta.
\end{align*}$$

Symplectic structure on level sets $\kappa^{-1}(k)$ include:

1. Weil-Petersson symplectic structure on Fricke spaces $k \leq -2$.
2. Narasimhan-Atiyah-Bott structure for $G = SU(2)$. 
3. $\kappa^{-1}(k)$ are the relative character varieties.
Invariant Poisson structure

- The boundary trace

\[ \kappa(\xi, \eta, \zeta) := \xi^2 + \eta^2 + \zeta^2 - \xi \eta \zeta - 2 \]

determines the Poisson structure on \( \mathbb{C}^3 \) defined by bivector

\[ d\kappa \cdot \partial_\xi \wedge \partial_\eta \wedge \partial_\zeta \]

\[ = (2\xi - \eta \zeta) \partial_\eta \wedge \partial_\zeta \]

\[ + (2\eta - \zeta \xi) \partial_\zeta \wedge \partial_\xi \]

\[ + (2\zeta - \xi \eta) \partial_\xi \wedge \partial_\eta. \]

- Symplectic structure on level sets \( \kappa^{-1}(k) \) include:
  - Weil-Petersson symplectic structure on Fricke spaces \( k \leq -2 \);
Invariant Poisson structure

The boundary trace

\[ \kappa(\xi, \eta, \zeta) := \xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta - 2 \]

determines the Poisson structure on \( \mathbb{C}^3 \) defined by bivector

\[
d\kappa \cdot \partial_\xi \wedge \partial_\eta \wedge \partial_\zeta \\
= (2\xi - \eta\zeta) \partial_\eta \wedge \partial_\zeta \\
+ (2\eta - \zeta\xi) \partial_\zeta \wedge \partial_\xi \\
+ (2\zeta - \xi\eta) \partial_\xi \wedge \partial_\eta.
\]

Symplectic structure on level sets \( \kappa^{-1}(k) \) include:

- Weil-Petersson symplectic structure on Fricke spaces \( k \leq -2 \);
- Narasimhan-Atiyah-Bott structure for \( G = SU(2) \).
Invariant Poisson structure

- The boundary trace

\[ \kappa(\xi, \eta, \zeta) := \xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta - 2 \]

determines the Poisson structure on \( \mathbb{C}^3 \) defined by bivector

\[
d\kappa \cdot \partial_\xi \wedge \partial_\eta \wedge \partial_\zeta \\
= (2\xi - \eta\zeta) \partial_\eta \wedge \partial_\zeta \\
+ (2\eta - \zeta\xi) \partial_\zeta \wedge \partial_\xi \\
+ (2\zeta - \xi\eta) \partial_\xi \wedge \partial_\eta.
\]

- Symplectic structure on level sets \( \kappa^{-1}(k) \) include:
  - Weil-Petersson symplectic structure on Fricke spaces \( k \leq -2 \);
  - Narasimhan-Atiyah-Bott structure for \( G = SU(2) \).

\( \kappa^{-1}(k) \) are the relative character varieties.
Vieta involutions

Nonlinear automorphisms of

\[ \kappa(\xi, \eta, \zeta) = \xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta - 2 = k \]
generated by involutions:

\[
\begin{pmatrix}
\xi \\
\eta \\
\zeta
\end{pmatrix} \mapsto
\begin{pmatrix}
\eta \zeta \\
-\xi \\
\eta \zeta
\end{pmatrix},
\]

\[
\begin{pmatrix}
\xi \\
\eta \\
\zeta
\end{pmatrix} \mapsto
\begin{pmatrix}
\xi \\
\xi \zeta \\
-\eta \zeta
\end{pmatrix},
\]

\[
\begin{pmatrix}
\xi \\
\eta \\
\zeta
\end{pmatrix} \mapsto
\begin{pmatrix}
\xi \\
\eta \\
\xi \eta \zeta \\
-\zeta
\end{pmatrix}
\]
Vieta involutions

▶ Nonlinear automorphisms of

\[ \kappa(\xi, \eta, \zeta) = \xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta - 2 = k \]

generated by involutions:

\[
\begin{bmatrix}
\xi \\
\eta \\
\zeta
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\eta\zeta - \xi \\
\eta \\
\zeta
\end{bmatrix},
\begin{bmatrix}
\xi \\
\eta \\
\zeta
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\xi \\
\xi\zeta - \eta \\
\zeta
\end{bmatrix},
\begin{bmatrix}
\xi \\
\eta \\
\zeta
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\xi \\
\eta \\
\xi\eta - \zeta
\end{bmatrix}
\]
Vieta involutions

