

THE COMPLEX-SYMPLECTIC GEOMETRY OF SL(2, C)-CHARACTERS OVER SURFACES

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Dedicated to M.S. Raghunathan on his sixtieth birthday

ABSTRACT. The SL(2, C)-character variety X of a closed surface M enjoys a natural complex-symplectic structure invariant under the mapping class group Γ of M . Using the ergodicity of Γ on the SU(2)-character variety, we deduce that every Γ -invariant meromorphic function on X is constant. The trace functions of closed curves on M determine regular functions which generate complex Hamiltonian flows. For simple closed curves, these complex Hamiltonian flows arise from holomorphic flows on the representation variety generalizing the Fenchel-Nielsen twist flows on Teichmüller space and the complex quakebend flows on quasi-Fuchsian space. Closed curves in the complex trajectories of these flows lift to paths in the deformation space $\mathbb{C}\mathbb{P}^1(M)$ of complex-projective structures between different $\mathbb{C}\mathbb{P}^1$ -structures with the same holonomy (grafting). If \mathcal{P} is a pants decomposition, then the trace map $\tau_{\mathcal{P}} : X \rightarrow \mathbb{C}^{\mathcal{P}}$ defines a holomorphic completely integrable system. Furthermore, if $\Gamma_{\mathcal{P}}$ is the subgroup of Γ preserving \mathcal{P} , then every $\Gamma_{\mathcal{P}}$ -invariant holomorphic function $X \rightarrow \mathbb{C}$ factors through $\tau_{\mathcal{P}}$. This holomorphic integrable system is related to the complex Fenchel-Nielsen coordinates on quasi-Fuchsian space $\mathcal{QF}(M)$ developed by Tan and Kourouniotis, and relate to recent formulas of Platis and Series on complex-length functions and complex twist flows on $\mathcal{QF}(M)$.

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INTRODUCTION

The $\mathrm{SL}(2, \mathbb{C})$ -character variety of a closed surface M with fundamental group π is the space of equivalence classes of representations ϕ of the fundamental group π of M into $\mathrm{SL}(2, \mathbb{C})$. The Teichmüller space $\mathfrak{T}(M)$, the moduli space X_U of irreducible flat unitary $\mathrm{SU}(2)$ -bundles, the deformation space $\mathbb{C}\mathbb{P}^1(M)$ of $\mathbb{C}\mathbb{P}^1$ -structures on M , and the quasi-Fuchsian space $\mathcal{QF}(M)$ all lie in the smooth stratum X of the $\mathrm{SL}(2, \mathbb{C})$ -character variety. These spaces all share several common features: a symplectic geometry derived from the topology of M , and a compatible action of the mapping class group Γ of M . This paper investigates these structures in terms of the Γ -invariant complex-symplectic structure on X .

A *complex-symplectic structure* on a complex manifold is a nondegenerate closed holomorphic exterior 2-form. The $\mathrm{SL}(2, \mathbb{C})$ -character variety is an affine variety defined over \mathbb{C} whose set of smooth \mathbb{C} -points is a complex-symplectic manifold of complex dimension $-3\chi(M)$ whose complex-symplectic structure is invariant under Γ .

In [46], Fenchel and Nielsen develop a set of coordinates for $\mathfrak{T}(M)$ based on hyperbolic geometry. This set of coordinates is based on a *pants decomposition* $\mathcal{P} = \{\alpha_1, \dots, \alpha_N\}$ on M , that is a set of disjoint simple closed curves α_i cutting M into three-holed spheres (“pants”). In a hyperbolic structure on M , the curves α_i are represented by disjoint simple closed geodesics and their lengths define a function

$$l_{\mathcal{P}} : \mathfrak{T}(M) \longrightarrow (\mathbb{R}_+)^N$$

which constitute half of the Fenchel-Nielsen coordinates on $\mathfrak{T}(M)$. According to Wolpert [48], these functions Poisson-commute and define a completely integrable Hamiltonian system. In the language of classical mechanics, these are the action variables, for which the Fenchel-Nielsen twist vector fields define angle variables. In particular $l_{\mathcal{P}}$ is the moment map for a Hamiltonian \mathbb{R}^N -action. Choosing a section σ to $l_{\mathcal{P}}$ determines a symplectomorphism

$$\begin{aligned} (\mathbb{R}_+)^N \times \mathbb{R}^N &\longrightarrow \mathfrak{T}(M) \\ (\lambda; t) &\longmapsto \xi_t \sigma(\lambda) \end{aligned}$$

where ξ_t is the Hamiltonian \mathbb{R}^N -action defined by $l_{\mathcal{P}}$. See Wolpert [46, 47, 48] for details.

This picture motivated the study of a symplectic geometry of moduli spaces $\mathrm{Hom}(\pi, G)/G$ developed in [9, 11]. The results there extend directly to the complex-symplectic geometry of deformation spaces $\mathrm{Hom}(\pi, G)/G$ where G is a *complex Lie group* with an Ad -invariant

complex-orthogonal structure \mathbb{B} on its Lie algebra. In this paper we consider only the special case $G = \mathrm{SL}(2, \mathbb{C})$ where \mathbb{B} is the trace form. Corresponding to an element $\alpha \in \pi$ is the function $f_\alpha : X \rightarrow \mathbb{C}$ associating to the equivalence class of ϕ the trace of $\phi(\alpha)$. When α is represented by a *simple* closed curve A , then the complex-Hamiltonian vector field $\mathrm{Ham}(f_\alpha)$ generates a flow which is covered by a *complex twist flow* on $\mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))$. We study this flow, computing its periods, and relating it to the action on X of the Dehn twist about A . Following [9, 11], we relate the complex-symplectic geometry and the Hamiltonian actions to the Fenchel-Nielsen twist flows on $\mathfrak{T}(M)$ and the complex earthquakes and bending deformations on $\mathbb{CP}^1(M)$ and $\mathcal{QF}(M)$.

In the presence of a conformal structure on M , the moduli spaces $\mathrm{Hom}(\pi, G)/G$ admit stronger structures (a Kähler structure when G is compact, and hyper-Kähler when G is complex), but these stronger structures fail to be Γ -invariant. However the symplectic (and complex-symplectic) structures are Γ -invariant. The symplectic structure defines a Γ -invariant measure. By [15] and Pickrell-Xia [38], the resulting measure is ergodic under Γ when G is a compact Lie group. This has the following consequence for the holomorphic geometry when G is complex:

Theorem. *There are no nonconstant Γ -invariant meromorphic functions on X .*

The proof uses the inclusion of the set X_U of irreducible unitary characters in X , and the ergodicity of the action of Γ on X_U ([15]). A key idea in the proof is the action of the subgroup $\Gamma_{\mathcal{P}}$ preserving a pants decomposition \mathcal{P} of M . The group $\Gamma_{\mathcal{P}}$ is a free abelian group freely generated by the Dehn twists about the curves in \mathcal{P} . This \mathbb{Z}^N -action lies in a Hamiltonian \mathbb{R}^N -action. The map which associates to $[\phi]$ the collection of traces

$$X_U \longrightarrow \mathbb{R}^N$$

is a moment map for the \mathbb{R}^N -action and is also the ergodic decomposition for the \mathbb{Z}^N -action. The holomorphic analog is:

Theorem. *Every $\Gamma_{\mathcal{P}}$ -invariant meromorphic function on X factors through the map*

$$X \longrightarrow \mathbb{C}^N$$

which associates to a character its values on the curves in \mathcal{P} .

The complex twist flows have been extensively studied in the *quasi-Fuchsian space* $\mathcal{QF}(M)$, which is the open subset of the $\mathrm{SL}(2, \mathbb{C})$ -character variety comprising equivalence classes of quasi-Fuchsian representations. In this case, the complex twist flows correspond geometrically to quake-bending pleated surfaces in quasi-Fuchsian hyperbolic 3-manifolds, that is, composing Fenchel-Nielsen twist flows (earthquakes) with bending deformations (Epstein-Marden [7]). These quakebends are defined more generally for geodesic laminations, although we only consider deformations supported on simple closed curves here. The complex Fenchel-Nielsen coordinates of Kourouniotis [32, 33, 34] and Tan [43] are holomorphic Darboux coordinates for the complex-symplectic structure. We recover results of Platis [39] expressing the symplectic duality between the complex twist flows and the complex length functions, and the formula of Series [42] for the derivative of a complex length function under a twist flow in terms of the complex-symplectic geometry of X .

More generally the complex twist flows are defined for \mathbb{CP}^1 -structures, that is, geometric structures with coordinates modelled on \mathbb{CP}^1 with coordinate changes in $\mathrm{PSL}(2, \mathbb{C})$. Let $\mathbb{CP}^1(M)$ denote the deformation space of \mathbb{CP}^1 -structures on M . Using the local biholomorphicity of the holonomy mapping

$$\mathrm{hol} : \mathbb{CP}^1(M) \longrightarrow \mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C})) // \mathrm{SL}(2, \mathbb{C}).$$

one obtains complex-symplectic structures, complex length functions, and Hamiltonian complex twist flows on $\mathbb{CP}^1(M)$. The complex twist flows on \mathbb{CP}^1 -structures can be described geometrically by inserting annuli into a \mathbb{CP}^1 -manifold split along a simple closed curve which is locally circular. This is a special case of the *grafting* construction considered in Tanigawa [44] and McMullen [36]. In particular closed curves in the complex trajectory of a complex twist flow lift to paths between different \mathbb{CP}^1 -structures with the same holonomy (Maskit and Hejhal, see Goldman [12]). The holomorphic properties of the grafting construction are discussed in McMullen [36], Tanigawa [44] and Scannell-Wolf [41].

1. REPRESENTATION VARIETIES AND CHARACTER VARIETIES

Let M be a closed oriented surface with fundamental group π . Let $\mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))$ denote the complex affine variety consisting of homomorphisms $\pi \longrightarrow \mathrm{SL}(2, \mathbb{C})$. (It is irreducible, by Goldman [13]; see Benyash-Krivets -Chernousov -Rapinchuk [2] for stronger more general results, and also Li [35].)

The group $\mathrm{SL}(2, \mathbb{C})$ acts by conjugation on the affine algebraic set $\mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))$. Let $\mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))/\mathrm{SL}(2, \mathbb{C})$ denote the set of $\mathrm{SL}(2, \mathbb{C})$ -orbits of $\mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))$. Denote the orbit of a representation ϕ by

$$[\phi] \in \mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C})).$$

1.1. Stable and semistable points. A homomorphism

$$\phi \in \mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))$$

is *irreducible* if it leaves invariant no proper linear subspace of \mathbb{C}^2 . Equivalently, ϕ is irreducible if the corresponding projective action fixes no point in \mathbb{CP}^1 . Irreducible homomorphisms are the *stable points* of $\mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))$ (with respect to the $\mathrm{SL}(2, \mathbb{C})$ -action). The subset of $\mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))^s \subset \mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))$ consisting of irreducible homomorphisms is Zariski-open and nonsingular. $\mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))^s$ is a complex manifold of complex dimension $6g - 3$ (see [9]).

