DEFORMATION SPACES OF CONVEX REAL-PROJECTIVE STRUCTURES AND HYPERBOLIC AFFINE STRUCTURES

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1. Introduction

A convex $\mathbb{RP}^2$-structure on a smooth manifold M is a representation of M as a quotient of a convex domain $\Omega \subset \mathbb{RP}^2$ by a discrete group Γ of collineations of $\mathbb{RP}^2$ acting properly on Ω. When M is a closed surface of genus $g > 1$, then the equivalence classes of such structures forms a moduli space $\mathfrak{P}(M)$ homeomorphic to an open cell of dimension $16(g-1)$ (Goldman [2]). This cell contains the Teichmüller space $\mathcal{T}(M)$ of M and it is of interest to know what of the rich geometric structure extends to $\mathfrak{P}(M)$. In [3], a symplectic structure on $\mathfrak{P}(M)$ is defined, which extends the symplectic structure on $\mathcal{T}(M)$ defined by the Weil-Petersson Kähler form.

We conjecture that $\mathfrak{P}(M)$ itself admits a Kähler geometry extending this symplectic structure and restricting to the Weil-Petersson geometry on $\mathcal{T}(M)$.

At the first step in this project we define a Riemannian metric on $\mathfrak{P}(M)$. This metric exists for structures in all dimensions. The basic technique is that a canonically defined Riemannian metric on Ω/Γ defines a Riemannian structure on the moduli space via the Hodge theory of harmonic forms. Then we define a bundle map J on the tangent bundle of $\mathfrak{P}(M)$ by using the previous Riemannian and symplectic structures. We show that J is an almost complex structure.

Another result is the following alternative description of $\mathfrak{P}(M)$. Fix $\lambda > 1$ and let Λ be the multiplicative group of integral powers of λ. For any manifold M, let M' denote the Cartesian product $M \times S^1$. Let $\frac{\partial}{\partial\theta}$ be the vector field on M' generating the flow $\Phi_t : (x, \theta) \mapsto (x, \theta + t)$. Let $d\theta$ the closed 1-form on M' tangent to the projection $M' \longrightarrow S^1$. The group $\mathcal{D}$ of diffeomorphisms of M isotopic to the identity acts on

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M' by $h : (x,\theta) \longrightarrow (h(x),\theta)$ and hence acts on the space of all affine connections on M'.

Theorem 1. *The deformation space* $\mathfrak{P}(M)$ *of convex* $\mathbb{RP}^{\nvDash}$*-structures on* M *identifies with the space of* $\mathcal{D}$*-orbits of flat torsionfree affine connections* ∇ *on* M' *such that:*

- $\frac{\partial}{\partial\theta}$ *is* radiant *with respect to* ∇*: for any vector field* $X \in \mathrm{Vect}(M')$*,* $\nabla_X \frac{\partial}{\partial\theta} = X$*;*
- $\nabla(d\theta) > 0$*;*
- *Each trajectory of* Φ *is an closed geodesic affinely isomorphic to a Hopf circle* $\mathbb{R}_+/\Lambda$*;*

2. Deformation space of convex $\mathbb{RP}^{\nvDash}$-structures

Let M be a smooth 2-manifold.A real projective structure on M is a maximal collection $\{(U_\alpha,\psi_\alpha)\}$, such that

- $\{U_\alpha\}$ is an open cover of M,
- For each α, $\psi_\alpha : U_\alpha \longrightarrow \mathbb{RP}^{\nvDash}$ is a surjective diffeomorphism,
- The change of coordinates are locally projective: If $\{(U_\alpha,\psi_\alpha)\}$ and $\{(U_\beta,\psi_\beta)\}$ are two such coordinate charts, then the restriction of $\psi_\beta \circ \psi_\alpha^{-1}$ to any connected component of $\psi_\beta^{-1}(\psi_\alpha(U_\alpha \cap U_\beta))$ is a projective transformation.

A manifold with an $\mathbb{RP}^{\nvDash}$-structure is called an $\mathbb{RP}^{\nvDash}$-manifold. A fundamental fact about $\mathbb{RP}^{\nvDash}$-structures is the following *Development Theorem,* apparently due to Ch. Ehresmann in 1936 ([1]).