- Nonlinear automorphisms of

\[ \kappa(\xi, \eta, \zeta) = \xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta - 2 = k \]

generated by *involutions*:

\[
\begin{pmatrix}
\xi \\
\eta \\
\zeta
\end{pmatrix}
\mapsto
\begin{pmatrix}
\eta\zeta - \xi \\
\eta \\
\zeta
\end{pmatrix},
\begin{pmatrix}
\xi \\
\eta \\
\zeta
\end{pmatrix}
\mapsto
\begin{pmatrix}
\xi \\
\xi\zeta - \eta \\
\zeta
\end{pmatrix},
\begin{pmatrix}
\xi \\
\eta \\
\zeta
\end{pmatrix}
\mapsto
\begin{pmatrix}
\xi \\
\eta \\
\xi\eta - \zeta
\end{pmatrix}
\]

- Coordinate projections \( \mathbb{C}^3 \to \mathbb{C}^2 \) branched double coverings; involutions are deck transformations.
Vieta involutions

- Nonlinear automorphisms of

\[ \kappa(\xi, \eta, \zeta) = \xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta - 2 = k \]

generated by involutions:

\[
\begin{bmatrix}
\xi \\
\eta \\
\zeta
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\eta\zeta - \xi \\
\eta \\
\zeta
\end{bmatrix},
\begin{bmatrix}
\xi \\
\eta \\
\zeta
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\xi \\
\xi\zeta - \eta \\
\zeta
\end{bmatrix},
\begin{bmatrix}
\xi \\
\eta \\
\zeta
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\xi \\
\eta \\
\xi\eta - \zeta
\end{bmatrix}
\]

- Coordinate projections \( \mathbb{C}^3 \rightarrow \mathbb{C}^2 \) branched double coverings; involutions are deck transformations.
  - Fixing \( \eta \) and \( \zeta \) yields quadratic equation in \( \xi \);

\[ \xi^2 - (\eta\zeta) \xi = k + 2 - \eta^2 - \zeta^2 \]

whose roots \( \xi \) and \( \xi' = \eta\zeta - \xi \) sum to linear coefficient \( \eta\zeta \).
Cayley cubic $\xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta = 4$
Cayley cubic \( \xi^2 + \eta^2 + \zeta^2 - \xi \eta \zeta = 4 \)

- Reducible representations correspond precisely to \( \kappa^{-1}(2) \).
Cayley cubic $\xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta = 4$

- Reducible representations correspond precisely to $\kappa^{-1}(2)$.
- Quotient of $\mathbb{C}^* \times \mathbb{C}^*$ by the involution

$$(a, b) \mapsto (a^{-1}, b^{-1}).$$

$$\xi = a + a^{-1}, \quad \eta = b + b^{-1}, \quad \zeta = ab + (ab)^{-1}$$
Cayley cubic $\xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta = 4$

- Reducible representations correspond precisely to $\kappa^{-1}(2)$.
- Quotient of $\mathbb{C}^* \times \mathbb{C}^*$ by the involution

$$(a, b) \mapsto (a^{-1}, b^{-1}).$$

$\xi = a + a^{-1}, \quad \eta = b + b^{-1}, \quad \zeta = ab + (ab)^{-1}$

- Homogeneous dynamics: $\text{GL}(2, \mathbb{Z})$-action on $(\mathbb{C}^* \times \mathbb{C}^*)/ (\mathbb{Z}/2)$. 

R-points: Unitary representations

- Points correspond to representations into $\mathbb{R}$-forms of $\text{SL}(2)$: either $\text{SL}(2, \mathbb{R})$ or $\text{SU}(2)$.

- Characters in $[-2, 2]$ with $\kappa \leq 2 \leftrightarrow \text{SU}(2)$-representations.
\( \mathbb{R} \)-points: Unitary representations

- \( \mathbb{R} \)-points correspond to representations into \( \mathbb{R} \)-forms of SL(2): either \( SL(2, \mathbb{R}) \) or \( SU(2) \).
\(\mathbb{R}\)-points: Unitary representations

- \(\mathbb{R}\)-points correspond to representations into \(\mathbb{R}\)-forms of SL(2): either SL(2, \(\mathbb{R}\)) or SU(2).
- Characters in \([-2, 2]^3\) with \(\kappa \leq 2\) \(\longleftrightarrow\) SU(2)-representations.
R-points: Hyperbolic structures on 3-holed spheres

Hyperbolic three-holed spheres parametrized by boundary lengths $\ell_X, \ell_Y, \ell_Z \geq 0$

$\xi := -2 \cosh(\ell_X/2) \leq -2$

$\eta := -2 \cosh(\ell_Y/2) \leq -2$

$\zeta := -2 \cosh(\ell_Z/2) \leq -2$

Necessarily $k = \kappa(\xi, \eta, \zeta) \geq 18 = \kappa(-2, -2, -2)$.