More generally, a homomorphism is *reductive* if every invariant subspace possesses an invariant complement. Equivalently, ϕ is reductive if it is either reducible or fixes a pair of distinct points on \mathbb{CP}^1 or is *central*. A central homomorphism maps into the center $\{\pm\mathbb{I}\}$ of $\mathrm{SL}(2, \mathbb{C})$ and acts trivially on \mathbb{CP}^1 . Letting $\mathrm{Fix}(\phi)$ denote the subset of \mathbb{CP}^1 fixed by ϕ , a homomorphism ϕ is reductive if and only if $\mathrm{Fix}(\phi)$ equals \emptyset , a pair of distinct points, or all of \mathbb{CP}^1 . Reductive homomorphisms are the *semistable points* of $\mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))$ (with respect to the $\mathrm{SL}(2, \mathbb{C})$ -action), comprising the subset

$$\mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))^{ss} \subset \mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C})).$$

A *semistable* (respectively *stable*) orbit is the orbit of a semistable (respectively stable) point.

Let $\mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))/\mathrm{SL}(2, \mathbb{C})$ denote the set of \mathbb{C} -points of the *categorical quotient* of the $\mathrm{SL}(2, \mathbb{C})$ -action on $\mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))$, that is, the variety whose coordinate ring is the ring of invariants of $\mathrm{SL}(2, \mathbb{C})$ acting on the coordinate ring $\mathbb{C}[\mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))]$ of $\mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))$. The natural map

$$\mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))/\mathrm{SL}(2, \mathbb{C}) \longrightarrow \mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))/\mathrm{SL}(2, \mathbb{C})$$

is surjective, but not injective. However $\mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))/\mathrm{SL}(2, \mathbb{C})$ may be identified with the set of semistable orbits. Thus the inclusion

$$\mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))^{ss} \hookrightarrow \mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))$$

induces a bijection

$$\mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))^{ss}/\mathrm{SL}(2, \mathbb{C}) \longleftrightarrow \mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))/\mathrm{SL}(2, \mathbb{C}).$$

The group $\mathrm{SL}(2, \mathbb{C})$ acts freely and properly on $\mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))^s$. The quotient $X := \mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))^s / \mathrm{SL}(2, \mathbb{C})$ is thus a $(6g-6)$ -dimensional complex manifold which embeds as a Zariski open subset of the categorical quotient $\mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C})) // \mathrm{SL}(2, \mathbb{C})$. Thus the map

$$X = \mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))^s / \mathrm{SL}(2, \mathbb{C}) \longrightarrow \mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C})) // \mathrm{SL}(2, \mathbb{C})$$

induced by $\mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))^s \hookrightarrow \mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))$ is an embedding onto an open subset. Thus X is a smooth irreducible complex quasi-affine variety which is dense in the quotient $\mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C})) // \mathrm{SL}(2, \mathbb{C})$.

1.2. Symplectic geometry of deformation spaces. By the general construction of [9], X has a natural *complex-symplectic structure* Ω , which is Γ -invariant. Furthermore Γ is algebraic in the sense that there exists an algebraic tensor field on $\mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))$ inducing Ω . (See [9] for an explicit formula.)

The Zariski tangent space $T_\phi \mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))$ is the space

$$Z^1(\pi, \mathfrak{sl}(2, \mathbb{C})_{\mathrm{Ad}\phi})$$

of *1-cocycles* $\pi \longrightarrow \mathfrak{sl}(2, \mathbb{C})_{\mathrm{Ad}\phi}$ and the tangent space $T_\phi(G \cdot \phi)$ of the G -orbit equals the subspace

$$B^1(\pi, \mathfrak{sl}(2, \mathbb{C})_{\mathrm{Ad}\phi}) \subset Z^1(\pi, \mathfrak{sl}(2, \mathbb{C})_{\mathrm{Ad}\phi})$$

of *1-coboundaries*. (Compare Raghunathan [40].) The quotient vector space is the cohomology

$$H^1(\pi, \mathfrak{sl}(2, \mathbb{C})_{\mathrm{Ad}\phi})$$

which, under the de Rham isomorphism is isomorphic to the cohomology $H^1(M; V_\phi)$ where V_ϕ denotes the flat vector bundle over M corresponding to the π -module $\mathfrak{sl}(2, \mathbb{C})_{\mathrm{Ad}\phi}$.

Let $\mathbb{B} : \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C}) \longrightarrow \mathbb{C}$ be the *trace form* of the standard representation on \mathbb{C}^2 : if $\alpha, \beta \in \mathfrak{sl}(2, \mathbb{C})$, then the inner product is defined as

$$\mathbb{B}(\alpha, \beta) := \mathrm{tr}(\alpha\beta).$$

Since \mathbb{B} is Ad -invariant, it defines a bilinear pairing of π -modules

$$\mathfrak{sl}(2, \mathbb{C})_{\mathrm{Ad}\phi} \times \mathfrak{sl}(2, \mathbb{C})_{\mathrm{Ad}\phi} \longrightarrow \mathbb{C}$$

or equivalently flat vector bundles

$$V_\phi \times V_\phi \longrightarrow \mathbb{C}.$$

Cup-product defines a bilinear pairing

$$\Omega_\phi : H^1(M; V_\phi) \times H^1(M; V_\phi) \longrightarrow \mathbb{C}$$

with coefficients paired by \mathbb{B} . Symmetry of \mathbb{B} implies that Ω_ϕ is skew-symmetric. Since \mathbb{B} is nondegenerate, Ω_ϕ is nondegenerate. It

follows from [9] that Ω_ϕ can be expressed as an algebraic tensor on $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$, and thus is a holomorphic exterior 2-form. By arguments of [9] or Karshon [29], Weinstein [45], Guruprasad-Huebschmann-Jeffrey-Weinstein [19], Ω is closed. Thus Ω is a nondegenerate closed holomorphic (2, 0)-form, that is a *complex-symplectic structure*.

1.3. The mapping class group. The automorphism group $\text{Aut}(\pi)$ of π acts algebraically on $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$ commuting with the action of $\text{SL}(2, \mathbb{C})$. The normal subgroup $\text{Inn}(\pi)$ of inner automorphisms acts trivially on the quotient $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))//\text{SL}(2, \mathbb{C})$. The quotient $\text{Out}(\pi) = \text{Aut}(\pi)/\text{Inn}(\pi)$, acts on $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))//\text{SL}(2, \mathbb{C})$, leaving invariant the subset X . Furthermore the mapping class group $\Gamma := \pi_0(\text{Diff}^+(M))$ of M is isomorphic to $\text{Out}(\pi)$ by Nielsen [37]. The algebraic complex-symplectic structure on X is Γ -invariant.

1.4. Ergodicity and its holomorphic analog. The subset X_U of unitary characters is invariant under Γ . Furthermore the complex-symplectic structure restricts to a (real) symplectic structure on X_U which is the Kähler form for a Kähler structure on X_U . In particular the Lebesgue measure class on X_U is invariant under Γ . The main result of [15] is that this action is *ergodic*, that is every Γ -invariant measurable function on X_U is constant almost everywhere.

Ergodicity no longer holds for the $\text{SL}(2, \mathbb{C})$ -character variety X (see §4.3). However ergodicity on X_U does imply the following property of holomorphic functions on X :

Theorem 1.4.1. *A Γ -invariant meromorphic function $X \xrightarrow{h} \mathbb{CP}^1$ is constant.*

Proof. The restriction of h to X_U is a Γ -invariant measurable function, and by the main result of Goldman [15], h must be constant almost everywhere. Since h is continuous, it is constant.

Now we argue that in local holomorphic coordinates, X_U is equivalent to $\mathbb{R}^n \subset \mathbb{C}^n$ and a holomorphic function constant on X_U must be globally constant on X . (Compare, for example, Lemma 1 of §2.3 of Platis [39]). For the reader's convenience, we supply a brief proof. Let \mathcal{U} be a nonempty open coordinate neighborhood with local holomorphic coordinates $z = (z^1, \dots, z^n)$. In local holomorphic coordinates, $X_U \cap \mathcal{U}$ is described by

$$z \in \mathbb{R}^n \subset \mathbb{C}^n$$

and h is given by a power series

$$h(z) = \sum_{k=0}^{\infty} a_k z^k$$

which converges in the nonempty open set $z(\mathcal{U}) \in \mathbb{C}^n$. Since the restriction of h to a $z(\mathcal{U}) \cap \mathbb{R}^n$ is constant, $a_k = 0$ for $k > 0$ and thus h is constant on \mathcal{U} . Since X is connected, analytic continuation implies that h must be constant. \square

We do not know whether $X - X_U$ admits Γ -invariant nonconstant meromorphic functions.

2. THE HAMILTONIAN VECTOR FIELD OF A CHARACTER FUNCTION

Corresponding to free homotopy classes α of closed curves on M are complex regular functions $f_\alpha : X \rightarrow \mathbb{C}$. The complex-symplectic structure associates to these functions complex Hamiltonian vector fields $\text{Ham}(f_\alpha)$, which generate holomorphic local flows on X . When α corresponds to a simple closed curve, we define *complex twist flows* on $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$ which cover these holomorphic local flows on X . We begin with an preliminary section on the traces in $\text{SL}(2, \mathbb{C})$.

2.1. The variation of the trace function on $\text{SL}(2, \mathbb{C})$. Let

$$f : \text{SL}(2, \mathbb{C}) \rightarrow \mathbb{C}$$

be the trace function $f(P) = \text{tr}(P)$ and \mathbb{B} the trace form

$$\begin{aligned} \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C}) &\rightarrow \mathbb{C} \\ (X, Y) &\mapsto \text{tr}(XY). \end{aligned}$$

As in Goldman [11] the differential of f and the orthogonal structure \mathbb{B} determines a variation function

$$F : \text{SL}(2, \mathbb{C}) \rightarrow \mathfrak{sl}(2, \mathbb{C})$$

characterized by the identity

$$\left. \frac{d}{dt} \right|_{t=0} f(P \exp(tX)) = \mathbb{B}(F(P), X).$$

This function is defined by:

$$F(P) = P - \frac{\text{tr}(P)}{2} \mathbb{I}$$

and corresponds to the composition of the inclusion

$$\text{SL}(2, \mathbb{C}) \hookrightarrow \mathfrak{gl}(2, \mathbb{C})$$

with orthogonal projection

$$\mathfrak{gl}(2, \mathbb{C}) \hookrightarrow \mathfrak{sl}(2, \mathbb{C})$$

(orthogonal with respect to the trace form \mathbb{B} on $\mathfrak{sl}(2, \mathbb{C})$). Invariance of the trace

$$f(QPQ^{-1}) = f(P)$$

and Ad-invariance of the orthogonal structure \mathbb{B}

$$\mathbb{B}(\text{Ad}(Q)X, \text{Ad}(Q)Y) = \mathbb{B}(X, Y)$$

implies Ad-equivariance of its variation:

$$(2.1) \quad F(QPQ^{-1}) = \text{Ad}(Q)F(P).$$

Taking $Q = P$ shows that $F(P)$ lies in the centralizer of P in $\mathfrak{sl}(2, \mathbb{C})$. In particular the complex one-parameter subgroup of $\text{SL}(2, \mathbb{C})$

$$(2.2) \quad \zeta_t := \exp(tF(P))$$

centralizes A in $\text{SL}(2, \mathbb{C})$.

