MODULI SPACES

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This article summarizes a series of talks presented at the workshop held at the Korea National University of Education at Ch'onju on July 11, 1994. The talks were divided into segments of 50, 40 and 20 minutes each. I decided to lecture on this material partially to provide background for Ronnie Lee's later lectures at this conference. In particular I wanted to describe some of the symplectic geometry which Professor Lee uses in his discussion of the Casson invariant.

For a more detailed survey of this material see my article "Geometric structures and varieties of representations," in Contemporary Mathematics 1990, published by the American Mathematical Society.

1. Formal properties and basic examples

The first lecture dealt with the formal properties of such moduli spaces and a few basic examples. Spaces of homomorphisms give rise to a way in which topology and geometry are wedded via algebra. Topology enters through a closed oriented surface S, or rather through its fundamental group π. Geometry enters through a classical homogeneous space X, or rather through its automorphism group G. The space $Hom(\pi, G)$ of homomorphisms is a natural object closely related to geometric structures on S and flat connections on G-bundles on M.

This general construction admits rich formal symmetries: $\mathrm{Aut}(\pi) \times \mathrm{Aut}(G)$ on $\mathrm{Hom}(\pi, G)$ by

$$\begin{array}{ccccccc} (\,\mathrm{Aut}(\pi) \times & \mathrm{Aut}(G)) \times & \mathrm{Hom}(\pi, G) & \longrightarrow & \mathrm{Hom}(\pi, G) \\ (\quad \phi, & h) \quad \cdot & \rho & \longmapsto & h \circ \rho \circ \phi^{-1} \end{array}$$

In practice we form the quotient by the inner automorphisms in $\mathrm{Aut}(G)$ to obtain a deformation space $\mathrm{Hom}(\pi, G)/G$ whose points parametrize flat principal G-bundles. The inner automorphisms of π act trivially on this quotient so the resulting action on $\mathrm{Hom}(\pi, G)/G$ factors through the action of the *outer automorphism group* $\mathrm{Out}(\pi)$.

A basic problem is to determine the geometric structure on $\mathrm{Hom}(\pi, G)/G$ which is invariant under $\mathrm{Out}(\pi)$.

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1.1. The basic examples. When π is the fundamental group of a closed Riemann surface M, then $\mathrm{Out}(\pi)$ is the mapping class group $\pi_0(Diff(M))$. In this case different choices of Lie group G produce diverse behavior of moduli space geometry.

$G = \mathbb{R}$: Maybe the simplest Lie group is the additive group of real numbers $\mathbb{R}$. In this case the moduli space is $H^1(M;\mathbb{R})$. Its natural structure is that of a *symplectic vector space* under the symplectic form given by:

$$\begin{aligned} H^1(M;\mathbb{R}) \times H^1(M;\mathbb{R}) &\longrightarrow \mathbb{R} \\ (\alpha, \qquad\qquad \beta) \qquad &\longmapsto (\alpha \cup \beta) \cap [M] \end{aligned}$$

The mapping class group acts (non-effectively) via a homomorphism

$$\mathrm{Out}(\pi) \longrightarrow \mathrm{Sp}(\mathbb{Z}^{2g}) \subset \mathrm{Sp}(\mathbb{R}^{2g}).$$

When M is given a complex structure, then the moduli space inherits the structure of a *Hermitian vector space:* we can represent elements of $H^1(M,\mathbb{R})$ by harmonic 1-forms and the Hodge star operator (which takes a harmonic 1-form to its harmonic conjugate 1-form) is a complex structure on the space of harmonic 1-forms. However, this structure is *not* invariant under the mapping class group.

$G = U(1)$: When G is the circle group, then the above remarks apply as well. Now the moduli space identifies with the Jacobi variety of M, when M is given the structure of a Riemann surface. This space parametrizes topologically trivial holomorphic line bundles over M and is a Kähler manifold. Although the Kähler structure (in particular the complex structure) varies with the complex structure on M, the *symplectic structure* is a topological invariant of M.