$(-2, -2, -2)$ corresponds to the complete finite-area 3-punctured sphere.

Homotopy-equivalences $\Sigma_1, 1 \Rightarrow \Sigma_0, 3$ (and other surfaces with $\pi_1 \sim F_2$) form wandering domains for $\Out(F_2)$-action.
Hyperbolic three-holed spheres parametrized by boundary lengths $\ell_X, \ell_Y, \ell_Z \geq 0$

$\xi := -2 \cosh \left( \frac{\ell_X}{2} \right) \leq -2$

$\eta := -2 \cosh \left( \frac{\ell_Y}{2} \right) \leq -2$

$\zeta := -2 \cosh \left( \frac{\ell_Z}{2} \right) \leq -2$

$\kappa(\xi, \eta, \zeta) \geq 18 = \kappa(-2, -2, -2)$

$(-2, -2, -2)$ corresponds to the complete finite-area 3-punctured sphere.

Homotopy-equivalences $\Sigma_1, 1 \Rightarrow \Sigma_0, 3$ (and other surfaces with $\pi_1 \sim F_2$) form wandering domains for $\text{Out}(F_2)$-action.
Hyperbolic three-holed spheres parametrized by boundary lengths $\ell_X, \ell_Y, \ell_Z \geq 0$

$$\xi := -2 \cosh \left( \frac{\ell_X}{2} \right) \leq -2$$
$$\eta := -2 \cosh \left( \frac{\ell_Y}{2} \right) \leq -2$$
$$\zeta := -2 \cosh \left( \frac{\ell_Z}{2} \right) \leq -2$$

Necessarily $k = \kappa(\xi, \eta, \zeta) \geq 18 = \kappa(-2, -2, -2)$. 

$(−2, −2, −2)$ corresponds to the complete finite-area 3-punctured sphere.
Hyperbolic three-holed spheres parametrized by boundary lengths $\ell_X, \ell_Y, \ell_Z \geq 0$

$$\xi := -2 \cosh \left( \frac{\ell_X}{2} \right) \leq -2$$
$$\eta := -2 \cosh \left( \frac{\ell_Y}{2} \right) \leq -2$$
$$\zeta := -2 \cosh \left( \frac{\ell_Z}{2} \right) \leq -2$$

Necessarily $k = \kappa(\xi, \eta, \zeta) \geq 18 = \kappa(-2, -2, -2)$.

$(-2, -2, -2)$ corresponds to the complete finite-area 3-punctured sphere.
Hyperbolic three-holed spheres parametrized by boundary lengths $\ell_X, \ell_Y, \ell_Z \geq 0$

\[
\xi := -2 \cosh \left( \frac{\ell_X}{2} \right) \leq -2
\]

\[
\eta := -2 \cosh \left( \frac{\ell_Y}{2} \right) \leq -2
\]

\[
\zeta := -2 \cosh \left( \frac{\ell_Z}{2} \right) \leq -2
\]

Necessarily $k = \kappa(\xi, \eta, \zeta) \geq 18 = \kappa(-2, -2, -2)$.

$(-2, -2, -2)$ corresponds to the complete finite-area 3-punctured sphere.

Homotopy-equivalences $\Sigma_{1,1} \sim \Sigma_{0,3}$ (and other surfaces with $\pi_1 \cong F_2$) form wandering domains for $\text{Out}(F_2)$-action.
Example: The Markoff surface $x^2 + y^2 + z^2 = xyz$

$\mathbb{R}^3 \cap \kappa^{-1}(-2)$ parametrizes hyperbolic structures on the punctured torus. The origin $(0, 0, 0)$ corresponds to the unique SU(2)-representation with $k = -2$. The famous Markoff triples correspond to triply symmetric hyperbolic punctured tori.
Fricke orbits define wandering domains for $k > 2$. For $k \leq 18$, action ergodic. For $k > 18$, action ergodic on complement. The level surface $k = 18$ extends to the famous Clebsch diagonal surface in $\mathbb{C}P^3$ defined by:

$$(X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5 = X_0 + X_1 + X_2 + X_3 + X_4 = 0)$$
Fricke orbits define wandering domains for $k > 2$