For example, if P is the diagonal matrix

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$$

then $f(P) = \lambda + \lambda^{-1}$ and

$$F(P) = \frac{\lambda - \lambda^{-1}}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

with corresponding complex one-parameter subgroup:

$$(2.3) \quad \zeta_t = \begin{bmatrix} e^{t(\lambda - \lambda^{-1})/2} & 0 \\ 0 & e^{-t(\lambda - \lambda^{-1})/2} \end{bmatrix}.$$

If P is \pm the unipotent matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

then $f(P) = \pm 2$ and

$$F(P) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

with corresponding complex one-parameter subgroup:

$$(2.4) \quad \zeta_t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

The following lemma, whose proof is immediate from the above calculations, will be needed in the sequel. Recall that for any subset $S \subset \text{SL}(2, \mathbb{C})$, $\text{Fix}(S)$ denotes the subset of \mathbb{CP}^1 fixed by S .

Lemma 2.1.1. *If $P \in \text{SL}(2, \mathbb{C})$, then $\text{Fix}(\zeta_t) = \text{Fix}(P)$ or $\zeta_t = \pm \mathbb{I}$*

2.2. Character functions. Let $\alpha \in \pi$. The *character function*

$$\begin{aligned} \text{Hom}(\pi, \text{SL}(2, \mathbb{C})) &\xrightarrow{\tilde{f}_\alpha} \mathbb{C} \\ \phi &\longmapsto f(\phi(\alpha)) \end{aligned}$$

is a regular function. Since f is a class function, \tilde{f}_α is invariant under $\text{SL}(2, \mathbb{C})$ and defines a regular function

$$f_\alpha : X \longrightarrow \mathbb{C}.$$

Let $[\phi] \in X$. Then the tangent space to X equals $H^1(M; V_\phi)$, so the Hamiltonian vector field $\text{Ham}(f_\alpha)$ associates to $[\phi] \in X$ a tangent vector in $H^1(M; V_\phi)$.

Cap product with the fundamental homology class $[M] \in H_2(M; \mathbb{C})$ defines the *Poincaré duality isomorphism*:

$$\cap[M] : H^1(M; V_\phi) \xrightarrow{\cong} H_1(M; V_\phi).$$

Choose a basepoint $x_0 \in M$, an isomorphism of the fiber of V_ϕ over x_0 with \mathbb{C}^2 , and a representative holonomy homomorphism

$$\phi_0 : \pi_1(M; x_0) \longrightarrow \text{SL}(2, \mathbb{C}).$$

Let $s_0 \in S^1$ be a basepoint. Let $\alpha_0 : (S^1, s_0) \longrightarrow (M, x_0)$ be a based loop in M corresponding to α . Let σ be the parallel section of the flat vector bundle $\alpha_0^* V_\phi$ over S^1 which equals $F(\phi_0(\alpha_0))$ at s_0 . Then σ defines a V_ϕ -valued 1-cycle in M , with homology class

$$[\sigma] \in H_1(M; V_\phi).$$

Lemma 2.2.1. *The value of the Hamiltonian vector field $\text{Ham}(f_\alpha)$ at a point $[\phi] \in X$ is the vector in*

$$T_{[\phi]}X \cong H^1(M; V_\phi)$$

corresponding to the Poincaré dual of the homology class of the V_ϕ -valued cycle σ :

$$\begin{aligned} \cap[M] : H^1(M; V_\phi) &\longrightarrow H_1(M; V_\phi) \\ \text{Ham}(f_\alpha) &\longmapsto [\sigma]. \end{aligned}$$

Using the above formula and the duality between cup-product and intersections of cycles, we obtain a formula for the Poisson bracket of trace functions ([11]):

Proposition 2.2.2. *Let α, β be oriented closed curves meeting transversely in double points p_1, \dots, p_k . For each p_i , choose representatives*

$$\phi_i : \pi_1(M; p_i) \longrightarrow \text{SL}(2, \mathbb{C}).$$

Let α_i and β_i be the elements of $\pi_1(M; p_i)$ representing α, β respectively. Then the Poisson bracket of functions f_α, f_β is:

$$\begin{aligned} \{f_\alpha, f_\beta\} &= \Omega(\text{Ham}(f_\alpha), \text{Ham}(f_\beta)) \\ &= \sum_{i=1}^k \epsilon(p_i; \alpha, \beta) \mathbb{B}(F(\phi_i(\alpha_i)), F(\phi_i(\beta_i))) \end{aligned}$$

where $\epsilon(p_i; \alpha, \beta)$ denotes oriented intersection number.

Corollary 2.2.3. *If α, β are disjoint, then f_α and f_β Poisson-commute.*

By arguments in [11] (based on a suggestion of S. Wolpert), the converse holds if one of α or β is simple. A purely topological proof of this fact has recently been given by Chas [5].

2.3. Complex twist flows. Suppose that α is represented by a simple closed curve A . Then a holomorphic \mathbb{C} -action on the representation variety $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$ covers the complex Hamiltonian flow on X . The complex twist flow τ_α is the holomorphic action of \mathbb{C} of $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$ defined as follows.

There are two cases, depending on whether A separates M or not. Denote by $M|A$ the compact surface with boundary whose interior is homeomorphic to the complement $M - A$. Denote the two components of $\partial(M|A)$ by A_+ and A_- . The quotient map $q : M|A \rightarrow A$ results from identifying these components by a homeomorphism $\eta : A_+ \rightarrow A_-$.

2.3.1. The nonseparating case. If A is nonseparating, then $M|A$ is connected, and choosing a basepoint in A_+ we express π as an HNN-extension

$$\pi_1(M|A) \star_\iota$$

as follows. Choose a basepoint $x_0 \in A$, and lift it to a basepoint $\tilde{x}_0 \in A_+ \subset M|A$. Let α_+ denote the element of $\pi_1(M|A; \tilde{x}_0)$ corresponding to A_+ . Let $\tilde{\beta}$ denote a simple arc joining \tilde{x}_0 to $\eta(\tilde{x}_0)$ in $M|A$, and α_- the based loop $\tilde{\beta}^{-1} \cdot \alpha_+ \cdot \tilde{\beta}$ at the basepoint \tilde{x}_0 . Then η corresponds to the isomorphism

$$\iota : \langle \alpha_+ \rangle \rightarrow \langle \alpha_- \rangle$$

between the two subgroups of $\pi_1(M|A)$. Let N be the normal closure of the element

$$\beta \alpha_+ \beta^{-1} (\alpha_-)^{-1}$$

of $\pi_1(M|A) \star \langle \beta \rangle$. Then the HNN-extension $\pi_1(M|A) \star_t$ is defined as the quotient

$$\pi \cong \left(\pi_1(M|A) \star \langle \beta \rangle \right) / N.$$

As in [11], for any $\phi \in \text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$, and $t \in \mathbb{C}$, define

$$(2.5) \quad \xi_t(\phi) : \gamma \mapsto \begin{cases} \phi(\gamma) & \text{if } \gamma \in \pi_1(M|A) \\ \phi(\beta) \zeta_t & \text{if } \gamma = \beta. \end{cases}$$

where

$$\zeta_t = \exp(tF(\phi(\alpha_+)))$$

is the one-parameter subgroup corresponding to $\phi(\alpha_+)$.

2.3.2. The separating case. If A separates M , then let M_+ and M_- denote the two components of $M|A$. Then π is the free product of subgroups corresponding to $\pi_+(M_+)$ and $\pi_1(M_-)$ respectively, amalgamated over the images $\langle \alpha \rangle \rightarrow \pi_1(M_\pm)$. Then as in [11], for any $\phi \in \text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$, $t \in \mathbb{C}$, define

$$\xi_t^\alpha(\phi) : \gamma \mapsto \begin{cases} \phi(\gamma) & \text{if } \gamma \in \pi_1(M_+) \\ \zeta_t \phi(\gamma) \zeta_{-t} & \text{if } \gamma \in \pi_1(M_-) \end{cases}$$

where

$$\zeta_t = \exp(tF(\phi(\alpha)))$$

is the one-parameter subgroup corresponding to $\phi(\alpha)$.

2.3.3. Irreducibility. To show that the Hamiltonian flows act on X , we need to show that the complex twist flows preserve the set of irreducible representations.

Lemma 2.3.1. *Suppose that $\phi \in \text{Hom}(\pi, \text{SL}(2, \mathbb{C}))^s$. Then*

$$\xi_t^\alpha(\phi) \in \text{Hom}(\pi, \text{SL}(2, \mathbb{C}))^s.$$

Proof. Let M' be a component of $M|A$. As above, choose basepoints and identifications to express π as an amalgamated free product or HNN construction with $\pi_1(M') \hookrightarrow \pi_1(M)$. Then

$$\xi_t^\alpha(\phi(\pi)) \supset \phi(\pi_1(M'))$$

implies that if $\phi_{\pi_1(M')}$ is irreducible, then so is $\xi_t^\alpha \phi$. Thus we may assume that $\phi_{\pi_1(M')}$ is reducible but ϕ is irreducible.

Suppose first that A is nonseparating. We assume $\xi_t^\alpha \phi$ is reducible and derive a contradiction. Let $\Phi = \text{Fix}(\phi(\pi_1(M|A)))$. Since

$$\phi(\alpha_+) \in \phi(\pi_1(M|A)),$$

the element $\phi(\alpha_+)$ fixes Φ , and Lemma 2.1.1 implies that ζ_t fixes Φ . Since $\xi_t^\alpha \phi(\pi)$ is reducible and is generated by $\phi(\pi_1(M|A))$ and $\phi(\beta)$, it follows that

$$\xi_t^\alpha \phi(\beta) = \phi(\beta)\zeta_t$$

fixes no element of Φ . Since ζ_t fixes Φ , this implies that $\phi(\beta)$ fixes Φ , a contradiction.

