MATH 431-2018 PROBLEM SET 2

DUE THURSDAY 20 SEPTEMBER 2018

(1) Find all the (possibly) complex solutions of the following equations:

$$z^2 + 1 = 0, \qquad z^2 - 2z + 2 = 0, \qquad z^3 + 1 = 0.$$

(2) We explore the relationship between the real vector space $\mathbb{R}^2$ and the field $\mathbb{C}$ of complex numbers. We first write complex numbers $z, w \in \mathbb{C}$ in terms of their *real* and *imaginary* parts x, y and u, v, respectively (where $x, y, u, v \in \mathbb{R}$):

$$\mathbf{z} = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad z = x + iy$$

$$\mathbf{w} = \begin{bmatrix} u \\ v \end{bmatrix}, \qquad w = u + iv.$$

That is, $\mathbf{z}, \mathbf{w}$ are *vectors* in $\mathbb{R}^2$ and z, w are the corresponding *complex scalars.* We write $\Phi(z) = \mathbf{z}$ and $\Phi(w) = \mathbf{w}$, where

$$\mathbb{C} \xrightarrow{\Phi} \mathbb{R}^2$$

is an isomorphism of *real* vector spaces. Φ takes the *usual basis* $(1, i)$ of $\mathbb{C}$ to the *coordinate basis* of $\mathbb{R}^2$:

$$1 \overset{\Phi}{\longmapsto} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$i \overset{\Phi}{\longmapsto} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(a) Express the vector sum $\mathbf{z} + \mathbf{w}$, the dot product $\mathbf{z} \cdot \mathbf{w}$ and

$$\mathbf{z} \wedge \mathbf{w} = \mathsf{Det}\big(\mathbf{z}, \mathbf{w}\big) = \mathsf{Det} \begin{bmatrix} x & u \\ y & v \end{bmatrix} := \begin{vmatrix} x & u \\ y & v \end{vmatrix}$$

in terms of the complex numbers $z, w, \overline{z}, \overline{w}$ and complex addition and multiplication.

(b) Multiplication by z is a linear map $\mathbb{C} \xrightarrow{\mathbf{Mult}_z} \mathbb{C}^2$. Express $\mathbf{Mult}_z$ as a 2×2 matrix $M_z \in \mathsf{Mat}_2$.

(c) Express $\mathsf{Det}(M_z)$ in terms of z and $\bar{z}$, and in terms of the absolute value (or length or magnitude) $|z|$.

(d) Prove that if $z \in \mathbb{C}$ is nonzero, then z^{-1} exists. That is, there is some $w \in \mathbb{C}$ such that $zw = wz = 1$. Show that w is uniqu.
(e) Show that matrix $M_{z^{-1}}$ corresponding to $z^{-1} \in \mathbb{C}$ is the inverse of the matrix M_z corresponding to z.
(f) What is Φ^{-1}?
(g) Relate the composition $\Phi \circ \mathbf{Mult}_z \circ \Phi^{-1}$ to M_z.

(3) Use Euler's formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

to prove the *addition formulas* in trigonometry:

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$
$$\sin(\alpha + \beta) = \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta)$$

from the *law of exponents:*

$$e^{x+y} = e^x e^y.$$

Interpret the complex product $e^{i\alpha}e^{i\beta}$ as a *matrix product.*

(4) Prove that the set of translations forms a *transformation group:*

(5) Which of the following $\mathbb{E}^2 \xrightarrow{T} \mathbb{E}^2$ are *isometries?* Which are *affine*? Determine which ones preserve or reverse orientation.
(a) $T(x, y) = (x, y + 1)$.
(b) $T(x, y) = (x + xy, y - x)$.
(c) $T(x, y) = (y, x)$.
(d) $T(x, y) = (3x, 4y)$
(e) $T(x, y) = (3x, -4y)$
(f) $T(x, y) = (-3x, -4y)$
(g) $T(x, y) = (4/5\ x\ -\ 3/5\ y,\ 3/5\ x\ +\ 4/5\ y\ -\ 6)$.
(h) $T(x, y) = (4/5\ x\ -\ 3/5\ y,\ 4/5\ x\ -\ 3/5\ y\ -\ 6)$.

(6) For which complex numbers $\zeta, \xi \in \mathbb{C}$ is the mapping $z \longmapsto \zeta z + \xi$ an isometry? For which $\zeta \in \mathbb{C}$ is the mapping $z \longmapsto \zeta\bar{z} + \xi$ an isometry?