$G = U(n)$: Study of the higher rank case was initiated by Narasimhan and Seshadri who showed the moduli space of flat $U(n)$-bundles over M is the moduli space parametrizing equivalence classes of semistable holomorphic $\mathbb{C}^n$-bundles over M. In particular they showed this moduli space has the structure of a complex projective variety, and this structure varies as one changes the complex structure on M. However, its symplectic structure is independent of the complex structure, and the mapping class group acts by symplectomorphisms on this moduli space. This space was reinterpreted by Atiyah and Bott as the moduli space of solutions of the Yang-Mills equations over a Riemann surface. Their work emphasizes the topological invariance of the symplectic structure. The mapping class group acts effectively on this space, preserving the symplectic structure (and hence a volume form). For $G = SU(2)$ this action is ergodic. Perhaps the mapping class group acts ergodically on $\mathrm{Hom}(\pi, G)/G$ for any compact Lie group G,.

$G = \mathrm{PSL}(2,\mathbb{R})$: Representations of the fundamental group arise from *uniformizations* of M — representations of M as quotients of the hyperbolic plane $\mathbf{H}^2$ by discrete groups. The relevant representations are embeddings of π onto discrete subgroups of the group $G \cong \mathrm{PSL}(2,\mathbb{R})$ of orientation-preserving isometries of $\mathbf{H}^2$. The moduli space of such representations is the Teichmüller space $\mathfrak{T}_M$ of M, a $6(g-1)$-dimensional cell constituting a connected component of $\mathrm{Hom}(\pi, G)/G$. The corresponding symplectic structure on $\mathfrak{T}_M$ is associated to the Weil-Petersson Kähler structure and is independent of the complex structure of M. The topological flavor of this symplectic structure is evident in Wolpert's work on the symplectic geometry of $\mathfrak{T}_M$: Wolpert showed that the Fenchel-Nielsen twist flows on $\mathfrak{T}_M$ were Hamiltonian (although they are not holomorphic). Furthermore the Fenchel-Nielsen coordinates on $\mathfrak{T}_M$ are canonical coordinates (global Darboux coordinates) for $\mathfrak{T}_M$ as a symplectic manifold.

$G = U(n,1)$: In this case the space $\mathrm{Hom}(\pi, G)/G$ contains both

$$\mathrm{Hom}(\pi, \mathrm{SU}(1,1))/\,\mathrm{SU}(1,1) \text{ and } \mathrm{Hom}(\pi, \mathrm{SO}(2,1))/\,\mathrm{SO}(2,1)$$

Both subspaces contain discrete embeddings. By recent joint work with Kapovich and Leeb, at least $2g-1$ components of $\mathrm{Hom}(\pi, G)/G$ contain discrete embeddings.

$G = SL(3,\mathbb{R})$: In this case Hitchin showed that a connected component of $\mathrm{Hom}(\pi, G)/G$ is diffeomorphic to a cell of dimension $16g - 16$. Joint work with Suhyoung Choi implies that this connected component consists of convex real projective structures, that is, uniformizations of M as quotients of convex domains in $\mathrm{P}^2(\mathbb{R})$ by discrete groups of collineations. $\mathrm{Hom}(\pi, G)/G$ has at least 3 connected components. We conjecture that $\mathrm{Hom}(\pi, G)/G$ has *exactly* three components.

2. Hyperbolic structures on surfaces

The second lecture considered the case $G = \mathrm{PSL}(2,\mathbb{R})$. Orientation-preserving discrete embeddings comprise one component of $\mathrm{Hom}(\pi, G)/G$, which may be identified with the Teichmüller space $\mathfrak{T}_S$ of S. This space has a Kähler geometry arising from a complex structure (discovered by Ahlfors and Teichmüller) and the Weil-Petersson metric. This Kähler structure is invariant under $\mathrm{Out}(\pi)$, which acts properly discontinuously. The quotient $\mathfrak{T}_S/Out(\pi)$ is the Riemann moduli space which parametrizes conformal equivalence classes of Riemann surfaces of genus g.

An important integer-valued invariant, which distinguishes the connected components of $\mathrm{Hom}(\pi, G)/G$ is the Euler class, discussed in detail in my doctoral dissertation. Associated to a representation $\rho : \pi \longrightarrow G$ is an oriented $\mathbf{H}^2$-bundle H_ρ.