Suppose finally that A separates M into two components M_\pm . By the remark above, we may assume that $\phi_{\pi_1(M_\pm)}$ is reducible. Let

$$\Phi_\pm = \text{Fix}(\phi(\pi_1(M_\pm))).$$

We may assume that each Φ_\pm is nonempty but $\Phi_+ \cap \Phi_- = \emptyset$. Now $\phi(\alpha)$ fixes each Π_\pm , and Lemma 2.1.1 implies that ζ_t fixes each Π_\pm . Since $\phi(\pi_1(M_+))$ and $\zeta_t \phi(\pi_1(M_-))\zeta_t^{-1}$ generate $\xi_t^\alpha \phi(\pi)$,

$$\begin{aligned} \text{Fix}(\xi_t^\alpha \phi(\pi)) &= \text{Fix}(\phi(\pi_1(M_+))) \cap \text{Fix}(\zeta_t \phi(\pi_1(M_-))\zeta_t^{-1}) \\ &= \text{Fix}(\phi(\pi_1(M_+))) \cap \zeta_t \text{Fix}(\phi(\pi_1(M_-))) \\ &= \Phi_+ \cap \zeta_t \Phi_- \\ &= \Phi_+ \cap \Phi_- = \emptyset \end{aligned}$$

as desired. □

2.4. Gauge-Theoretic Interpretation. These twist flows admit an interpretation in terms of flat connections. The character variety X identifies with the quotient of the space $\mathcal{F}^{irr}(E)$ of irreducible flat $\text{SL}(2, \mathbb{C})$ -connections on a principal $\text{SL}(2, \mathbb{C})$ -bundle E over M by the group $\mathcal{G}(E)$ of gauge transformations of E . (Since a principal $\text{SL}(2, \mathbb{C})$ -bundle over M which admits a flat connection is necessarily trivial, we may assume E is the product bundle.)

Let $A \subset M$ be a simple closed curve, and choose a basepoint $a_0 \in A$. Choose an orientation on A and let α be a based loop on M corresponding to the orientation on A .

Pull E back to a principal $\text{SL}(2, \mathbb{C})$ -bundle q^*E over $M|A$ by the quotient map $q : M \rightarrow M|A$. Choose a point e_0 in the fiber of E over a_0 . Let a_\pm be the two elements of $q^{-1}(a_0)$ in A_\pm respectively and e_\pm the elements in the fibers of q^*E over a_\pm corresponding to e_0 .

Define the twist flow $\tilde{\xi}_t$ on $\mathcal{F}^{irr}(E)$ as follows. Let $\nabla \in \mathcal{F}^{irr}(E)$ be a flat connection on E . Parallel transport of e_0 along α with respect to ∇ defines a holonomy transformation $\phi_0(\alpha_0)$ as above. Let ζ_t denote the corresponding one-parameter subgroup of $\text{SL}(2, \mathbb{C})$. There is a one-parameter family of gauge-transformations g_t of q^*E supported in a collar neighborhood N of $A_+ \subset M|A$ assuming the “value” ζ_t on A_+ .

Explicitly, choose a smooth embedding

$$\psi : [0, 1] \times S^1 \longrightarrow N \hookrightarrow M|A$$

mapping $\{1\} \times S^1$ diffeomorphically to A_+ . Let $r : [0, 1] \longrightarrow S^1$ denote the quotient mapping identifying $0, 1 \in [0, 1]$. Parallel transport of e_+ defines a trivialization of the pullback of E to $[0, 1] \times [0, 1]$ so that ψ^*E identifies with the quotient of

$$[0, 1] \times [0, 1] \times \mathbf{SL}(2, \mathbb{C})$$

by the identification

$$(s, 0, h) \longleftarrow (s, 1, \phi_0(\alpha_0)h).$$

Define g_t as the gauge transformation which is the identity map on the complement of N in $M|A$, and equals

$$(s, \theta, h) \longrightarrow (s, \theta, \zeta_{s,t}h)$$

in this trivialization.

Since ζ_t centralizes the holonomy of $q^*\nabla$ along A_+ , the identification map $\eta : A_- \longrightarrow A_+$ identifies the restrictions of $(g_t)^*(q^*\nabla)$ to A_{\pm} . Thus a unique flat connection $\tilde{\xi}_t(\nabla)$ exists, satisfying

$$q^*(\tilde{\xi}_t(\nabla)) = (g_t)^*(q^*\nabla).$$

This is the orbit of the twist flow on flat connections. Clearly the gauge transformation g_t does not arise from a gauge transformation of E , unless the holonomy along A is $\pm\mathbb{I}$. The orbit covers the orbit of the holonomy of ∇ under the twist flow in $X \cong \mathcal{F}^{irr}(E)/\mathcal{G}(E)$.

The flow $\tilde{\xi}$ on $\mathcal{F}^{irr}(E)$ depends only on the choice of the collar neighborhood $N \subset M|A$ and $\psi : [0, 1] \times S^1 \longrightarrow N$.

2.5. Periods. The subspace $\mathbf{Hom}(\pi, \mathbf{SU}(2))^s$ maps to a (real) symplectic submanifold $X_U \subset X$, and the corresponding real flows define Hamiltonian systems which have been studied in [11], and are closely related to periodic flows studied by Jeffrey-Weitsman [24, 25, 26, 27]. The orbits of these flows are all closed, although the period of the orbit varies with the value of f .

Proposition 2.5.1. *Let $[\phi] \in X_U$ and let $\alpha \in \pi$ be represented by a simple loop A . If $f_\alpha([\phi]) \neq \pm 2$, the period of the trajectory of the flow of $\mathbf{Ham}(f_\alpha)$ at $[\phi]$ equals*

$$\frac{4\pi}{\sqrt{4 - f_\alpha([\phi])^2}}$$

if A is nonseparating and

$$\frac{2\pi}{\sqrt{4 - f_\alpha([\phi])^2}}$$

if A separates. If $f_\alpha([\phi]) = \pm 2$, then $[\phi]$ is fixed under the flow of $\text{Ham}(f_\alpha)$.

Proof. Let $P \in \text{SU}(2)$. Apply an inner automorphism of $\text{SU}(2)$ to assume that P is diagonal:

$$(2.6) \quad P = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}.$$

Then

$$f(P) = 2 \cos(\theta), \quad F(P) = \begin{bmatrix} i \sin(\theta) & 0 \\ 0 & -i \sin(\theta) \end{bmatrix}$$

with corresponding one-parameter subgroup

$$\zeta_t = \exp(tF(P)) = \begin{bmatrix} e^{i \sin(\theta)t} & 0 \\ 0 & e^{-i \sin(\theta)t} \end{bmatrix}$$

Since the $\text{PSL}(2, \mathbb{C})$ -action on $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))^s$ is proper and free, the quotient map is a principal $\text{PSL}(2, \mathbb{C})$ -fibration:

$$\text{PSL}(2, \mathbb{C}) \longrightarrow \text{Hom}(\pi, \text{SL}(2, \mathbb{C}))^s \longrightarrow X.$$

Thus the trajectory of the flow ξ_t^α projects diffeomorphically to the trajectory of the flow of $\text{Ham}(f_\alpha)$ on X . The period equals the infimum of all $t > 0$ such that

$$\xi_t^\alpha(\phi) = \phi.$$

If α is nonseparating, then (2.5) implies that T equals the infimum of all $t > 0$ such that

$$\zeta_t \phi(\beta) = \phi(\beta).$$

By (2.2),

$$T = \frac{2\pi}{\sin(\theta)} = \frac{4\pi}{\sqrt{4 - f_\alpha([\phi])^2}}$$

If α separates, then (2.6) implies that T equals the infimum of all $t > 0$ such that

$$\zeta_t \phi(\beta) \zeta_{-t} = \phi(\beta).$$

for all $\beta \in \pi_1(M_-)$. If

$$\phi(\beta) \in \{\pm \mathbb{I}\} = \text{center}(\text{SU}(2))$$

for all $\beta \in \pi_1(M_-)$, then $\phi(\alpha) = 1$. Since $\phi \in \text{Hom}(\pi, \text{SL}(2, \mathbb{C}))^s$, there exists $\beta \in \pi$ such that $\phi(\beta) \neq 1$. The period T is the infimum of all $t > 0$ such that $\zeta_t = \pm \mathbb{I}$. By (2.2),

$$T = \frac{\pi}{\sin(\theta)} = \frac{2\pi}{\sqrt{4 - f_\alpha([\phi])^2}}$$

□

2.6. Dehn twists. Closely related to the twist flows are the maps of X_U induced by Dehn twists. Recall that the *Dehn twist* about a simple loop $A \subset M$ is the diffeomorphism supported in a tubular neighborhood of A . If

$$\begin{aligned} \psi : [0, 1] \times S^1 &\hookrightarrow M \\ \left\{ \frac{1}{2} \right\} \times S^1 &\hookrightarrow A \end{aligned}$$

is such a tubular neighborhood, then the Dehn twist τ_A about A is ψ -related to the diffeomorphism of $[0, 1] \times S^1$ (restricting to the identity on the boundary) defined by:

$$(s, e^{i\theta}) \longmapsto (s, e^{2\pi is} e^{i\theta}).$$

The action of τ_A on the fundamental group is given by:

$$(\tau_A)_* (\phi) : \gamma \longmapsto \begin{cases} \phi(\gamma) & \text{if } \gamma \in \pi_1(M|A) \\ \phi(\beta) \phi(\alpha) & \text{if } \gamma = \beta. \end{cases}$$

if A is nonseparating, and by:

$$(\tau_A)_* (\phi) : \gamma \longmapsto \begin{cases} \phi(\gamma) & \text{if } \gamma \in \pi_1(M_+) \\ \phi(\alpha) \phi(\gamma) \phi(\alpha)^{-1} & \text{if } \gamma \in \pi_1(M_-) \end{cases}$$

if A separates.

Let P be as in (2.6). Comparing (2.6) with (2.2), $\zeta_t = P$ for

$$\theta = \sin(\theta)t,$$

that is,

$$(2.7) \quad t = \frac{\theta}{\sin(\theta)} = \frac{\cos^{-1}(f(P)/2)}{2\pi\sqrt{4 - f(P)^2}}.$$

Combining (2.7), (2.7) with (2.7) implies:

Proposition 2.6.1. *Let $\alpha \in \pi$ correspond to a simple closed curve $A \subset M$. The map $(\tau_A)_*$ on $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$ induced by Dehn twist about A equals the time- t map of the flow generated by $\text{Ham}(f_\alpha)$, where*

$$t = \frac{2 \cos^{-1}(f_\alpha([\phi])/2)}{\sqrt{4 - f_\alpha([\phi])^2}}.$$

In terms of a parametrization $\mathbb{R}/\mathbb{Z} \rightarrow X_U$ of this trajectory, $(\tau_A)_*$ acts by translation of

$$\frac{\theta}{2\pi} = \frac{\cos^{-1}(f(P)/2)}{2\pi}$$

which has infinite order for almost every value of $f(A)$.