Theorem 2. *Let* M *be an* $\mathbb{RP}^{\nvDash}$*-manifold and denote its a universal covering space by* $p : \tilde{M} \longrightarrow M$*. Let* π *be the corresponding group of covering transformations.*

1. *There exist a projective map* $\mathrm{dev} : \tilde{M} \longrightarrow \mathbb{RP}^{\nvDash}$ *and a homomorphism* $h : \pi \longrightarrow \mathrm{PGL}(3,\mathbb{R})$ *such that for each* $\gamma \in \pi$*, the following diagram commutes:*

$$\begin{tikzcd}
\tilde{M} \arrow[r, "\mathrm{dev}"] \arrow[d, "\gamma"'] & \mathbb{RP}^{\nvDash} \arrow[d, "h(\gamma)"] \\
\tilde{M} \arrow[r, "\mathrm{dev}"'] & \mathbb{RP}^{\nvDash}
\end{tikzcd}$$

2. *If* (dev',h') *is another such pair, then there exists a projective transformation* $g \in PGL(3,\mathbb{R}$ *such that* $\mathrm{dev}' = g \circ \mathrm{dev}$ *and* $h' = i_g \circ h$ *where where* $i_g : \mathrm{PGL}(3,\mathbb{R}) \longrightarrow \mathrm{PGL}(\nvDash,\mathbb{R})$ *denotes the inner automorphism defined by* g [4].

An $\mathbb{RP}^{\nvDash}$-structure on M is called *convex* if dev is a diffeomorphism of $\tilde{M}$ onto a convex domain in $\mathbb{RP}^{\nvDash}$. .

Let S denote a compact surface . Define

$$\mathcal{E} = \{(\{, \mathcal{M}) \mid \{ : \mathcal{S} \longrightarrow \mathcal{M} \text{ is a diffeomorphism and } \mathcal{M} \text{ is an } \mathbb{RP}^{\nvDash}\text{-manifold}\}.$$

Two elements $(f, M), (f', M') \in \mathcal{E}$ are equivalent if and only if there exists a projective isomorphism $h : M \longrightarrow M'$ such that $h \circ f$ is isotopic to f'.The set of equivalence classes (denoted by $\mathbb{RP}^{\nvDash}(\mathbb{S})$) has a natural topology making it locally equivalent to $\operatorname{Hom}(\pi, \operatorname{PGL}(3, \mathbb{R}))/\operatorname{PGL}(\nVDash, \mathbb{R})$. .In other words there exists a map

$$\operatorname{hol} : \mathbb{RP}^{\nvDash}(\mathbb{S}) \longrightarrow \operatorname{Hom}(\pi, \operatorname{PGL}(\nVDash, \mathbb{R}))/\operatorname{PGL}(\nVDash, \mathbb{R})$$

which is a local diffeomorphism. Let $\mathfrak{P}(M)$ denote the subset of $\mathbb{RP}^{\nvDash}(\mathbb{S})$ corresponding to convex $\mathbb{RP}^{\nvDash}$-structures. $\mathfrak{P}(M)$ is an open set and the restriction of hol to $\mathfrak{P}(M)$ is an embedding of $\mathfrak{P}(M)$ onto an open set of $\operatorname{Hom}(\pi, \operatorname{PGL}(3, \mathbb{R}))/\operatorname{PGL}(\nVDash, \mathbb{R})$ ([2]). The Zariski tangent space to $\operatorname{Hom}(\pi, \operatorname{PGL}(3, \mathbb{R}))/\operatorname{PGL}(\nVDash, \mathbb{R})$ at $\{\phi\}$ is isomorphic to $H^1(\pi, \mathfrak{sl}(3, \mathbb{R})_{\operatorname{Ad}\approx})$ which by De Rham's theorem is isomorphic to $H^1(S, \xi)$ where ξ is the flat $\mathfrak{sl}(3, \mathbb{R})$-bundle over S with holonomy representation Adh). (See [5].)