- Homotopy equivalences $\Sigma_{1,1} \sim \Sigma_{0,3}$ define embeddings $\tilde{\mathcal{S}}(\Sigma_{0,3})_k \hookrightarrow \kappa^{-1}(k)$ for $k > 18$;
Fricke orbits define wandering domains for $k > 2$

- Homotopy equivalences $\Sigma_{1,1} \sim \Sigma_{0,3}$ define embeddings $\tilde{\mathfrak{G}}(\Sigma_{0,3})_k \hookrightarrow \kappa^{-1}(k)$ for $k > 18$;
- For $k \leq 18$, action ergodic.
Fricke orbits define wandering domains for $k > 2$

- Homotopy equivalences $\Sigma_{1,1} \sim \Sigma_{0,3}$ define embeddings $\mathcal{F}(\Sigma_{0,3})_k \leftrightarrow \kappa^{-1}(k)$ for $k > 18$;
- For $k \leq 18$, action ergodic.
- For $k > 18$, action ergodic on complement.

The level surface $k = 18$ extends to the famous Clebsch diagonal surface in $\mathbb{C}P^3$ defined by:

$$\left(\frac{X_0}{X_1}\right)^5 + \left(\frac{X_2}{X_1}\right)^5 + \left(\frac{X_3}{X_1}\right)^5 + \left(\frac{X_4}{X_1}\right)^5 = \frac{X_0}{X_1} + \frac{X_2}{X_1} + \frac{X_3}{X_1} + \frac{X_4}{X_1} = 0$$

in homogeneous coordinates.
Fricke orbits define wandering domains for $k > 2$

- Homotopy equivalences $\Sigma_{1,1} \sim \Sigma_{0,3}$ define embeddings $\mathcal{F}(\Sigma_{0,3}) \hookrightarrow \kappa^{-1}(k)$ for $k > 18$;
- For $k \leq 18$, action ergodic.
- For $k > 18$, action ergodic on complement.
- The level surface $k = 18$ extends to the famous *Clebsch diagonal surface* in $\mathbb{CP}^3$ defined by:

$$
(X_0)^5 + (X_1)^5 + (X_2)^5 + (X_3)^5 + (X_4)^5 = X_0 + X_1 + X_2 + X_3 + X_4 = 0
$$

in homogeneous coordinates.
\[ x^2 + y^2 + z^2 - xyz = 20 \]
Ergodicity for compact/noncompact groups

Mod(Σ)-action ergodic on each component \( \text{Rep}(\pi, G) \) with respect to the symplectic measure \( \nu \). (G-, Pickrell-Xia)

Ergodic: Only vectors in \( L^2(\text{Rep}(\pi, G)) \) fixed by Mod(Σ) are constants.

Weak-mixing: Only finite-dimensional Mod(Σ)-invariant subspaces on \( L^2(\text{Rep}(\pi, G)) \) are constants.

Other examples of chaotic dynamics occur, even when \( G \) is noncompact: (Marché-Wolff 2016) For \( G = \text{PSL}(2, \mathbb{R}) \) and in genus 2, three types of components:

- Euler class \( \pm 2 \) (maximal): Fuchsian representations, proper Mod(Σ)-action;
- Euler class \( \pm 1 \) ergodic Mod(Σ)-action;
- Euler class 0 component singular; two ergodic components.

Main technique for proving ergodicity uses dynamics of Dehn twists in Mod(Σ).
Ergodicity for compact/noncompact groups

- Mod(Σ)-action ergodic on each component $\text{Rep}(\pi, G)_{\tau}$ with respect to the symplectic measure $\nu$. (G-, Pickrell-Xia)
Ergodicity for compact/noncompact groups

- Mod(Σ)-action ergodic on each component \( \text{Rep}(\pi, G)_\tau \) with respect to the symplectic measure \( \nu \). (G-, Pickrell-Xia)
Ergodicity for compact/noncompact groups

- Mod(Σ)-action ergodic on each component Rep(π, G)τ with respect to the symplectic measure ν. (G-, Pickrell-Xia)
  - Ergodic: Only vectors in $L^2(\text{Rep}(\pi, G)τ)$ fixed by Mod(Σ) are constants.

- Other examples of chaotic dynamics occur, even when G is noncompact: (Marché-Wolff 2016) For $G = \text{PSL}(2, \mathbb{R})$ and in genus 2, three types of components:
  - Euler class $±2$ (maximal): Fuchsian representations, proper Mod(Σ)-action;
  - Euler class $±1$ ergodic Mod(Σ)-action;
  - Euler class 0 component singular; two ergodic components.