By reparametrizing the flow, we obtain Hamiltonian flows whose time-one map is the identity map or the Dehn twist. Jeffrey-Weitsman [25] (§5.1), consider flows of Hamiltonians given by invariant functions

$$\theta(P) = \cos^{-1}\left(\frac{f(P)}{2}\right)$$

which define S^1 -actions, but are undefined at $P = \pm\mathbb{I}$. Thus their flows are only defined on the dense open subset where $\phi(P)$ is not central. On the other hand their flows are periodic with period 2π if A is nonseparating, and period π if A separates. (Actually they work with the symplectic form which is $1/(4\pi^2)$ of ours, to obtain a 2-form with integral cohomology class. Thus their periods are $1/(2\pi)$ if A is nonseparating and $1/(4\pi)$ if A separates.)

2.6.1. *Orbits of complex twist flows.* On the $\text{SL}(2, \mathbb{C})$ -character variety X , the Hamiltonian vector field $\text{Ham}(f_\alpha)$ generates a holomorphic \mathbb{C} -action. (2.4) implies that if $f(A) = \pm 2$ and $A \neq \pm\mathbb{I}$ (that is, A is \pm -unipotent), then the trajectory defines an injective holomorphic map

$$(2.8) \quad \begin{aligned} \mathbb{C} &\longrightarrow X \\ t &\longmapsto \xi_t^\alpha([\phi]). \end{aligned}$$

If $f(A) \neq \pm 2$, then (2.3) implies that $\zeta_t = 1$ whenever $t \in t_0\mathbb{Z}$ where

$$(2.9) \quad t_0 := \frac{4\pi i}{\lambda - \lambda^{-1}} = \frac{4\pi i}{(f(A)^2 - 4)^{1/2}}$$

(which is well-defined only up to sign). In this case the trajectory of $[\phi]$ is the image of the holomorphic embedding

$$(2.10) \quad \begin{aligned} \mathbb{C}^* &\longrightarrow X \\ e^z &\longmapsto \xi_{(2z / (\lambda - \lambda^{-1}))}^\alpha([\phi]). \end{aligned}$$

2.6.2. *Dehn twists.* At a point $[\phi] \in X$ where $f_\alpha(\phi) = \pm 2$, then either

- $\phi(\alpha) = \pm \mathbb{I}$ is a central element and $[\phi]$ is fixed under the entire \mathbb{C} -action, or
- $\phi(\alpha)$ is \pm a parabolic element and (2.4) implies $\phi(\alpha) = \pm \exp F(\alpha)$.

In the latter case the embedding (2.8) is equivariant with respect to the action of \mathbb{Z} by translation on \mathbb{C} and the \mathbb{Z} -action generated by $(\tau_A)_*$ on X .

Suppose that $f_\alpha([\phi]) \neq \pm 2$. In that case $\zeta_t = P$ precisely when $t \equiv t_1 \pmod{t_0}$, where

$$t_1 = \frac{\log(\lambda)}{\lambda - \lambda^{-1}} = \frac{\log((f(A) \pm (f(A)^2 - 4)^{1/2})/2)}{(f(A)^2 - 2)^{1/2}}.$$

(The two choices for $(f(A)^2 - 4)^{1/2}$ differ by sign but determine equal values for t_1 .) In that case the embedding (2.10) is equivariant with respect to the actions generated by multiplication by

$$\lambda = \frac{(f(A) \pm (f(A)^2 - 4)^{1/2})}{2}$$

on \mathbb{C}^* and by $(\tau_A)_*$ on X .

2.6.3. *Fenchel-Nielsen twist flows for $G = \mathbf{SL}(2, \mathbb{R})$.* Since the complex trace form \mathbb{B} on $\mathfrak{sl}(2, \mathbb{C})$ restricts to the trace form on $\mathfrak{sl}(2, \mathbb{R})$, the complex-symplectic structure Ω on X restricts to the symplectic structure on $\mathfrak{T}(M)$ defined by the trace form of $\mathbf{SL}(2, \mathbb{R})$. By [9] this symplectic structure equals (-2) the Weil-Petersson Kähler form.

Inside the complex twist flows are the *Fenchel-Nielsen twist flows* when the holonomy is hyperbolic. Suppose that $\phi \in \mathbf{Hom}(\pi, \mathbf{SL}(2, \mathbb{R}))$ is a discrete embedding (that is, a *Fuchsian representation*). The subset of X consisting of equivalence classes of Fuchsian representations corresponds bijectively to the space of marked hyperbolic structures on M , that is, the *Teichmüller space* $\mathfrak{T}(M)$. In that case, for every $\mathbb{I} \neq \alpha \in \pi$, the element $\phi(\alpha) \in \mathbf{SL}(2, \mathbb{R})$ is hyperbolic. Geometrically it is a *transvection* along a geodesic γ , and is conjugate to a diagonal matrix

$$P = \pm \begin{bmatrix} e^{l/2} & 0 \\ 0 & e^{-l/2} \end{bmatrix}.$$

l is the distance $\phi(\alpha)$ moves points along γ . In the hyperbolic structure on M , the element α corresponds to a homotopy class of closed loops; γ corresponds to the unique closed geodesic A freely homotopic to a free loop in α and l is the length of this geodesic.

The character function is related to the length function by:

$$f = 2 \cosh(l/2).$$

Thus the invariant function l (which is only defined on the subset of $SL(2, \mathbb{R})$ consisting of hyperbolic elements) defines functions

$$l_\alpha : \mathfrak{T}(M) \longrightarrow \mathbb{R}_+.$$

As in [11], the corresponding variation is

$$L(P) = \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

with corresponding one-parameter subgroup

$$\zeta_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix},$$

the one-parameter subgroup of transvections moving at speed 2 along the geodesic.

The inner product of these infinitesimal transvections can be computed as follows. The matrices

$$L_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, L_2 = \begin{bmatrix} -\sin(\psi) & \cos(\psi) \\ \cos(\psi) & \sin(\psi) \end{bmatrix}$$

represent infinitesimal transvections about axes which intersect with angle ψ and

$$\mathbb{B}(L_1, L_2) = \text{tr}(L_1 L_2) = 2 \cos(\psi).$$

Thus (as in §3.9 of [11]), Wolpert's cosine formula 2.6.2, follows from Proposition 2.2.2.

If α corresponds to a simple closed curve A , then the corresponding twist flow η_t^α corresponds to the *Fenchel-Nielsen twist flow* on $\mathfrak{T}(M)$ defined geometrically as follows. (Compare Abikoff [1], Buser [4] and Harvey [20].) Represent A by the (necessarily simple) closed geodesic. Split M along A , and identify the boundary components A_+, A_- of $M|A$ by a translation of length t in the positively oriented direction. This is a well-defined path of marked hyperbolic structures in $\mathfrak{T}(M)$; see Wolpert [46] for details. In [46], Wolpert proves that the Fenchel-Nielsen twist flow is Hamiltonian with respect to the Weil-Petersson Kähler form on $\mathfrak{T}(M)$, with Hamiltonian potential the length function l_α [47].

The Fenchel-Nielsen twist flow η^α and the twist flow ξ^α are reparametrizations of one another. Specifically,

$$\xi_t^\alpha = \eta_{\sinh(t)}^\alpha.$$

As in [9] (see also [11]), the Weil-Petersson Kähler form equals $-1/2$ the symplectic structure defined by the trace form \mathbb{B} restricted to $\mathrm{SL}(2, \mathbb{R})$, implying the Fenchel-Nielsen flow is Hamiltonian. Since the Fenchel-Nielsen twist flow is Hamiltonian for the length function l_α , the derivative of the length function along the twist vector field equals the Poisson bracket $\{l_\alpha, l_\beta\}$, which can be computed by Proposition 2.2.2. We thus obtain Wolpert's Derivative Formula:

Theorem 2.6.2 ((Wolpert [47, 46])). *Let $\alpha, \beta \in \pi$ where α is represented by a simple closed curve. Then the derivative of the length function l_β with respect to the Fenchel-Nielsen twist flow equals the sum*

$$\sum_{i=1}^k \cos(\theta_i)$$

where α and β are represented by closed geodesics, p_1, \dots, p_k are their intersection points, and θ_i is the angle from α to β at p_i .

In particular the action of the Dehn twist τ_A equals the time- t map of the Fenchel-Nielsen twist flow at time $t = l/2$.

3. ABELIAN HAMILTONIAN ACTIONS

The classical Fenchel-Nielsen coordinates on $\mathfrak{T}(M)$ can be interpreted as a moment map arising from a *pants decomposition*, that is, a maximal collection of disjoint simple closed curves $\mathcal{P} = \{\alpha_1, \dots, \alpha_N\}$ on M which are each homotopically nontrivial and mutually nonhomotopic. If M has genus g , then $N = 3g - 3$ and the length functions define a map

$$l_{\mathcal{P}} = (l_{\alpha_1}, \dots, l_{\alpha_N}) : \mathfrak{T}(M) \longrightarrow (\mathbb{R}_+)^N$$

which is the moment map for a Hamiltonian \mathbb{R}^N -action. This \mathbb{R}^N -action is proper and free, and choosing a cross-section (a left-inverse to $l_{\mathcal{P}}$)

$$\sigma : (\mathbb{R}_+)^N \longrightarrow \mathfrak{T}(M)$$

defines a diffeomorphism

$$(3.1) \quad (\mathbb{R}_+)^N \times \mathbb{R}^N \longrightarrow \mathfrak{T}(M) \\ (\lambda, t) \longmapsto \eta_{t_1}^{\alpha_1} \dots \eta_{t_N}^{\alpha_N}(\sigma(\lambda))$$

The map $l_{\mathcal{P}}$ defines the *action variables* λ while the coordinates $t \in \mathbb{R}^N$ are the *angle variables*. Indeed, Wolpert shows [48] that this completely integrable system defines a symplectomorphism with $(\mathbb{R}_+)^N \times \mathbb{R}^N$: the Fenchel-Nielsen coordinates are Darboux coordinates with respect to the symplectic structure ω_{WP} defined by the Weil-Petersson Kähler form:

$$\omega_{WP} = \sum_{i=1}^N d\lambda_i \wedge dt_i$$

This section extends this theory from $SL(2, \mathbb{R})$ to $SL(2, \mathbb{C})$.

3.1. Pants decompositions. By Corollary 2.2.3, the complex Hamiltonian vector fields of f_α and f_β Poisson commute if α and β are represented by disjoint curves. Thus for a family of disjoint simple closed curves

$$\mathcal{P} = \{A_1, \dots, A_N\}$$

the corresponding complex twist flows of $\mathbf{Ham} f_{\alpha_i}$ generate a \mathbb{C}^N -action. Suppose these curves are each homotopically nontrivial and mutually nonhomotopic. The resulting map

$$\tau_{\mathcal{P}} : X \longrightarrow \mathbb{C}^N$$

is a moment map for a complex-Hamiltonian action of \mathbb{C}^N on the complex-symplectic manifold X .