3. A WEIL-PETERSSON METRIC ON $\mathfrak{P}(M)$

Let S be a compact surface with $\chi(S) < 0$ and let (f, M) correspond to a convex $\mathbb{RP}^{\nvDash}$-structure as above. In fact M is a quotient Ω/Γ where $\Omega \subset \mathbb{RP}^{\nvDash}$ is a convex domain and $\Gamma \subset \operatorname{PGL}(3, \mathbb{R})$ is a discrete group acting properly on Ω. Let $\Omega' \subset \mathbb{R}^{\nVDash}$ be the corresponding cone in affine space $E = \mathbb{R}^{\nVDash}$. The *dual cone* Ω^* is the subset of the dual vector space E^* consisting of linear functionals $\psi : E \longrightarrow \mathbb{R}$ which are positive on Ω'. Recall the Koszul-Vinberg characteristic function: For $x \in \Omega'$, define

$$f(x) = \int_{\Omega^*} e^{-\psi(x)} d\psi$$

Then f satisfies

$$f(\gamma x) = \det(\gamma)^{-1} f(x) \tag{1}$$

for any $\gamma \in \operatorname{Aut}(\Omega')$ and the Hessian $d^2 \log f$ is a positive definite symmetric bilinear form on E invariant under $\operatorname{Aut}(\Omega')$.

Now consider the section

$$k : \Omega \longrightarrow \Omega'$$

$$k([p]) \longmapsto f(p)^{1/3} p$$

By (1), k is well-defined and $k(\Omega) = f^{-1}(1)$. $k^*(d^2 \log f)$ is a Riemannian metric on Ω invariant under Γ. Hence $k^*(d^2 \log f)$ defines a Riemannian metric on Ω/Γ. Corresponding to every convex $\mathbb{RP}^{\nvDash}$-structure on S, there exists a canonical metric on S, so there exists a Weil-Petersson metric on $\mathfrak{P}(M)$, defined as follows.

Let $[M] \in \mathfrak{P}(S)$. From 2 , we know that

$$T_{[M]}\mathfrak{P}(S) = H^1(M;\xi)$$

The space $\mathcal{A}^\infty(\mathcal{M},\xi)$ of all ξ-valued 1-forms on M consists of sections of the vector bundle $\mathrm{Hom}(TM,\xi)$. For every $x \in f^{-1}(1)$, the symmetric bilinear form $(d^2 \log f)_x$ defines a metric on $\mathrm{Hom}(\mathbb{R}^{\nvDash}, \mathbb{R}^{\nvDash}) \cong \mathbb{R}^{\nvDash} \otimes (\mathbb{R}^{\nvDash})^*$ and therefore induces a Riemannian metric on the bundle ξ. If ϕ, ψ are sections of ξ, then the induced inner product is

$$d^2 \log f(\phi,\psi) = \mathrm{trace}(\phi \circ \tilde{\psi})$$

where $\tilde{\psi}$ is the adjoint of ψ with respect to $d^2 \log f$ ([7]).

On the other hand the Hodge star operator associated to the metric on M defines a metric on $\mathcal{A}^\infty(\mathcal{M})$. The metric on M induces a metric on $\mathcal{A}^\infty(\mathcal{M}, \mathfrak{sl}(\ni,\mathbb{R})_{\mathrm{Ad}\,\approx})$ such that if $\sigma_1 \otimes \phi_1$ and $\sigma_2 \otimes \phi_2$ belong to $\mathcal{A}^\infty(\mathcal{M}, \mathfrak{sl}(\ni,\mathbb{R}))$ then (see [14])

$$(\sigma_1 \otimes \phi_1, \sigma_2 \otimes \phi_2) = \int_M (\sigma_1 \wedge *\sigma_2)\, \mathrm{trace}(\phi_1 \circ \tilde{\phi}_2)$$

This metric induces a metric g on the cohomology $H^1(M;\xi)$ as follows:

Consider the operator δ adjoint to exterior differential d with respect to this inner product, and the corresponding Laplacian Δ on 1-forms:

$$\Delta = d\delta + \delta d$$

The kernel $\mathcal{H}^\infty(\mathcal{M};\xi)$ of Δ and the images of $d : \mathcal{A}'(\mathcal{M};\xi) \longrightarrow \mathcal{A}^\infty(\mathcal{M};\xi)$ and $\delta : \mathcal{A}^\in(\mathcal{M};\xi) \longrightarrow \mathcal{A}^\infty(\mathcal{M};\xi)$ decompose the vector space of 1-forms as an orthogonal direct sum

$$\mathcal{A}^\infty(\mathcal{M};\xi) = \mathcal{H}^\infty(\mathcal{M};\xi) \oplus \lceil\mathcal{A}'(\mathcal{M};\xi) \oplus \delta\mathcal{A}^\in(\mathcal{M};\xi)$$

Consequently each De Rham cohomology class contains a unique harmonic representative. Define the pairing

$$g : \mathcal{H}^\infty(\mathcal{M};\xi) \times \mathcal{H}^\infty(\mathcal{M};\xi) \longrightarrow \mathbb{R}$$

as the tensor product of the inner product on exterior differential forms and the metric on ξ induced from $x^* d^2 \log f$.

4. A SYMPLECTIC FORM ON $\mathfrak{P}(M)$

As in §2, the restriction of $\mathrm{hol} : \mathbb{RP}^{\nvDash}(\mathbb{S}) \longrightarrow \mathrm{Hom}(\pi, \mathrm{PGL}(\nvDash, \mathbb{R}))/\,\mathrm{PGL}(\nvDash, \mathbb{R})$ to $\mathfrak{P}(S)$ embeds $\mathfrak{P}(S)$ as an open subset of $\mathrm{Hom}(\pi, \mathrm{PGL}(3, \mathbb{R}))/\,\mathrm{PGL}(\nvDash, \mathbb{R})$. On the other hand the trace form

$$B : \mathfrak{sl}(3, \mathbb{R}) \times \mathfrak{sl}(\nvDash, \mathbb{R}) \longrightarrow \mathbb{R}$$
$$B(X, Y) = \mathrm{trace}(XY)$$

is an Ad-invariant bilinear form, so defines a bundle pairing $\xi \times \xi \longrightarrow \mathbb{R}$ The natural dual pairing

$$\omega : H^1(M; \xi) \times H^1(M; \xi) \longrightarrow H^2(M; \mathbb{R}) \cong \mathbb{R}$$

defined by the cup-product on M and with B as a coeficient pairing defines a symplectic form on $H^1(M; \xi)$. The induced symplectic structure on

$$\mathrm{Hom}(\pi, \mathrm{PGL}(3, \mathbb{R}))/\,\mathrm{PGL}(\nvDash, \mathbb{R})$$

gives a symplectic structure on $\mathbb{RP}^{\nvDash}(\mathbb{S})$ and in particular one on $\mathfrak{P}(S)$. (See [5] and, for a more analytic treatment, [3].)

As in §2, identify $H^1(\pi; \mathfrak{sl}(3, \mathbb{R}))$ with $H^1(M; \xi)$. Let $\alpha, \beta \in H^1(M; \xi)$ and let $\sum_{i=1}^{k} \sigma_i \otimes \phi_i$ and $\sum_{i=1}^{l} \sigma'_i \otimes \phi'_i$ be harmonic forms representing α, β respectively. Then

$$\sum_{1 \le i \le k, 1 \le j \le l} (\sigma_i \wedge \sigma'_j) \otimes B(\phi_i, \phi_j)$$

is an exterior 2-form and its integral defines the symplectic structure on $\mathfrak{P}(S)$:

$$\omega(\alpha, \beta) = \int_M \sum_{1 \le i \le k, 1 \le j \le l} (\sigma_i \wedge \sigma'_j) \otimes B(\phi_i, \phi_j)$$

