(1) Find all the (possibly) complex solutions of the following equations:

\[ z^2 + 1 = 0, \quad z^2 - 2z + 2 = 0, \quad z^3 + 1 = 0. \]

(2) We explore the relationship between the real vector space \( \mathbb{R}^2 \) and the field \( \mathbb{C} \) of complex numbers. We first write complex numbers \( z, w \in \mathbb{C} \) in terms of their real and imaginary parts \( x, y, u, v \) (where \( x, y, u, v \in \mathbb{R} \)):

\[
\begin{align*}
z &= \begin{bmatrix} x \\ y \end{bmatrix}, & z &= x + iy \\
w &= \begin{bmatrix} u \\ v \end{bmatrix}, & w &= u + iv.
\end{align*}
\]

That is, \( z, w \) are vectors in \( \mathbb{R}^2 \) and \( z, w \) are the corresponding complex scalars. We write \( \Phi(z) = z \) and \( \Phi(w) = w \), where

\[
\mathbb{C} \xrightarrow{\Phi} \mathbb{R}^2
\]

is an isomorphism of real vector spaces. \( \Phi \) takes the usual basis \( (1, i) \) of \( \mathbb{C} \) to the coordinate basis of \( \mathbb{R}^2 \):

\[
\begin{align*}
1 & \xrightarrow{\Phi} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
i & \xrightarrow{\Phi} \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{align*}
\]

(a) Express the vector sum \( z + w \), the dot product \( z \cdot w \) and

\[
z \wedge w = \text{Det}(z, w) = \text{Det} \begin{bmatrix} x & u \\ y & v \end{bmatrix} := \begin{vmatrix} x & u \\ y & v \end{vmatrix}
\]

in terms of the complex numbers \( z, w, \bar{z}, \bar{w} \) and complex addition and multiplication.

(b) Multiplication by \( z \) is a linear map \( \mathbb{C} \xrightarrow{\text{Mult}_z} \mathbb{C}^2 \). Express \( \text{Mult}_z \) as a \( 2 \times 2 \) matrix \( M_z \in \text{Mat}_2 \).

(c) Express \( \text{Det}(M_z) \) in terms of \( z \) and \( \bar{z} \), and in terms of the absolute value (or length or magnitude) \( |z| \).
(d) Prove that if \( z \in \mathbb{C} \) is nonzero, then \( z^{-1} \) exists. That is, there is some \( w \in \mathbb{C} \) such that \( zw = wz = 1 \). Show that \( w \) is unique.

(e) Show that matrix \( M_{z^{-1}} \) corresponding to \( z^{-1} \in \mathbb{C} \) is the inverse of the matrix \( M_z \) corresponding to \( z \).

(f) What is \( \Phi^{-1} \)?

(g) Relate the composition \( \Phi \circ \text{Mult}_z \circ \Phi^{-1} \) to \( M_z \).

(3) Use Euler’s formula

\[ e^{i\theta} = \cos(\theta) + i\sin(\theta) \]

to prove the addition formulas in trigonometry:

\[
\begin{align*}
\cos(\alpha + \beta) &= \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta) \\
\sin(\alpha + \beta) &= \cos(\alpha) \sin(\beta) + \sin(\alpha) \cos(\beta)
\end{align*}
\]

from the law of exponents:

\[ e^{x+y} = e^x e^y. \]

Interpret the complex product \( e^{i\alpha} e^{i\beta} \) as a matrix product.

(4) Prove that the set of translations forms a transformation group:

(5) Which of the following \( \mathbb{E}^2 \xrightarrow{T} \mathbb{E}^2 \) are isometries? Which are affine? Determine which ones preserve or reverse orientation.

(a) \( T(x, y) = (x, y + 1) \).

(b) \( T(x, y) = (x + xy, y - x) \).

(c) \( T(x, y) = (y, x) \).

(d) \( T(x, y) = (3x, 4y) \)

(e) \( T(x, y) = (3x, -4y) \)

(f) \( T(x, y) = (-3x, -4y) \)

(g) \( T(x, y) = (4/5 x - 3/5 y, 3/5 x + 4/5 y - 6) \).

(h) \( T(x, y) = (4/5 x - 3/5 y, 4/5 x - 3/5 y - 6) \).

(6) For which complex numbers \( \zeta, \xi \in \mathbb{C} \) is the mapping \( z \mapsto \zeta z + \xi \) an isometry? For which \( \zeta \in \mathbb{C} \) is the mapping \( z \mapsto \zeta \overline{z} + \xi \) an isometry?