- Main technique for proving ergodicity uses dynamics of Dehn twists in Mod(Σ).
Ergodicity for compact/noncompact groups

- Mod(\(\Sigma\))-action ergodic on each component \(\text{Rep}(\pi, G)_\tau\) with respect to the symplectic measure \(\nu\). (G-, Pickrell-Xia)
  - Ergodic: Only vectors in \(L^2(\text{Rep}(\pi, G)_\tau)\) fixed by Mod(\(\Sigma\)) are constants.
  - Weak-mixing: Only finite-dimensional Mod(\(\Sigma\))-invariant subspaces on \(L^2(\text{Rep}(\pi, G)_\tau)\) are constants.

- Other examples of chaotic dynamics occur, even when \(G\) is noncompact: (Marché-Wolff 2016) For \(G = \text{PSL}(2, \mathbb{R})\) and in genus 2, three types of components:
  - Euler class \(\pm 2\) (maximal): Fuchsian representations, proper Mod(\(\Sigma\))-action;
  - Euler class \(\pm 1\) ergodic Mod(\(\Sigma\))-action;
  - Euler class 0 component singular; two ergodic components.

- Main technique for proving ergodicity uses dynamics of Dehn twists in Mod(\(\Sigma\)).
Ergodicity for compact/noncompact groups

- Mod(Σ)-action ergodic on each component \( \text{Rep}(\pi, G)_\tau \) with respect to the symplectic measure \( \nu \). (G-, Pickrell-Xia)
  - **Ergodic:** Only vectors in \( L^2(\text{Rep}(\pi, G)_\tau) \) fixed by \( \text{Mod}(\Sigma) \) are constants.
  - **Weak-mixing:** Only finite-dimensional \( \text{Mod}(\Sigma) \)-invariant subspaces on \( L^2(\text{Rep}(\pi, G)_\tau) \) are constants.

- Other examples of chaotic dynamics occur, even when \( G \) is noncompact: (Marché-Wolff 2016) For \( G = \text{PSL}(2, \mathbb{R}) \) and in genus 2, three types of components:
  - Euler class \( \pm 2 \) (maximal): Fuchsian representations, proper \( \text{Mod}(\Sigma) \)-action;
  - Euler class \( \pm 1 \) ergodic \( \text{Mod}(\Sigma) \)-action;
  - Euler class 0 component singular; two ergodic components.

- Main technique for proving ergodicity uses dynamics of Dehn twists in \( \text{Mod}(\Sigma) \).
Ergodicity for compact/noncompact groups

- Mod(Σ)-action ergodic on each component $\text{Rep}(\pi, G)_\tau$ with respect to the symplectic measure $\nu$. (G-, Pickrell-Xia)
  - Ergodic: Only vectors in $L^2(\text{Rep}(\pi, G)_\tau)$ fixed by $\text{Mod}(\Sigma)$ are constants.
  - Weak-mixing: Only finite-dimensional $\text{Mod}(\Sigma)$-invariant subspaces on $L^2(\text{Rep}(\pi, G)_\tau)$ are constants.

- Other examples of chaotic dynamics occur, even when $G$ is noncompact: (Marché-Wolff 2016) For $G = \text{PSL}(2, \mathbb{R})$ and in genus 2, three types of components:
  - Euler class $\pm 2$ (maximal): Fuchsian representations, proper $\text{Mod}(\Sigma)$-action;
Ergodicity for compact/noncompact groups

- Mod(Σ)-action ergodic on each component $\text{Rep}(\pi, G)_{\tau}$ with respect to the symplectic measure $\nu$. (G-, Pickrell-Xia)
  - Ergodic: Only vectors in $L^2(\text{Rep}(\pi, G)_{\tau})$ fixed by Mod(Σ) are constants.
  - Weak-mixing: Only finite-dimensional Mod(Σ)-invariant subspaces on $L^2(\text{Rep}(\pi, G)_{\tau})$ are constants.

- Other examples of chaotic dynamics occur, even when $G$ is noncompact: (Marché-Wolff 2016) For $G = \text{PSL}(2, \mathbb{R})$ and in genus 2, three types of components:
  - Euler class $\pm 2$ (maximal): Fuchsian representations, proper Mod(Σ)-action;
  - Euler class $\pm 1$ ergodic Mod(Σ)-action;
Ergodicity for compact/noncompact groups

- Mod(Σ)-action ergodic on each component $\text{Rep}(\pi, G)_{\tau}$ with respect to the symplectic measure $\nu$. (G-, Pickrell-Xia)
  - **Ergodic**: Only vectors in $L^2(\text{Rep}(\pi, G)_{\tau})$ fixed by Mod(Σ) are constants.
  - **Weak-mixing**: Only finite-dimensional Mod(Σ)-invariant subspaces on $L^2(\text{Rep}(\pi, G)_{\tau})$ are constants.