5. AN ALMOST COMPLEX STRUCTURE ON $\mathfrak{P}(M)$

In this section we compute the almost complex structure defined on $\mathfrak{P}(M)$, by using the previous Riemannian and symplectic structures. In fact an operator J is defined by

$$\omega(\alpha, \beta) = g(\alpha, J\beta) \tag{2}$$

Lemma 3. *: The action of J on 1-forms level is given by:*

$$J(\sigma \otimes \phi) = - * \sigma \otimes \tilde{\phi}$$

*where * is the Hodge star operator .*

Proof. For finding the action of J , we follow a formal discussion. We guess

$$J(\sigma \otimes \phi) = *\sigma \otimes K(\phi)$$

and determine K such that the operator J satisfies in (1) . We have

$$\omega(\sigma_1 \otimes \phi_1, \sigma_2 \otimes \phi_2) = \int_S \sigma_1 \wedge \sigma_2 \operatorname{trace}(\phi_1 \circ \phi_2) \tag{3}$$

and

$$\begin{aligned} g(\sigma_1 \otimes \phi_1, J(\sigma_2 \otimes \phi_2)) &= g(\sigma_1 \otimes \phi_1, *\sigma_2 \otimes K\phi_2)) \\ &= \int \sigma_1 \wedge *(*\sigma_2) \operatorname{trace}(\phi_1 \circ \widetilde{K\phi_2}) \\ &= -\int \sigma_1 \wedge \sigma_2 \operatorname{trace}(\phi_1 \circ \widetilde{K\phi_2}) \qquad (4) \end{aligned}$$

By formally comparing (3) and (4):

$$\operatorname{trace}(\phi_1 \circ \phi_2) = -\operatorname{trace}(\phi_1 \circ \widetilde{K\phi_2})$$

or $-\widetilde{K\phi_2} = \phi_2$, so $K(\phi_2) = -\tilde{\phi}_2$. Then $J(\sigma \otimes \phi) = - * \sigma \otimes \tilde{\phi}$ because J is uniquely determined by (2). ☐

Lemma 4. *$g(Ja, Jb) = g(a, b)$*

Proof. It suffices to consider $a = \sigma_1 \otimes \phi_1$ and $b = \sigma_2 \otimes \phi_2$. Then

$$\begin{aligned} g(Ja, Jb) &= g(Jb, Ja) \\ &= g(J(\sigma_2 \otimes \phi_2), J(\sigma_1 \otimes \phi_1)) \\ &= g(- * \sigma_2 \otimes \tilde{\phi}_2), - * \sigma_1 \otimes \tilde{\phi}_1) \\ &= \int - * \sigma_2 \wedge *(- * \sigma_1) \operatorname{trace}(\tilde{\phi}_2 \circ \phi_1) \\ &= \int \sigma_1 \wedge *\sigma_2 \operatorname{trace}(\tilde{\phi}_1 \circ \phi_2) \end{aligned}$$

☐

Lemma 5. *$\mathcal{H}^\infty(\mathcal{M}, \xi) = \mathcal{Z}^\infty(\mathcal{M}; \xi) \cap \mathcal{J}\mathcal{Z}^\infty(\mathcal{M}; \xi)$.*

Proof. A differential form is harmonic if and only if it is closed and co-closed. The almost complex structure J preserves the harmonic 1-forms ☐

6. Hessian manifolds

Let M be a (flat) affine manifold . A Riemannian metric g on M is said to be *Hessian* if for each point p there exists a function f defined on a neighborhood of p such that $\nabla df > 0$. A flat affine manifold provided with a Hessian metric is called *a Hessian manifold.* [11] , [12]. The following theorem is due to Koszul [10] :

Theorem 6. *Let M be a connected Hessian manifold with Hessian metric G . Suppose M admits a closed 1-form such that . If there exists a subgroup G of affine transformations of M such that is quasi-compact and that G leaves invariant , then the universal covering manifold of M is a convex domain in a real affine space not containing any full straight line .*

7. Affine connections

For each $\lambda \in \mathbb{R}^+$, let h_λ denote the homothety

$$S \times \mathbb{R}^+ \longrightarrow S \times \mathbb{R}^+$$
$$(s,t) \longmapsto (s, \lambda t)$$

Let dt denote the 1-form on $S \times \mathbb{R}^+$ pulled back from dt on $\mathbb{R}^+$ by projection $t : S \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$. The $t^{-1}dt$ is a 1-form on $S \times \mathbb{R}^+$ invariant under the homotheties above.