Suppose that \mathcal{P} is *maximal*, that is $N = 3g - 3$, in which case each component

$$P \subset M | (\cup_{i=1}^N A_i)$$

is homeomorphic to a 3-holed sphere. In that case Fricke's theorem [13, 16] implies that the generic inverse image $(\tau_{\mathcal{P}})^{-1}(z)$ is a single \mathbb{C}^N -orbit. Specifically, let $(\mathbb{C}^N)^s$ denote the subset consisting of $z \in \mathbb{C}^N$ such that, for every (i, j, k) for which $\alpha_i, \alpha_j, \alpha_k$ bound a 3-holed sphere P_{ijk} ,

$$z_{\alpha_i}^2 + z_{\alpha_j}^2 + z_{\alpha_k}^2 - z_{\alpha_i} z_{\alpha_j} z_{\alpha_k} \neq 4.$$

This expresses the condition that the restriction $\phi|_{\pi_1(P_{ijk})}$ is irreducible; Fricke's theorem asserts the triple

$$(f(\phi(\alpha_i)), f(\phi(\alpha_j)), f(\phi(\alpha_k))) \in \mathbb{C}^3$$

determines the equivalence class of such an irreducible representation. If $z \in (\mathbb{C}^N)^s$, the fiber $(\tau_{\mathcal{P}})^{-1}(z)$ is a single \mathbb{C}^N -orbit.

The subgroup $\Gamma_{\mathcal{P}}$ of Γ preserving the pants decomposition is the free abelian group generated by the Dehn twists τ_{α_i} for $i = 1, \dots, N$.

The resulting \mathbb{Z}^n -action lies in the \mathbb{C}^N -action by the formulas in §2.6. Namely, at a point $[\phi] \in X$ the orbit is an embedded product

$$\prod_{i=1}^N G_i$$

where

$$G_i = \begin{cases} \{1\} & \text{if } \alpha_i([\phi]) = \pm\mathbb{I} \\ \mathbb{C} & \text{if } f_{\alpha_i}([\phi]) = \pm 2 \text{ and } \alpha_i([\phi]) \neq \pm\mathbb{I} \\ \mathbb{C}^* & \text{if } f_{\alpha_i}([\phi]) \neq \pm 2 \end{cases}$$

For each i with $f_{\alpha_i}(\phi) \neq 2$, let

$$\lambda_i = \frac{f_{\alpha_i}(\phi) \pm (f_{\alpha_i}(\phi)^2 - 4)^{1/2}}{2}.$$

Then $(k_1, \dots, k_N) \in \mathbb{Z}^N$ acts by translation by k_i on the i -th factor $G_i \approx \mathbb{C}$ for each i with $f_{\alpha_i}(\phi) = \pm 2$ and by multiplication by λ_i on each factor $G_i \approx \mathbb{C}^*$ with $f_{\alpha_i}(\phi) \neq \pm 2$.

3.2. The moment map as an ergodic decomposition and its holomorphic extension. The action of $\Gamma_{\mathcal{P}}$ on X_U is discussed in [15]. In this case the restriction

$$f_{\mathcal{P}} : X_U \longrightarrow [-2, 2]^N$$

is a moment map for a Hamiltonian \mathbb{R}^n -action whose orbits are all closed. These orbits are tori, invariant under the $\Gamma_{\mathcal{P}} \cong \mathbb{Z}^N$ -action. Almost every fiber has dimension N , and the action is equivalent to an \mathbb{Z}^N -action on T^N by translation. For almost all level sets, this action is ergodic. It follows that the moment map $f_{\mathcal{P}}$ for the Hamiltonian \mathbb{R}^N -action is also the *ergodic decomposition* for the \mathbb{Z}^N -action ([15], Theorem 2.2):

Proposition 3.2.1. *Let $h : X_U \longrightarrow \mathbb{R}$ be a $\Gamma_{\mathcal{P}}$ -invariant measurable function. Then there exists a measurable function $\psi : [-2, 2]^N \longrightarrow \mathbb{R}$ such that $h = \psi \circ f_{\mathcal{P}}$ almost everywhere.*

Theorem 3.2.2. *Let $h : X \longrightarrow \mathbb{C}\mathbb{P}^1$ be a meromorphic function which is invariant under $\Gamma_{\mathcal{P}}$. Then there exists a meromorphic function $H : \mathbb{C}^{\mathcal{P}} \longrightarrow \mathbb{C}\mathbb{P}^1$ such that $h = H \circ \tau_{\mathcal{P}}$.*

Proof. Let G denote the complex-Hamiltonian $\mathbb{C}^{\mathcal{P}}$ -action on X and let G_U denote the Hamiltonian $\mathbb{R}^{\mathcal{P}}$ -action on X_U . By Proposition 3.2.1, the measurable function h must be almost everywhere constant on the G_U -orbits on X_U . Thus for each $t \in \tau_{\mathcal{P}}(X_U)$, the function h is constant on the preimage $\tau_{\mathcal{P}}^{-1}(t) \cap X_U$ and:

- The preimages of the restriction of $\tau_{\mathcal{P}}$ to X_U are the G_U -orbits in X_U ;
- Each G_U -orbit in $\tau_{\mathcal{P}}^{-1}(t)$ is \mathbb{C} -Zariski dense in its G -orbit;
- G acts transitively on $\tau_{\mathcal{P}}^{-1}(t)$.

Hence h is constant on each $\tau_{\mathcal{P}}^{-1}(t)$. for $t \in \tau_{\mathcal{P}}(X_U)$.

Now we find a holomorphic section of $\tau_{\mathcal{P}}$. There exists a connected open neighborhood W of X_U , a Zariski-closed subset $Z \subset X$ of positive codimension, and a holomorphic map

$$W - Z \xrightarrow{\sigma} \tau_p p^{-1}(W - Z)$$

so that the composition

$$W - Z \xrightarrow{\sigma} \tau_p p^{-1}(W - Z) \xrightarrow{\tau_{\mathcal{P}}} W - Z$$

equals the identity. Then the meromorphic function

$$h - h \circ \sigma \circ \tau_{\mathcal{P}}$$

on $\tau_p p^{-1}(W - Z)$ vanishes on each fiber $\tau_p p^{-1}(t)$ for $t \in W - Z \cap X_U$. Since $W - Z \cap X_U$ is totally real, it follows that $h - h \circ \sigma \circ \tau_{\mathcal{P}}$ vanishes on all of $W - Z$. Since $W - Z$ is dense in W , this function vanishes on W . Since W is nonempty and open, this function vanishes on all of X . Thus h is constant on the fibers of $\tau_{\mathcal{P}}$ as desired. \square

4. DEFORMATION SPACES OF \mathbb{CP}^1 -STRUCTURES

A \mathbb{CP}^1 -*structure* on M is a geometric structure with local coordinate charts mapping to \mathbb{CP}^1 with coordinate changes in the group $\mathrm{PSL}(2, \mathbb{C})$. The holonomy mapping hol maps the deformation space $\mathbb{CP}^1(M)$ locally biholomorphically to the character variety $X - X_U$. Therefore Ω induces a complex-symplectic structure $\mathrm{hol}^* \Omega$ on $\mathbb{CP}^1(M)$. On the other hand, $\mathbb{CP}^1(M)$ is classically the total space of an affine bundle over $\mathfrak{X}(M)$ whose associated vector bundle is the cotangent bundle $T^* \mathfrak{X}(M)$. The canonical exact complex-symplectic structure on $T^* \mathfrak{X}(M)$ defines a complex-symplectic structure on $\mathbb{CP}^1(M)$. Kawai [30] has proved this complex-symplectic structure equals $\mathrm{hol}^* \Omega$:

Theorem. (Kawai) *The complex-symplectic structure on $\mathbb{CP}^1(M)$ induced from the holomorphic cotangent bundle structure on $T^* \mathfrak{X}(M)$ equals the complex-symplectic structure induced from the complex-symplectic structure Ω on X by*

$$\mathbb{CP}^1(M) \xrightarrow{\mathrm{hol}} \mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C})) // \mathrm{SL}(2, \mathbb{C}).$$

The complex twist flows on X lift to complex twist flows on $\mathbb{CP}^1(M)$. We relate the complex twist geometry on the open subset $\mathcal{QF}(M) \subset \mathbb{CP}^1(M)$ comprising *quasi-Fuchsian structures* to the quakebends studied by Epstein-Marden [7], Kourouniotis [32, 33, 34], McMullen [36], Platis [39], Series [42], and Tanigawa [44].

4.1. \mathbb{CP}^1 -structures. Let M be a smooth surface. A \mathbb{CP}^1 -atlas on M consists of

- An open covering \mathcal{U} of M by *coordinate patches*;
- For each coordinate patch $U \in \mathcal{U}$, a *coordinate chart*

$$\psi_U : U \longrightarrow \mathbb{CP}^1$$

which is a diffeomorphism of U onto its image $\psi_U(U)$;

- For each connected open subset $C \subset U \cap V$ of the intersection of coordinate patches $U, V \in \mathcal{U}$, a transformation $g_C \in \mathrm{PSL}(2, \mathbb{C})$ such that

$$g_C \circ \psi_U|_{U \cap V} = \psi_V|_{U \cap V}.$$

The *coordinate change* g_C is unique.

Such atlases are partially ordered by inclusion. A \mathbb{CP}^1 -structure on M is a maximal \mathbb{CP}^1 -atlas. A \mathbb{CP}^1 -manifold is a manifold with a \mathbb{CP}^1 -structure.

Suppose M and M' are \mathbb{CP}^1 -manifolds. A mapping $\phi : M \longrightarrow M'$ is a \mathbb{CP}^1 -mapping if and only if for each coordinate chart (U, ψ_U) in M and each coordinate chart $(U', \psi'_{U'})$ in M' the composition

$$(\psi'_{U'}) \circ \phi \circ (\psi_U)^{-1} : \psi_U(U \cap \phi^{-1}(U')) \longrightarrow \psi'_{U'}(\phi(U) \cap U')$$

extends to an element of $\mathrm{PSL}(2, \mathbb{C})$ on each component of $\psi_U(U \cap \phi^{-1}(U'))$. A \mathbb{CP}^1 -mapping is necessarily a local diffeomorphism.

If $f : M \longrightarrow M'$ is a local diffeomorphism, then every \mathbb{CP}^1 -structure on M' determines a unique \mathbb{CP}^1 -structure on M such that f is a \mathbb{CP}^1 -mapping. In particular every covering space of a \mathbb{CP}^1 -manifold is a \mathbb{CP}^1 -manifold, and its group of covering transformations is realized by a group of \mathbb{CP}^1 -automorphisms.

If M is a simply-connected \mathbb{CP}^1 -manifold, then any chart on M extends to a globally defined *developing map*

$$\mathrm{dev} : M \longrightarrow \mathbb{CP}^1$$

which is a \mathbb{CP}^1 -mapping. This mapping is unique up to left-composition with transformations in $\mathrm{PSL}(2, \mathbb{C})$.

We denote by Σ the Riemann surface whose underlying manifold is M for which dev is holomorphic.