- Other examples of chaotic dynamics occur, even when $G$ is noncompact: (Marché-Wolff 2016) For $G = \text{PSL}(2, \mathbb{R})$ and in genus 2, three types of components:
  - Euler class $\pm 2$ (maximal): Fuchsian representations, proper Mod(Σ)-action;
  - Euler class $\pm 1$ ergodic Mod(Σ)-action;
  - Euler class 0 component singular; two ergodic components.

- Main technique for proving ergodicity uses dynamics of Dehn twists in Mod(Σ).
Ergodicity for compact/noncompact groups

- Mod(Σ)-action ergodic on each component $\text{Rep}(\pi, G)_{\tau}$ with respect to the symplectic measure $\nu$. (G-, Pickrell-Xia)
  - **Ergodic:** Only vectors in $L^2(\text{Rep}(\pi, G)_{\tau})$ fixed by Mod(Σ) are constants.
  - **Weak-mixing:** Only finite-dimensional Mod(Σ)-invariant subspaces on $L^2(\text{Rep}(\pi, G)_{\tau})$ are constants.

- Other examples of chaotic dynamics occur, even when $G$ is noncompact: (Marché-Wolff 2016) For $G = \text{PSL}(2, \mathbb{R})$ and in genus 2, three types of components:
  - Euler class $\pm 2$ (maximal): Fuchsian representations, proper Mod(Σ)-action;
  - Euler class $\pm 1$ ergodic Mod(Σ)-action;
  - Euler class 0 component singular; two ergodic components.

- Main technique for proving ergodicity uses dynamics of Dehn twists in Mod(Σ).
The universal moduli space over Teichmüller space.
The universal moduli space over Teichmüller space.

- Flat bundle $E_G(\Sigma)$ over $M(\Sigma)$ with fibers $\text{Rep}(\pi, G)$ parametrizes these structures as Riemann surface $M$ varies:

$$E_G(\Sigma) := (\mathcal{T}(\Sigma) \times \text{Rep}(\pi, G))/\text{Mod}(\Sigma)$$

$$M(\Sigma) := \mathcal{T}(\Sigma)/\text{Mod}(\Sigma)$$
The universal moduli space over Teichmüller space.

- Flat bundle $\mathcal{E}_G(\Sigma)$ over $\mathcal{M}(\Sigma)$ with fibers $\text{Rep}(\pi, G)$ parametrizes these structures as Riemann surface $M$ varies:

$$\mathcal{E}_G(\Sigma) := (\mathcal{T}(\Sigma) \times \text{Rep}(\pi, G)) / \text{Mod}(\Sigma)$$

$$\mathcal{M}(\Sigma) := \mathcal{T}(\Sigma) / \text{Mod}(\Sigma)$$

- Leaves of horizontal foliation $\mathcal{F}_G(\Sigma) := [\mathcal{T}(\Sigma) \times \{[\rho]\}]$ correspond to $\text{Mod}(\Sigma)$-orbit $\text{Mod}(\Sigma)[\rho]$ on $\text{Rep}(\pi, G)$. 
The universal moduli space over Teichmüller space.

- Flat bundle $\mathcal{E}_G(\Sigma)$ over $\mathcal{M}(\Sigma)$ with fibers $\text{Rep}(\pi, G)$ parametrizes these structures as Riemann surface $M$ varies:

$$\mathcal{E}_G(\Sigma) := (\mathcal{T}(\Sigma) \times \text{Rep}(\pi, G))/\text{Mod}(\Sigma)$$

$$\mathcal{M}(\Sigma) := \mathcal{T}(\Sigma)/\text{Mod}(\Sigma)$$

- Leaves of horizontal foliation $\mathcal{F}_G(\Sigma) := [\mathcal{T}(\Sigma) \times \{[\rho]\}]$ correspond to $\text{Mod}(\Sigma)$-orbit $\text{Mod}(\Sigma)[\rho]$ on $\text{Rep}(\pi, G)$.

- Dynamics of $\mathcal{F}_G(\Sigma)$ equivalent to dynamics of action of discrete group $\text{Mod}(\Sigma)$. 
Extending Teichmüller geodesic flow

Replace dynamics of $\text{Mod}(\Sigma)$ on $\text{Rep}(\pi, G)$ by a measure-preserving flow $\Phi$ on flat bundle $U_E G(\Sigma)$.