Lemma 7. *: Let S be a closed surface with convex real projective structure. Then there exit a radiant affine manifold M, a diffeomorphism to $f : S \times \mathbb{R}^+ \longrightarrow \mathbb{M}$ and and an exact 1-form α_M on M. Let $\alpha_S = (f^{-1})^* \alpha_M$ be the corresponding 1-form on $S \times \mathbb{R}^+$. Then $\alpha_S = t^{-1}dt$ and*

$$(h_\lambda)^* \alpha_S = \lambda \alpha_S$$

Proof. The convex $\mathbb{RP}^{\nvDash}$-structure on S induces a convex $\mathbb{RP}^{\nvDash}$-structure on $\tilde{S}$. Let x be a base-point in S, $\Pi : \tilde{S} \longrightarrow S$ be the corresponding universal covering space and π the corresponding group of covering transformations. By the Development Theorem there exist a projective map dev and a homomorphism ρ such that dev is equivariant with respect to ρ. Let Ω' be the corresponding affine cone. Projectivization defines the structure of a principal $\mathbb{R}^+$-bundle $\Omega' \longrightarrow \Omega$. By definition, dev is a diffeomorphism onto a convex domain Ω. By pulling back this bundle via dev, we obtain a principal $\mathbb{R}^+$-bundle. The open cone Ω' whose projectivization is Ω is the total space of a principal $\mathbb{R}^+$-bundle. Pulling back this bundle via dev produces a principal $\mathbb{R}^+$-principal bundle over $\tilde{M}$. The affine structure on Ω', induced from $\mathbb{R}^{\nVDash} - \{\nvdash\}$, induces an affine structure on S'. There exists a lift $\tilde{h} : \pi_1(S) \longrightarrow \mathrm{SL}(3,\mathbb{R})$ of the homomorphism $h : \pi_1(S) \longrightarrow \mathrm{PGL}(3,\mathbb{R})$ so that $\pi_1(S)$ acts affinely on S' . It is clear that this action is proper and free, obtaining a radiant affine structure on the total space $\hat{S} = S'/\pi_1(S) \approx S \times \mathbb{R}^+$ of a principal $\mathbb{R}^+$-bundle over S with holonomy representation $\tilde{h}$. . The radiant vector field $\rho_{\hat{S}}$ generates the (fiberwise) affine action of $\mathbb{R}^+$ on $\hat{S}$, which is given locally in coordinates by homotheties . On the other hand every principal $\mathbb{R}^+$-bundle is trivial. Choose any $\lambda > 1$. The cyclic group $\rangle\lambda\langle \subset \mathbb{R}^+$ acts properly and freely by affine transformations on $\hat{S}$. The resulting affine manifold $\hat{S}/\rangle\lambda\langle$ is homeomorphic to $S \times S^1$. Corresponding to a convex real projective structure on S is a whole family of radiant affine structures on $S \times S^1$, one for each λ. The radiant vector field is, in fact, the vector field on $S \times S^1$ in the direction of S^1, which we denote by $\frac{\partial}{\partial\theta}$. Consider the characteristic function $f : \Omega' \longrightarrow \mathbb{R}$ of Ω'. The logarithmic differential $d\log f$ is a closed 1-form on Ω' which is:

- positive definite;
- invariant under $\mathrm{Aff}(\Omega')$

Then by the above construction, there exists a closed 1-form $\tilde{\alpha}$ on S' such that $\nabla\tilde{\alpha} > 0$ and is invariant under $\pi_1(S)$ and $\langle\lambda\rangle$. Consequently, there exists a closed 1-form α on $S \times S^1$ such that $\nabla\alpha > 0$. ☐

With notation in lemma 1 , we have :

Lemma 8. *α represents the cohomology class of S^1, i.e. if $\pi_2 : S \times S^1 \longrightarrow S^1$ denotes projection then $[\alpha] = \pi_2^*[S^1]$ where $[S^1] \in H^1(S^1)$ denotes a generator.*