In general the universal covering space \tilde{M} of a $\mathbb{C}\mathbb{P}^1$ -manifold admits a developing map into $\mathbb{C}\mathbb{P}^1$. The group $\pi_1(M)$ of covering transformations of $\tilde{M} \rightarrow M$ admits a representation $\rho : \pi_1(M) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ such that the diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\mathrm{dev}} & \mathbb{C}\mathbb{P}^1 \\ \gamma \downarrow & & \downarrow \rho(\gamma) \\ \tilde{M} & \xrightarrow{\mathrm{dev}} & \mathbb{C}\mathbb{P}^1 \end{array}$$

commutes for every $\gamma \in \pi_1(M)$. The representation ρ is called the *holonomy representation*. The *developing pair* (dev, ρ) is unique up to the action of $\mathrm{PSL}(2, \mathbb{C})$ (by left-composition with dev and conjugation on ρ). The developing map globalizes the coordinate charts and the holonomy representation globalizes the coordinate changes.

4.1.1. *Deformation spaces.* Let S be a fixed topological surface with fundamental group $\pi = \pi(S)$. A *marked $\mathbb{C}\mathbb{P}^1$ -structure* on S consists of a $\mathbb{C}\mathbb{P}^1$ -manifold M and a homeomorphism $f_M : S \rightarrow M$. Two marked $\mathbb{C}\mathbb{P}^1$ -structures (M, f_M) and $(M', f_{M'})$ are *equivalent* if and only if there exists a $\mathbb{C}\mathbb{P}^1$ -isomorphism $\phi : M \rightarrow M'$ such that

$$\phi \circ f_M \simeq f_{M'}.$$

Let $\mathbb{C}\mathbb{P}^1(S)$ denote the set of equivalence classes of marked $\mathbb{C}\mathbb{P}^1$ -structures on S . Holonomy defines a mapping

$$\mathrm{hol} : \mathbb{C}\mathbb{P}^1(S) \rightarrow \mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C})) // \mathrm{SL}(2, \mathbb{C}).$$

As in [14] (see also Earle [6], Hubbard [22] and Kapovich [28]), there is a natural topology on $\mathbb{C}\mathbb{P}^1(S)$ such that hol is a local homeomorphism. Furthermore Gallo-Kapovich-Marden [8] proved that the image of hol is $X - X_U - X_D$, where $X_D \subset X - X_U$ is the closed subset consisting of equivalence classes $[\phi]$ of representations leaving invariant a pair of points of $\mathbb{C}\mathbb{P}^1$ (these correspond to irreducible representations for which an index-two subgroup acts reducibly).

4.2. **Holonomy and the conformal structure.** Since $\mathrm{PSL}(2, \mathbb{C})$ acts holomorphically on $\mathbb{C}\mathbb{P}^1$, a $\mathbb{C}\mathbb{P}^1$ -atlas on M is a *holomorphic atlas*, that is, an atlas for a complex structure on M . Thus underlying every $\mathbb{C}\mathbb{P}^1$ -manifold is a Riemann surface. Recording the complex structure underlying a $\mathbb{C}\mathbb{P}^1$ -structure is a map

$$\mathbb{C}\mathbb{P}^1(S) \xrightarrow{\Pi} \mathfrak{T}(M).$$

A *projective structure on a Riemann surface* Σ is a $\mathbb{C}\mathbb{P}^1$ -structure whose underlying complex structure is Σ .

Lemma 4.2.1. Π is holomorphic.

Proof. We recall the classical description of $\mathbb{C}\mathbb{P}^1(S)$ as an affine bundle $\mathcal{Q}(M)$ over $\mathfrak{T}(M)$ whose associated vector bundle equals the cotangent bundle $T^*(\mathfrak{T}(M))$. (See for example §2.3 of Earle [6], Hubbard [22] or Gunning [18]). Fix a Riemann surface Σ homeomorphic to M and a projective structure s_0 on Σ . Given any other projective structure on Σ , its developing map is a holomorphic map from the universal covering $\tilde{\Sigma}$ to $\mathbb{C}\mathbb{P}^1$. Its Schwarzian derivative (in the local projective coordinates defined by s_0) is a holomorphic quadratic differential which is invariant under π , and therefore defines a holomorphic quadratic differential on Σ . Furthermore this quadratic differential completely determines the developing map up to an element of $\mathrm{PSL}(2, \mathbb{C})$ of $\mathbb{C}\mathbb{P}^1$. Thus the fiber $\Pi^{-1}([\Sigma])$ admits a simply transitive action of the complex vector space $H^0(\Sigma; \mathcal{O}(\kappa^2))$ comprising holomorphic quadratic differentials on Σ . The composition law for the Schwarzian derivative implies that changing the origin s_0 effects a translation of $Q(\Sigma)$, so $\Pi^{-1}(\Sigma)$ is an affine space modelled on $H^0(\Sigma; \mathcal{O}(\kappa^2))$, which is the cotangent space $T_{[\Sigma]}^*\mathfrak{T}(M)$ of $\mathfrak{T}(M)$ at $[\Sigma]$.

In particular $\mathbb{C}\mathbb{P}^1(S)$ inherits a complex structure, making it a holomorphic affine bundle over $\mathfrak{T}(M)$. In particular the projection Π is holomorphic in this complex structure.

By Lemma 2 of Earle [6] or Hubbard [22], the holonomy mapping hol is a local biholomorphism, and therefore this complex structure on $\mathbb{C}\mathbb{P}^1$ induced by hol is the above complex structure. Thus Π is holomorphic with respect to the complex structure induced by hol . \square

Kawai's theorem implies that Π is a holomorphic Lagrangian fibration. As described in Hubbard [22], the differential $d\Pi$ identifies with the map

$$T_{s_0}\mathbb{C}\mathbb{P}^1(S) \cong H^1(\Sigma; \mathfrak{sl}(2, \mathbb{C})) \longrightarrow H^1(\Sigma; \mathcal{O}(\kappa^{-1})) \cong T_{[\Sigma]}\mathfrak{T}(M)$$

induced by the short exact sequence of sheaves

$$(4.1) \quad 0 \longrightarrow \mathfrak{sl}(2, \mathbb{C}) \xrightarrow{i} \mathcal{O}(\kappa^{-1}) \xrightarrow{D_3} \mathcal{O}(\kappa^2) \longrightarrow 0.$$

Here D_3 is the map of holomorphic sheaves given in local holomorphic coordinates by

$$h(z) \frac{\partial}{\partial z} \longmapsto h'''(z) dz^2.$$

Its kernel $\mathfrak{sl}(2, \mathbb{C})$ is the locally constant sheaf of locally projective vector fields (whose coefficients in affine coordinates are quadratic). The

monomorphism i regards a locally projective vector field as a holomorphic vector field. Then (4.1) induces the exact sequence

$$0 \longrightarrow H^0(\Sigma, \mathcal{O}(\kappa^2)) \xrightarrow{\delta} H^1(\Sigma; \mathfrak{sl}(2, \mathbb{C})) \xrightarrow{i_*} H^1(\Sigma; \mathcal{O}(\kappa^{-1})) \longrightarrow 0.$$

If $\alpha \in H^0(\Sigma, \mathcal{O}(\kappa^2))$ is a holomorphic quadratic differential and $\beta \in H^1(\Sigma; \mathfrak{sl}(2, \mathbb{C}))$, then

$$(4.2) \quad \langle \delta(\alpha), \beta \rangle = \alpha \cdot i_*(\beta)$$

where the first pairing \langle, \rangle is the pairing on $H^1(\Sigma; \mathfrak{sl}(2, \mathbb{C}))$ induced by the complex Killing form \mathbb{B} on $\mathfrak{sl}(2, \mathbb{C})$ and the second pairing is the Serre duality pairing

$$H^0(\Sigma, \mathcal{O}(\kappa^2)) \times H^1(\Sigma, \mathcal{O}(\kappa^{-1})) \longrightarrow H^1(\Sigma, \mathcal{O}(\kappa)) \cong \mathbb{C}.$$

This discussion implies that Π is a Lagrangian fibration. Suppose

$$\beta, \gamma \in \ker(d\Pi) = \delta(H^0(\Sigma, \mathcal{O}(\kappa^2))) = \ker(i_*)$$

Then $\gamma = \delta(\alpha)$ for some $\alpha \in H^0(\Sigma, \mathcal{O}(\kappa^2))$, and (4.2) implies $\langle \gamma, \beta \rangle = 0$ since $i_*\beta = 0$. Therefore $\ker(d\Pi)$ is isotropic. Since its dimension is half of that of $H^1(\Sigma; \mathfrak{sl}(2, \mathbb{C}))$, the fibers of Π are Lagrangian. For further details, see Kawai [30].

4.2.1. Meromorphic functions on $\mathbb{CP}^1(S)$.

Proposition 4.2.2. *There exist nonconstant Γ -invariant meromorphic functions on $\mathbb{CP}^1(S)$.*

Proof. The Riemann moduli space of curves is the quotient $\mathfrak{T}(M)/\Gamma$, which by Knudsen [31], is a quasiprojective variety. Thus $\mathfrak{T}(M)/\Gamma$ admits nonconstant meromorphic functions and $\mathfrak{T}(M)$ admits nonconstant Γ -invariant meromorphic functions.

Let ψ be a nonconstant Γ -invariant meromorphic function on $\mathfrak{T}(M)$. Composing with the projection

$$\Pi : \mathbb{CP}^1(S) \longrightarrow \mathfrak{T}(M)$$

provides a nonconstant Γ -invariant meromorphic function $\psi \circ \Pi$ on $\mathbb{CP}^1(S)$. \square

4.3. Quasi-Fuchsian space. A \mathbb{CP}^1 -structure is *Fuchsian* if and only if a developing map embeds \tilde{M} in a hyperbolic plane $\mathbb{H}_{\mathbb{C}}^1 \subset \mathbb{CP}^1$. Necessarily the holonomy representation is conjugate to a *Fuchsian representation*, that is, a discrete embedding $\pi \hookrightarrow \text{PU}(1, 1)$ (where $\text{PU}(1, 1)$ is the stabilizer in $\text{PSL}(2, \mathbb{C})$ of the Poincaré disc $\mathbb{H}_{\mathbb{C}}^1$). However by a construction of Maskit and Hejhal (see also Goldman [12]) there exist \mathbb{CP}^1 -structures with Fuchsian holonomy with surjective developing

maps. A *quasi-Fuchsian* \mathbb{CP}^1 -structure is a \mathbb{CP}^1 -structure topologically conjugate to a Fuchsian structure, that is, there exists a homeomorphism h of \mathbb{CP}^1 such that

$$(h \circ \mathbf{dev}, h \circ \rho \circ h^{-1})$$

is the developing pair for a Fuchsian structure. The limit set Λ of $\rho(\pi)$ is the Jordan curve $h(\partial\mathbb{H}_{\mathbb{C}}^1)$. The developing image $\mathbf{dev}(\tilde{M})$ is one component of the complement $\mathbb{CP}^1 - \Lambda$.