Teichmüller unit sphere bundle $U_M(\Sigma)$ over $M(\Sigma)$ with Teichmüller geodesic flow $\phi$: $U_M(\Sigma) := U_T(\Sigma)/\text{Mod}(\Sigma)$ invariantly stratified; strata fall into components $U_\sigma M(\Sigma)$.

Insert $\text{Rep}(\pi, G)$ as the fiber: $U_\sigma \tau E G(\Sigma) := (U_\sigma T(\Sigma) \times \text{Rep}(\pi, G)) / \text{Mod}(\Sigma)$ and horizontally lift $\phi$ to flow $\Phi$ on $U_\sigma \tau E G(\Sigma)$.

Mod($\Sigma$)-dynamics on $\text{Rep}(\pi, G)$ replaced by equivalent action of more tractable (continuous) groups $R$ and $\text{SL}(2, R)$.

(Forni – G) For $G$ compact, $\Phi$ strongly mixing on $U_\sigma \tau E G(\Sigma)$:

$\mu(g t (A) \cap B) \to \mu(A) \mu(B)$ for $A$, $B$ measurable and $g t \to \infty$. 

Extending Teichmüller geodesic flow

- Replace dynamics of $\text{Mod}(\Sigma)$ on $\text{Rep}(\pi, G)_T$ by a measure-preserving flow $\Phi$ on flat bundle $U\mathcal{E}_G(\Sigma)$.
Extending Teichmüller geodesic flow

- Replace dynamics of $\text{Mod}(\Sigma)$ on $\text{Rep}(\pi, G)_\tau$ by a measure-preserving flow $\Phi$ on flat bundle $U\mathcal{E}_G(\Sigma)$.

- **Teichmüller unit sphere bundle** $U\mathcal{M}(\Sigma)$ over $\mathcal{M}(\Sigma)$ with Teichmüller geodesic flow $\phi$:

  $$ U\mathcal{M}(\Sigma) := U\mathcal{T}(\Sigma)/\text{Mod}(\Sigma) $$
  $$ \mathcal{M}(\Sigma) := \mathcal{T}(\Sigma)/\text{Mod}(\Sigma) $$

  invariantly stratified; strata fall into components $U^\sigma\mathcal{M}(\Sigma)$. 

- Mod$(\Sigma)$-dynamics on $\text{Rep}(\pi, G)_\tau$ replaced by equivalent action of more tractable (continuous) groups $\mathbb{R}$ and $\text{SL}(2, \mathbb{R})$.

- (Forni – G) For $G$ compact, $\Phi$ strongly mixing on $U\sigma\mathcal{M}(\Sigma)$:

  $$ \mu(gt(A) \cap B) \to \mu(A) \mu(B) $$ for $A, B$ measurable and $g_t \to \infty$. 

Extending Teichmüller geodesic flow

- Replace dynamics of \( \text{Mod}(\Sigma) \) on \( \text{Rep}(\pi, G)_\tau \) by a measure-preserving flow \( \Phi \) on flat bundle \( \mathcal{U} \mathcal{E}_G(\Sigma) \).

- **Teichmüller unit sphere bundle** \( \mathcal{U} \mathcal{M}(\Sigma) \) over \( \mathcal{M}(\Sigma) \) with Teichmüller geodesic flow \( \phi \):

  \[
  \mathcal{U} \mathcal{M}(\Sigma) := \mathcal{U} \mathcal{I}(\Sigma) / \text{Mod}(\Sigma)
  
  \mathcal{M}(\Sigma) := \mathcal{I}(\Sigma) / \text{Mod}(\Sigma)
  \]

  invariantly stratified; strata fall into components \( \mathcal{U}^\sigma \mathcal{M}(\Sigma) \).

- Insert \( \text{Rep}(\pi, G)_\tau \) as the fiber:

  \[
  \mathcal{U}^\sigma \mathcal{E}_G(\Sigma) := (\mathcal{U}^\sigma \mathcal{I}(\Sigma) \times \text{Rep}(\pi, G)_\tau) / \text{Mod}(\Sigma)
  \]

  and horizontally lift \( \phi \) to flow \( \Phi \) on \( \mathcal{U}^\sigma \mathcal{E}_G(\Sigma) \).
Extending Teichmüller geodesic flow

- Replace dynamics of $\text{Mod}(\Sigma)$ on $\text{Rep}(\pi, G)_\tau$ by a measure-preserving flow $\Phi$ on flat bundle $U\mathcal{E}_G(\Sigma)$.
  - *Teichmüller unit sphere bundle* $U\mathcal{M}(\Sigma)$ over $\mathcal{M}(\Sigma)$ with Teichmüller geodesic flow $\phi$:

    $$U\mathcal{M}(\Sigma) := U\mathcal{I}(\Sigma)/\text{Mod}(\Sigma)$$
    $$\mathcal{M}(\Sigma) := \mathcal{I}(\Sigma)/\text{Mod}(\Sigma)$$

    invariantly stratified; strata fall into components $U^\sigma\mathcal{M}(\Sigma)$.