The characteristic function on Ω' induces a function on S' which we again denote by $\log f$. The fundamental group of S naturally acts on S' by $\tilde{h}$. For all $\gamma \in \Gamma = \tilde{h}(\pi_1(S'))$,

$$\log f \circ \gamma = \log f - \log\det(\gamma)$$

But for $\gamma \in \pi_1(S')$, $\det(\gamma) = 1$, so $\log f = \log f \circ \gamma$. It follows that $\log f$ defines a function $l : \hat{S} \longrightarrow \mathbb{R}$ such that $\alpha = dl$ is exact. Let $\Pi_\lambda : \hat{S} \longrightarrow S_\lambda = \hat{S}/\langle\lambda\rangle$ denote projection and let $\hat{\alpha} = \Pi_\lambda^*\alpha$. There is a function l_λ related to l on S_λ such that $\hat{\alpha} = dl_\lambda$. Consider the generator of the cyclic group $\langle\lambda\rangle$ i.e.

$$D_\lambda : z \longmapsto \lambda z$$

and

$$\log(f \circ D_\lambda) = \log f - 3\log\lambda$$

Then α is a 1-form on $S \times S^1$. For every $\gamma \in \pi_1(S \times S^1) \cong \pi_1(S) \times \mathbb{Z}$,

$$\int_\gamma \alpha = \int_{\tilde{\gamma}} d \log f = \log f(\tilde{\gamma}\tilde{p}) - \log f(\tilde{p}) = -\log \det(\gamma)$$

where $\tilde{\gamma}$ is the lifting of γ and $\tilde{p}$ is an arbitrary point in S' whose projection by $S' \approx S \times S^1 \longrightarrow S^1$ is the base point of S^1. Now

$$S'/\Gamma = \hat{S} \approx S \times \mathbb{R}^+$$

and $\pi_1(S) \cong \pi_1(\hat{S}) \cong \Gamma$ and for all $\gamma \in \Gamma$, the period of α around γ is zero. Also $\langle \lambda \rangle \subset \pi_1(S \times S^1)$, and $\forall \gamma \in \Gamma$ the period of α is zero. So by using the Hurewicz isomorphism

$$\begin{aligned} H^1(W, \mathbb{R}) &\cong \mathrm{Hom}(H_1(W, \mathbb{Z}), \mathbb{R}) \\ &\cong \mathrm{Hom}(pi_1(W), \mathbb{R}) \end{aligned}$$

for $S \times S^1$, we have $[\alpha] = \pi_2^*[S^1]$.

Let $\frac{\partial}{\partial\theta}$ be the vector field on $S \times S^1$ in the direction of S^1 , i.e. the infinitesimal generator of the flow :

$$\Theta_t : (s, u) \longmapsto (s, u + t)$$

for $u \in \mathbb{R}/\mathbb{Z}$. Let $d\theta$ be the 1-form dual to $\frac{\partial}{\partial\theta}$, i.e. $d\theta(\frac{\partial}{\partial\theta}) = 1$. Let $\mathcal{C}$ denote the set of all affine connections ∇ on $S \times S^1$ such that:

- ∇ is flat and torsionfree;
- $\frac{\partial}{\partial\theta}$ is radiant with respect to ∇, i.e.

  $$\nabla_X(\frac{\partial}{\partial\theta}) = X$$

  for all vector fields X on $S \times S^1$.
- $\nabla d\theta > 0$.

Before the main theorem , we prove the following lemma, which asserts that a "radiant vector field is affine."

Lemma 9. *Let ∇ be a flat torsionfree connection on N. Suppose ρ is a vector field on N and $\{\Theta_t\}_{t\in\mathbb{R}}$ be its flow. If ρ is radiant with respect to ∇, then ∇ is invariant under Θ_t.*

Proof. Define a derivation $A_\rho = L_\rho - \nabla_\rho$, where L_ρ is Lie derivative with respect to ρ. So

$$\begin{aligned} A_\rho(X) = L_\rho(X) - \nabla_\rho(X) &= [\rho, X] - (\nabla_X\rho + [\rho, X] + T(\rho, X)) \\ &-\nabla_X\rho - T(\rho, X) \end{aligned}$$

since ρ is radiant and the torsion $T = 0$. Then $A_\rho(X) = -X$, that is, $A_\rho = -I$. By Prop. 2.6 of page 235 of [9], the vector field ρ is an infinitesimal affine transformation if and only if for all vector fields Y on N

$$\nabla_Y(A_\rho) = R(\rho, Y)$$

where R is the curvature tensor. But $\nabla_Y(A_\rho) = \nabla_Y(-I) = 0$ (again since ∇ is torsionfree) and $R = 0$ (since ∇ is flat). Thus ρ is an infinitesimal affine transformation. Now by prop. 1.4. page 228 of [9], ∇ is invariant with respect to Θ_t as desired. □