Oriented $\mathbf{H}^2$-bundles are classified by the Euler class. The resulting map

$$e : \mathrm{Hom}(\pi, G)/G \longrightarrow \mathbb{Z}$$

expresses the topological type of a flat $\mathrm{PSL}(2,\mathbb{R})$-bundle over S. By a classical result of Milnor and Wood, the Euler number satisfies

$$|e(\rho)| \leq |\chi(S)| = 2g-2$$

and every such integer occurs. Moreover the connected components are the $4g-3$ inverse images $e^{-1}(j)$ for $j = 2-2g, \dots, 0, \dots, 2g-2$.

These other components relate to moduli spaces of *singular hyperbolic structures.* Let S be a surface and let $p_1, \dots, p_k \in S$ be points. Let $\theta_i > 0$ for $i = 1, \dots, k$. We consider geometric structures on M which are locally modelled on $\mathbf{H}^2$ on $S - \{p_1, \dots, p_k\}$ and such that each p_i has a coordinate neighborhood isometric to a hyperbolic cone of cone angle θ_i.

Let M be a Riemann surface and choose singular data $(p_1, \theta_1), \dots, (p_k, \theta_k) \in M \times \mathbb{R}_+$ as above. Then there exists a singular conformal structure on S which agrees with the original conformal structure on $M - \{p_1, \dots, p_k\}$ and has conical singularities at p_i of cone angle θ_i. If

$$e(M; \theta_1, \dots, \theta_k) := \chi(M) - \sum_{i=1}^{k} \frac{\theta}{2\pi} - 1 < 0$$

then by a result of McOwen and Troyanov, there exists a singular hyperbolic structure on S with cone angles θ_i at each p_i in this singular conformal structure. Since the hyperbolic structure on $M - \{p_1, \dots, p_k\}$ is nonsingular, there is a holonomy homomorphism $\pi_1(M - \{p_1, \dots, p_k\}) \longrightarrow \mathrm{PSL}(2,\mathbb{R})$ sending a simple loop encircling p_i to an elliptic element of rotation angle θ_i. In particular if all the θ_i are multiples of 2π, this homomorphism extends to $\pi_1(M)$ and has Euler number $e(M; \theta_1, \dots, \theta_k)$.

For $j = 1, \dots, 2g-2$, let Σ_j be the bundle over $\mathfrak{T}_M$ whose fiber over the Riemann surface M is the j-th symmetric product $\mathrm{Symm}^j(M)$ of M; it is a complex manifold of complex dimension $3g-3+j$ which parametrizes singular conformal structures with cone angles $\theta_i = 2\pi(1+m_i)$ at p_i, where the positive integer m_i is the *multiplicity* of p_i in the corresponding effective divisor in $\mathrm{Symm}^j(M)$. The above singular uniformization theorem gives a map

$$h : \Sigma_j \longrightarrow e^{-1}(2g-2+j)$$

where $\sum_{i=1}^k m_i = j$. It follows from the theory developed by Hitchin that this map is a homotopy- equivalence.

The uniformization map h is equivariant with respect to the action of the mapping class group, which acts properly on Σ^j (since it factors through the proper action on Teichmüller space). In contrast, the action on components other than Teichmüller space ($e^{-1}(j)$ for $|j| < 2g-2$) is more dynamically interesting. One may try to investigate this action by viewing h as a "resolution" of this action, studying instead the dynamical properties of the foliation induced by h on the quotient of $\Sigma_j / \operatorname{Out}(\pi)$.

2.1. Examples of hyperbolic structures with conical singularities. Here are two constructions of singular hyperbolic structures on a closed oriented surface Σ with a single cone point. These constructions agree when Σ has genus two and the cone point has cone angle 4π.