  - Insert $\text{Rep}(\pi, G)_\tau$ as the fiber:

    $$U^\sigma_\tau\mathcal{E}_G(\Sigma) := \left( U^\sigma\mathcal{I}(\Sigma) \times \text{Rep}(\pi, G)_\tau \right)/\text{Mod}(\Sigma)$$

    and horizontally lift $\phi$ to flow $\Phi$ on $U^\sigma_\tau\mathcal{E}_G(\Sigma)$.

- $\text{Mod}(\Sigma)$-dynamics on $\text{Rep}(\pi, G)$ replaced by equivalent action of more tractable (continuous) groups $\mathbb{R}$ and $\text{SL}(2, \mathbb{R})$. 
Extending Teichmüller geodesic flow

- Replace dynamics of $\text{Mod}(\Sigma)$ on $\text{Rep}(\pi, G)_{\tau}$ by a measure-preserving flow $\Phi$ on flat bundle $U\mathcal{E}_G(\Sigma)$.

  - *Teichmüller unit sphere bundle* $U\mathcal{M}(\Sigma)$ over $\mathcal{M}(\Sigma)$ with Teichmüller geodesic flow $\phi$:

    \[
    U\mathcal{M}(\Sigma) := U\mathcal{I}(\Sigma)/\text{Mod}(\Sigma)
    \]
    \[
    \mathcal{M}(\Sigma) := \mathcal{I}(\Sigma)/\text{Mod}(\Sigma)
    \]

    invariantly stratified; strata fall into components $U^\sigma\mathcal{M}(\Sigma)$.

  - Insert $\text{Rep}(\pi, G)_{\tau}$ as the fiber:

    \[
    U^\sigma_\tau \mathcal{E}_G(\Sigma) := (U^\sigma\mathcal{I}(\Sigma) \times \text{Rep}(\pi, G)_{\tau})/\text{Mod}(\Sigma)
    \]

    and horizontally lift $\phi$ to flow $\Phi$ on $U^\sigma_\tau \mathcal{E}_G(\Sigma)$.

  - $\text{Mod}(\Sigma)$-dynamics on $\text{Rep}(\pi, G)$ replaced by equivalent action of more tractable (continuous) groups $\mathbb{R}$ and $\text{SL}(2, \mathbb{R})$.

  - (Forni – G) For $G$ compact, $\Phi$ *strongly mixing* on $U^\sigma_\tau \mathcal{E}_G(\Sigma)$:
Extending Teichmüller geodesic flow

- Replace dynamics of Mod(Σ) on $\text{Rep}(\pi, G)_{\tau}$ by a measure-preserving flow $\Phi$ on flat bundle $U\mathcal{E}_G(\Sigma)$.
  - Teichmüller unit sphere bundle $U\mathcal{M}(\Sigma)$ over $\mathcal{M}(\Sigma)$ with Teichmüller geodesic flow $\phi$:
    
    $$U\mathcal{M}(\Sigma) := U\mathcal{T}(\Sigma)/\text{Mod}(\Sigma)$$
    $$\mathcal{M}(\Sigma) := \mathcal{T}(\Sigma)/\text{Mod}(\Sigma)$$

    invariantly stratified; strata fall into components $U^\sigma\mathcal{M}(\Sigma)$.
  - Insert $\text{Rep}(\pi, G)_{\tau}$ as the fiber:
    
    $$U^\sigma\mathcal{E}_G(\Sigma) := (U^\sigma\mathcal{T}(\Sigma) \times \text{Rep}(\pi, G)_{\tau})/\text{Mod}(\Sigma)$$

    and horizontally lift $\phi$ to flow $\Phi$ on $U^\sigma\mathcal{E}_G(\Sigma)$.
  - Mod(Σ)-dynamics on $\text{Rep}(\pi, G)$ replaced by equivalent action of more tractable (continuous) groups $\mathbb{R}$ and $\text{SL}(2, \mathbb{R})$.
  - (Forni – G) For $G$ compact, $\Phi$ strongly mixing on $U^\sigma\mathcal{E}_G(\Sigma)$:
    
    $$\mu(g_t(A) \cap B) \to \mu(A)\mu(B)$$ for $A, B$ measurable and $g_t \to \infty$. 
Happy birthday, Giovanni!