Quasi-Fuchsian space $\mathcal{QF}(M)$ is the subset of $\mathbb{CP}^1(S)$ corresponding to quasi-Fuchsian structures. The holonomy representation ρ is a *quasi-Fuchsian representation*. The holonomy mapping

$$\mathbf{hol} : \mathcal{QF}(M) \longrightarrow X$$

embeds $\mathcal{QF}(M)$ as the open subset of X consisting of conjugacy classes of quasi-Fuchsian representations. Quasi-Fuchsian space $\mathcal{QF}(M)$ is both the open subset of $\mathbb{CP}^1(S)$ consisting of quasi-Fuchsian structures and the open subset of $X - X_U$ consisting of characters of quasi-Fuchsian representations.

The conformal structures of the quotients of $\mathbb{CP}^1 - \Lambda$ by $\phi(\pi)$ determine an ordered pair $\mathfrak{T}(M) \times \bar{\mathfrak{T}}(M)$; the first parameter is the marked conformal structure underlying the \mathbb{CP}^1 -structure. The *simultaneous uniformization theorem* of Bers [3] asserts that the corresponding map

$$\mathcal{QF}(M) \xrightarrow{\sim} \mathfrak{T}(M) \times \bar{\mathfrak{T}}(M)$$

is a biholomorphism. It is evidently Γ -equivariant.

Since $\mathcal{QF}(M) \hookrightarrow \mathbb{CP}^1(S)$, Corollary 4.2.2 implies that nonconstant Γ -invariant meromorphic functions exist on $\mathcal{QF}(M)$.

4.4. Twist flows of \mathbb{CP}^1 -structures and grafting. Let $\alpha \in \pi$. Then the composition

$$\mathbb{CP}^1(S) \xrightarrow{\mathbf{hol}} X \xrightarrow{f_\alpha} \mathbb{C}$$

defines a holomorphic function on $\mathbb{CP}^1(S)$ and its complex-Hamiltonian $\mathbf{Ham}(f_\alpha \circ \mathbf{hol})$ is a holomorphic vector field on $\mathbb{CP}^1(S)$. If α corresponds to a simple closed curve A , then there is a *complex twist flow* $\tilde{\xi}_t^\alpha$ on \mathbb{CP}^1 covering the complex twist flow generated by the vector field $\mathbf{Ham}(f_\alpha)$ on X .

This twist flow is defined geometrically on Fuchsian structures as follows (see [12], Kourouniotis [32, 33, 34]). Let s_0 be a Fuchsian \mathbb{CP}^1 -structure, so a developing map embeds \tilde{M} as a geometric disc $\Delta \subset \mathbb{CP}^1$. Then the simple closed curve $A \subset M$ can be represented by a simple closed geodesic with respect the Poincaré metric induced from that of Δ . Let $\ell(A)$ denote the Poincaré length of this geodesic. A lift \tilde{A} of A to

\tilde{M} develops to a circular arc orthogonal to $\partial \text{dev}(\tilde{M})$. The split surface $M|A$ inherits a \mathbb{CP}^1 -structure whose boundary components develop to circular arcs. To define $s_t := \tilde{\xi}^\alpha_t(s_0)$, insert an annulus A_θ with \mathbb{CP}^1 -structure into $M|A$. The angular parameter θ is the imaginary part $\text{Im}(t)$. The annulus is a θ -annulus in the sense of §2.12 of [12]. Choose a holomorphic universal covering map

$$E : \mathbb{C} \longrightarrow \mathbb{CP}^1 - \text{Fix}(\phi(\alpha))$$

which is periodic with period $2\pi i$. (When $\text{Fix}(\phi(\alpha)) = \{0, \infty\}$, then this is just the exponential map $\exp : \mathbb{C} \longrightarrow \mathbb{C}^*$. For general A , we compose \exp with a projective transformation taking $\{0, \infty\}$ to $\text{Fix}(\phi(\alpha))$.)

Define the θ -strip

$$S_\theta := \mathbb{R} + i[0, \theta] \subset \mathbb{C}$$

with \mathbb{CP}^1 -structure induced by E . The developing map E for S_θ is equivariant with respect to the \mathbb{Z} -action on S_θ generated by translation by $\ell(A)$ and the \mathbb{Z} -action on $\mathbb{CP}^1 - \text{Fix}(\phi(\alpha))$. The θ -annulus A_θ is defined as the quotient \mathbb{CP}^1 -manifold of S_θ .

The *grafted manifold* $M(t)$ is obtained by inserting a θ -annulus into $M|A$. Choose one component of $A_- \subset \partial(M|A)$ and attach A_θ to A_- to obtain a \mathbb{CP}^1 -manifold homeomorphic to $M|A$ with one boundary component A'_+ corresponding to the component A_+ of $\partial(M|A)$ and another boundary component A' corresponding to the other component of $\partial(A_\theta)$. Identify A'_+ to A' by a transvection of displacement $\text{Re}(t)$.

When $\theta = 0$, this is just the Fenchel-Nielsen twist deformation, obtained from $M|A$ by identifying the two components of $\partial(M|A)$ by a transvection of displacement $\text{Re}(t)$.

Let $P \in \text{SL}(2, \mathbb{C})$. These holomorphic flows on $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$ cover holomorphic flows on X which have been extensively studied on quasi-Fuchsian space $\mathcal{QF}(M) \subset X$. On $\mathcal{QF}(M)$ these flows geometrically correspond to the *quakebends* or *complex earthquakes* discussed by Epstein-Marden [7], Kourouniotis [32, 33, 34], McMullen [36], Platis [39], Series [42], and Tanigawa [44]. In particular we obtain the following result of Platis [39]:

Theorem 4.4.1 ((Platis [39], Theorem 7, §2.2). *The complex twist vector field $\tilde{\xi}$ on $\mathcal{QF}(M)$ is Hamiltonian with respect to the complex length function $l^{\mathbb{C}}$.*

Just as the geodesic length function l on the hyperbolic subset of $\text{SL}(2, \mathbb{R})$ and the angle function θ on $\text{SU}(2) - \{\pm \mathbb{I}\}$ are more geometrically natural than the trace functions, we consider the *complex length*

“function” on $\mathrm{SL}(2, \mathbb{C})$ defined by:

$$2 \cosh \left(\frac{l^{\mathbb{C}}(P)}{2} \right) = f(P)$$

or equivalently

$$l^{\mathbb{C}}(P) = 2 \log \left(\frac{(f(P) \pm (f(P)^2 - 4)^{1/2})}{2} \right).$$

(This differs from Tan [43] whose complex length is half of ours. Our definition is consistent with Kourouniotis [32, 33, 34], Platis [39] and Series [42].) This function takes values in $(\mathbb{C}/4\pi i\mathbb{Z})/\{\pm 1\}$, since the logarithm is well-defined only modulo $4\pi i$ and the choice of square root introduces an ambiguity of sign.

Choose the branch $\tilde{l}_{\mathbb{C}}$ of the complex length which is positive on the subset of hyperbolic elements in $\mathrm{SL}(2, \mathbb{R})$; since $\mathcal{QF}(M)$ is simply connected

$$[\phi] \mapsto (\tilde{l}^{\mathbb{C}}(\phi(\alpha_1)), \dots, \tilde{l}^{\mathbb{C}}(\phi(\alpha_N)))$$

defines a single-valued function

$$l^{\mathbb{C}}|_{\mathcal{P}} : \mathcal{QF}(M) \longrightarrow \mathbb{C}^N.$$

The restriction of $l_{\mathcal{P}}^{\mathbb{C}}$ to $\mathfrak{T}(M)$ is the length function $l_{\mathcal{P}} : \mathfrak{T}(M) \longrightarrow (\mathbb{R}_+)^N$ defined by (3.1). The fibers of $l^{\mathbb{C}}|_{\mathcal{P}}$ are orbits of the complex-Hamiltonian \mathbb{C}^N -action having $l^{\mathbb{C}}|_{\mathcal{P}}$ as moment map. Tan [43] and Kourouniotis [34] show that there exists a section $\sigma : \mathbb{C}^N \longrightarrow \mathcal{QF}(M)$ and a neighborhood of $(\mathbb{R}_+)^N \times \{0\}$ in $\mathbb{C}^N \times \mathbb{C}^N$ which maps biholomorphically to $\mathcal{QF}(M)$.

Kourouniotis [33] and Series [42] derive formulas for the derivative of the complex length functions along quakebend flows. We briefly sketch how their formulas can be derived from Proposition 2.2.2 as a Poisson bracket, referring to [33, 42] for details.

Let $P \in \mathrm{SL}(2, \mathbb{C})$ be loxodromic with invariant axis a_P . Then the variation function $L^{\mathbb{C}}$ associated to the invariant function $l^{\mathbb{C}}$ and the complex-orthogonal structure $\mathbb{B}(X, Y) = \mathrm{tr}(XY)$ maps a diagonal matrix to

$$D_0 := \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(which is not uniquely defined without restrictions to make the function $l^{\mathbb{C}}$ single-valued). By (2.1), the element of $\mathrm{PSL}(2, \mathbb{C})$ corresponding to $L^{\mathbb{C}}(P)$ is the involution fixing a_P . If $P_1, P_2 \in \mathrm{PSL}(2, \mathbb{C})$ are loxodromic, with complex distance z between their invariant axes, then a simple calculation implies that $\mathbb{B}(L^{\mathbb{C}}(P_1), L^{\mathbb{C}}(P_2))$ equals $\cosh(z)$. Applying Proposition 2.2.2 to the complex length functions $l_{\beta}^{\mathbb{C}}$ and $l_{\alpha}^{\mathbb{C}}$, where α

corresponds to a simple closed curve, we obtain the following extension of Theorem 2.6.2:

Theorem 4.4.2 ((Korouniotis [33], Series [42]). *Let $\alpha, \beta \in \pi$ where α is represented by a simple closed curve A . Then the derivative of the complex length function l_β^C on $\mathcal{QF}(M)$ with respect to the quakebend flow with respect to A equals the sum*

$$\sum_{i=1}^k \cosh(d_i)$$

where α and β are represented by closed geodesics, p_1, \dots, p_k are their intersection points, and d_i the complex distance between the axes of $\phi_i(\alpha_i)$ to $\phi_i(\beta_i)$ at p_i , where ϕ_i is a representative homomorphism from $\pi_1(M; p_i)$ and α_i, β_i are elements of $\pi_1(M; p_i)$ representing α and β .

See Korouniotis [33] and Series [42] for details.

Tan [43] and Kourouniotis [32, 33, 34] extend Fenchel-Nielsen coordinates to complex Fenchel-Nielsen coordinates on $\mathcal{QF}(M)$. (See also Series [42] for another exposition.)

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