The first construction uses a branched covering and identifies the holonomy representation as a non-injective homomorphism onto a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$. Consider the involution $f : \Sigma \longrightarrow \Sigma$ with two fixed points whose quotient orbifold O is a torus T with two branch points p_1, p_2 of order two. The fundamental group of O is

$$\pi_1(O) \cong \langle A, B, C_1, C_2 \mid ABA^{-1}B^{-1}C_1C_2 = C_1^2 = C_2^2 = I \rangle$$

and $\chi(O) = 2 - 2g - \sum_j (1 - (n_j)^{-1}) = -1$. (Here the generator C_i corresponds to the loop which encircles the singular point p_i.) Corresponding to the quotient map $q : \Sigma \longrightarrow O$ (a covering space of orbifolds) is an inclusion $q_* : \pi_1(\Sigma) \longrightarrow \pi_1(O)$ onto an index-two subgroup.

Consider a hyperbolic structure O' on T with a cone point of angle π at p_1 and no singularity at p_2 (cone angle 2π). Then there is a homeomorphism $h : O \longrightarrow O'$ which induces a singular hyperbolic structure on O. Then

$$\pi_1(O') \cong \langle A, B, C \mid ABA^{-1}B^{-1}C = C^2 = I \rangle \subset \mathrm{PSL}(2,\mathbb{R})$$

and we arrange that $h_*(C_1) = C$ and the kernel of $h_* : \pi_1(O) \longrightarrow \pi_1(O')$ is normally generated by C_2. We pull back the singular hyperbolic structure on O induced by $h : O \longrightarrow O'$ by the double covering $q : \Sigma \longrightarrow O$ to obtain a singular hyperbolic structure on Σ with a cone angle of 2π at h^{p_1} (a smooth point) and cone angle of 4π at h^{p_2}. We see that for this hyperbolic structure, the Euler class is the negative of

$$\operatorname{area}/2\pi = 2g - 2 + \sum_j (1 - \alpha_j/2\pi) = 2(2) - 2 + (1 - 2) = 1$$

as expected. By varying the hyperbolic structure on the orbifold O' we obtain a family of representations of $\pi_1(\Sigma)$ with Euler class -1 parametrized by $\mathbb{R}^2$.

Here is another construction of singular hyperbolic structures on a surface of genus g with a single cone point for each Euler number $2 - 2g \le e < 0$, which agrees with one of the above structures for $g = 2, e = -1$. Begin by choosing an angle

$$0 < \theta < \frac{4g-2}{4g}\pi$$

(For an integer $2 - 2g \le j < 0$,

$$\theta = 2\pi \frac{j - 1 + 2g}{4g}$$

will produce an example on a surface of genus g with Euler number j.) Now let $P = P(g,\theta)$ be a regular $4g$-gon with angle θ. (To construct P, choose a center O and choose rays $l_1,\dots,l_{4g}$ at O at angles $\pi/(2g)$ from each other. For each $i = 1,\dots,4g$, let v_i be the point at distance

$$\cosh^{-1}\left(\cot(\frac{\theta}{2})\cot(\frac{\pi}{4g})\right)$$

from O along l_i. Then $v_1,\dots,v_{4g}$ are the vertices of P.)

For each $i = 1,\dots,g$ there exist unique isometries $A_i, B_i \in \mathrm{PSL}(2,\mathbb{R})$ such that A_i takes the edge with endpoints

$$v_{4g-4+i+3}, \quad v_{4g-4+i+4}$$

to the edge with endpoints

$$v_{4g-4+i+2}, \quad v_{4g-4+i+1}$$

and B_i takes the edge with endpoints

$$v_{4g-4+i+2}, \quad v_{4g-4+i+3}$$

to the edge with endpoints

$$v_{4g-4+i+4}, \quad v_{4g-4+i+5}$$

(where $v_j = v_k$ for $j \equiv k(\mod 4g)$). Then $A_1, B_1, \dots, A_g, B_g$ satisfy the standard relation

$$A_1B_1A_1^{-1}B_1^{-1}\dots A_gB_gA_g^{-1}B_g^{-1} = I$$

and define identifications on ∂P such that the quotient $Q = P/\approx$ is a singular hyperbolic structure. Each point in the interior of P, or in the interior of an edge of ∂P, is a smooth point of Q, but the point of Q corresponding to a vertex of P is a singularity of cone angle:

$$\alpha = 4g\theta$$

The Euler class of the holonomy representation is given by the area formula:

$$(2-2g) + (1 - \frac{\alpha}{2\pi}) = j$$

where $j = 2-2g,\dots,-2,-1$. For $j = 2-2g$, the structure is nonsingular and corresponds to the Fuchsian representation.