Let S be a closed surface with $\chi(S) < 0$ and fix a basepoint $p \in S$. Let $\mathcal{E}$ denote the set of of all pairs (f, M) where $f : S \longrightarrow M$ is a diffeomorphism and M is a convex $\mathbb{RP}^{\nvDash}$-manifold. Define a homomorphism

$$T : \mathrm{Diff}^0(S) \longrightarrow \mathrm{Diff}(S \times S^1)$$

by $T(h)(s, u) = (h(s), u)$. If h fixes the basepoint $p \in S$, then $T(h)$ fixes the basepoint $(p, 0) \in S \times S^1$.

Theorem 10. *The natural map*

$$\Phi : \mathcal{E} \longrightarrow \mathcal{C}$$

is equivariant with respect to T and induces an isomorphism

$$\mathfrak{P}(\mathfrak{S}) \longrightarrow \mathcal{C}/\mathcal{T}(\mathrm{Diff}'(\mathcal{S})))$$

By Lemma 1, corresponding to every $(f, M) \in \mathcal{E}$, there exists an element of $\mathcal{C}$. Conversely , let $\nabla \in \mathcal{C}$ and

$$\Omega_p \subset T_{(p,0)}(S \times S^1)$$

be the domain of the exponential map . By Koszul's theorem, Ω_p is a sharp convex cone and $\exp : \Omega_p \longrightarrow S \times S^1$ is a covering map. Consider the projection map $\Pi : S \times S^1 \longrightarrow S^1$. It is clear that $\frac{\partial}{\partial\theta}$ is transverse to level sets of Π. Lift the flow of $\frac{\partial}{\partial\theta}$ to Ω_p (denoting it by $\frac{\tilde{\partial}}{\partial\theta}$). The 1-form $\alpha = \exp^*(d\theta)$ is closed. $H^1(\Omega) = 0$ implies that there exists a function $\phi : \Omega \longrightarrow \mathbb{R}$ such that $\alpha = d\phi$. Level sets of ϕ are transverse the flow of $\frac{\tilde{\partial}}{\partial\theta}$ because the tangent space of a level set of an arbitrary point x is the kernel of $d\phi = \alpha$ at x and

$$1 = (d\theta)(\frac{\partial}{\partial\theta}) = (d\theta)(\exp_* \frac{\tilde{\partial}}{\partial\theta}) = (\exp^* d\theta)(\frac{\tilde{\partial}}{\partial\theta}) = d\phi(\frac{\tilde{\partial}}{\partial\theta})$$

Thus each level set $\phi^{-1}(c)$ is a cross-section of $\frac{\tilde{\partial}}{\partial\theta}$. On the other hand, Ω_p is a convex cone and the projectivization of $\phi^{-1}(c)$ is a convex domain in $\mathbb{RP}^{\nvDash}$. Now $\pi_1(S \times S^1) \cong \pi_1(S) \times \mathbb{Z}$ and $\pi_1(S) \hookrightarrow \mathrm{SL}(3, \mathbb{R})$ so $\phi^{-1}(c)/\Gamma$ is a convex $\mathbb{RP}^{\nvDash}$-structure on S. The commutativity of the diagram:

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\Phi"] \arrow[d, "h"'] & \mathcal{C} \arrow[d, "T(h)"] \\
\mathcal{E} \arrow[r, "\Phi"'] & \mathcal{C}
\end{tikzcd}$$

is obvious from the above construction and the proof of lemma 1, i.e. Φ is equivariant with respect to T, and induces the isomorphism

$$\mathfrak{P}(\mathfrak{S}) \longrightarrow \mathcal{C}/\mathcal{T}(\mathrm{Diff}'(\mathcal{S})))$$

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