MATH 431-2018 PROBLEM SET 1

DUE THURSDAY 6 SEPTEMBER 2018

(1) For each of the following vectors

$$\mathbf{v} = \begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}3\\4\\0\end{bmatrix}, \begin{bmatrix}0\\5\\6\end{bmatrix}, \begin{bmatrix}1\\3\\-1\end{bmatrix},$$

compute the intersection of the line $\mathbb{R}\mathbf{v}$ with the planes $z = 1$, the plane $x = 1$, and the plane $x + y + z = 1$.

(2) Compute all the common solutions of the following pairs of (inhomogeneous) linear equations:

(a) $3x + 1 = 0, \qquad 2y - 4 = 0.$

(b) $x + y = 1, \qquad x - y = 3.$

(c) $x + 3y = 1/2, \qquad x/3 + y = 1/6.$

(d) $x + 3y = 1/2, \qquad 2x + 6y = -1.$

(3) Compute all nine of the cross-products of the following (standard basic) vectors:

$$\mathbf{i} := \begin{bmatrix}1\\0\\0\end{bmatrix}, \qquad \mathbf{j} := \begin{bmatrix}0\\1\\0\end{bmatrix}, \qquad \mathbf{k} := \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

(4) Consider the three vectors and the 3×3 matrix having the vectors as columns:

$$\mathbf{a} = \begin{bmatrix}1\\2\\0\end{bmatrix}, \ \mathbf{b} = \begin{bmatrix}3\\4\\0\end{bmatrix}, \ \mathbf{c} = \begin{bmatrix}0\\5\\6\end{bmatrix},$$

$$\mathbf{X} = \begin{bmatrix}\mathbf{a} & \mathbf{b} & \mathbf{c}\end{bmatrix} = \begin{bmatrix}1 & 3 & 0\\2 & 4 & 5\\0 & 0 & 6\end{bmatrix}.$$

(a) Compute the cross products $\mathbf{a} \times \mathbf{b}$, $\mathbf{b} \times \mathbf{c}$, $\mathbf{c} \times \mathbf{a}$.

(b) Compute the dot products $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$, $(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}$, $(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}$.

(c) Compute the determinant $\mathsf{Det}(\mathbf{X})$.

(5) Let $\mathbf{v}, \mathbf{w}, \mathbf{u} \in \mathbb{R}^3$ be vectors. Show that both mappings

$$(\mathbf{v}, \mathbf{w}, \mathbf{u}) \longmapsto (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u}$$
$$(\mathbf{v}, \mathbf{w}, \mathbf{u}) \longmapsto \mathsf{Det}(\mathbf{v}, \mathbf{w}, \mathbf{u})$$

are alternating, mapping which agree on the standard basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Are they equal?

(6) If X is a $m \times n$ matrix, its *transpose* $X^\dagger$ is the $n \times m$ matrix whose rows are the columns of X, and whose columns are the rows of X. A matrix $X \in \mathsf{Mat}_n$ is *symmetric* if and only if $X = X^\dagger$, *skew-symmetric* if and only if $X = -X^\dagger$. It is *invertible* if it has (necessarily unique) inverse X^{-1} such that $XX^{-1} = X^{-1}X = \mathbf{I}$. It is *orthogonal* if $XX^\dagger = \mathbf{I}$, that is, if $X^{-1} = X^\dagger$.

Which of the following matrices are invertible, symmetric, skew-symmetric or orthogonal?

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

(7) Prove that $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if

$$\mathsf{Det}(M) := ad - bc$$

is *nonzero*. In that case its inverse is:

$$M^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

(8) Let $A \in \mathsf{Mat}_n$ be an *invertible* $n \times n$ matrix. Consider the transformation

$$\mathsf{Mat}_n \xrightarrow{\phi} \mathsf{Mat}_n$$
$$X \longmapsto AXA^{-1}$$

(a) Suppose that $X, Y \in \mathsf{Mat}_n$ and $s \in \mathbb{R}$. Show that:
  (i) $\phi(X + Y) = \phi(X) + \phi(Y)$.
  (ii) $\phi(sX) = s\phi(X)$.
  (iii) $\phi(XY) = \phi(X)\phi(Y)$.

(b) For another invertible matrix B, define:

$$\mathsf{Mat}_n \xrightarrow{\psi} \mathsf{Mat}_n$$
$$X \longmapsto BXB^{-1}.$$

Show that the composition $\phi \circ \psi$ maps $X \longmapsto CXC^{-1}$ for some invertible C. Compute C and C^{-1} in terms of A, B and their inverses.

(9) For which of the following matrices A does $A^2 = -\mathbf{I}$ (where $\mathbf{I}$ denotes the identity matrix)?

(a) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$,

(b) $\begin{bmatrix} 0 & 1/2 \\ -2 & 0 \end{bmatrix}$,

(c) $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

(d) $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$,

(e) $\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$.

(10) For each integer $n \geq 0$, determine $a(n), b(n), c(n), d(n)$ where:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}^n = \begin{bmatrix} a(n) & b(n) \\ c(n) & d(n) \end{bmatrix}$$

$\Big($Hint: first do the case for the diagonal matrix $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.$\Big)$

(11) Let

$$M := \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \mathbf{J} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

(a) Determine the condition that $M\mathbf{J} = \mathbf{J}M$ in terms of a, b, c, d.
(b) Relate the matrix $M^\dagger \mathbf{J} M$ to $\mathsf{Det}(M)$.

(12) Let α, β be linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$ defined by matrices

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

respectively. Compute the matrices AB and BA and the compositions $\alpha \circ \beta$ and $\beta \circ \alpha$. Identify which composition corresponds to AB and which composition corresponds to BA.

(13) Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$ be an arbitrary vector in $\mathbb{R}^3$.

Which of the following functions $\mathbb{R}^3 \xrightarrow{f} \mathbb{R}$ are *covectors,* that is, linear functionals? For each covector, write down the row vector ϕ to which it corresponds (that is,

$$f(\mathbf{v}) = \phi \mathbf{v} = \phi^\dagger \cdot \mathbf{v},$$

matrix multiplication of the row vector ϕ by the column $\mathbf{v}$.

(a) $f(x, y, z) = x + 1$;
(b) $f(x, y, z) = y$;
(c) $f(x, y, z) = x - z$;
(d) $f(x, y, z) = xy$;
(e) $f(x, y, z) = x + y + e^z$.
(f) $f(x, y, z) = x + y + z$.

(14) Which of the following functions $\mathbb{R}^2 \times \mathbb{R}^2 \xrightarrow{B} \mathbb{R}$ are *bilinear*? For those, find the corresponding 2×2 matrix $\mathcal{B}$ such that

$$B(\mathbf{v}, \mathbf{w}) = \mathbf{v}^\dagger \mathcal{B} \mathbf{w}.$$

(a) $B(\mathbf{v}, \mathbf{w}) = v_1 w_2 - v_2 w_1$;
(b) $B(\mathbf{v}, \mathbf{w}) = v_1 w_1$;
(c) $B(\mathbf{v}, \mathbf{w}) = v_1 (w_1)^2$;
(d) $B(\mathbf{v}, \mathbf{w}) = v_1 v_2$;
(e) $B(\mathbf{v}, \mathbf{w}) = v_1 w_1 + 2 v_2$;
(f) $B(\mathbf{v}, \mathbf{w}) = v_1 w_1 + w_2 v_2$;
(g) $B(\mathbf{v}, \mathbf{w}) = v_1 + 1$;
(h) $B(\mathbf{v}, \mathbf{w}) = v_1 - w_1$.