3. The symplectic structure

In the last part of my talk, I discussed the general construction of the symplectic structure and gave an application to 3-manifolds. Let $\mathfrak{g}$ be the Lie algebra of G. Suppose there exists a nondegenerate symmetric bilinear form $B : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$ invariant under the adjoint representation. (For example if G is a reductive matrix group, such as any of the above examples, then $B(X, Y) = \operatorname{tr}(XY)$ is such a pairing.) If $\rho \in \operatorname{Hom}(\pi, G)$, then there is a flat vector bundle $\mathfrak{g}_{\operatorname{Ad}\rho}$ whose fiber is $\mathfrak{g}$. Then B defines a pairing of local systems $\mathfrak{g}_{\operatorname{Ad}\rho} \times \mathfrak{g}_{\operatorname{Ad}\rho} \longrightarrow \mathbb{R}$

Recall that the "Zariski tangent space" to $\operatorname{Hom}(\pi, G)/G$ at an equivalence class $[\rho]$ identifies with the cohomology group $H^1(M; \mathfrak{g}_{\operatorname{Ad}\rho})$. The corresponding pairing

$$\begin{aligned} H^1(M; \mathfrak{g}_{\operatorname{Ad}\rho}) \times H^1(M; \mathfrak{g}_{\operatorname{Ad}\rho}) &\longrightarrow \mathbb{R} \\ (\alpha, \qquad\qquad \beta) &\longmapsto B_*(\alpha \cup \beta) \cap [M] \end{aligned}$$

defines a nondegenerate alternating bilinear mapping $\omega_{[\rho]}$ on each tangent space $T_{[\rho} \operatorname{Hom}(\pi, G)/G$. The corresponding exterior 2-form ω on $\operatorname{Hom}(\pi, G)/G$ is *closed,* and hence defines a symplectic structure on $\operatorname{Hom}(\pi, G)/G$ which includes all the symplectic structures mentioned above.

As a simple application of this construction, suppose that W^3 is a compact 3-manifold with boundary $\partial W = S$. The inclusion $i : S \hookrightarrow W$ induces a map

$$i_\# : \operatorname{Hom}(\pi_1(W), G) \longrightarrow \operatorname{Hom}(\pi, G)/G$$

which is *isotropic,* that is, $(i_\#)^*\omega = 0$. In other words, the representations which extend to W form an isotropic subvariety of $\operatorname{Hom}(\pi, G)/G$. (In particular this subspace has dimension at most half that of $\operatorname{Hom}(\pi, G)/G$.) To prove this assertion, let $\alpha, \beta \in H^1(W; \mathfrak{g}_{\operatorname{Ad}\rho})$. Then

$$\begin{aligned} \omega_{[\rho]}((i_\#)_*\alpha), (i_\#)_*\beta)) &= B_*((i_\#)_*\alpha), (i_\#)_*\beta)) \cap [M] \\ &= B_*(\alpha \cup \beta) \cap i^*[M] = 0 \end{aligned}$$

since $i^*[M] = \partial[W]$ is the zero cohomology class on W.

Particularly interesting is the case when W is a solid handlebody of genus g. Consider the case that $G = \mathrm{SU}(2)$. Then $\operatorname{Hom}(\pi, G)/G$ has dimension $6g - 6$ and since $\pi_1(W)$ is free of rank $g - 1$, $\operatorname{Hom}(\pi_1(W), G) \cong G^{g-1}$ and $\operatorname{Hom}(\pi_1(W), G)/G$ has dimension $3g - 3$. Furthermore the mapping $i_\# : \operatorname{Hom}(\pi_1(W), G)/G \longrightarrow \operatorname{Hom}(\pi, G)/G$ is an embedding. It follows that its image is a *Lagrangian subvariety* of $\operatorname{Hom}(\pi, G)/G$, a fact which has played an important role in the construction of recent 3-manifold invariants, such as those discussed in Ronnie Lee's lectures at this conference.

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