THE $\mathrm{PGL}(3, \mathbf{R})$-TEICHMÜLLER COMPONENTS OF 2-ORBIFOLDS OF NEGATIVE EULER CHARACTERISTIC

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Abstract. Let G be a Lie group acting on a space X. We show that the space of isotopy-equivalence classes of (G, X)-structures on an orbifold Σ maps locally homeomorphic to the space of representations of the fundamental group of Σ to G. Next, we define the Teichmüller component of the space of conjugacy equivalence classes of representations to $\mathrm{PGL}(3, \mathbf{R})$ of the fundamental groups of 2-orbifolds of negative Euler characteristic generalizing that for closed surfaces. This component is a component containing the conjugacy classes of $\mathrm{PO}(1, 2)$-representations of orbifolds corresponding to hyperbolic structures, i.e., the Teichmüller spaces. We identify this component with the deformation space of convex real projective structures on orbifolds. We thus show that this space is homeomorphic to a cell of certain dimensions computable from the Euler characteristic of the underlying space and the number of certain singular points.

An orbifold is a topological space with neighborhoods modeled on the orbit-spaces of finite group actions on open balls. Often an orbifold arises as a quotient space of a manifold by a proper action of a discrete group. They are so-called good orbifolds. For example, quotients of hyperbolic spaces by discrete subgroups of isometries are orbifolds (especially, when the group has torsion elements.) Very good orbifolds are orbifolds which are quotients of manifolds by finite group actions. The good orbifolds are as good as manifolds since they admit universal covering manifolds. (However, we are far from knowing when the orbifolds are good or bad.) Compact 2-dimensional orbifolds were classified by Thurston in Chapter 5 of his lecture notes [19]. One can define the term orbifold-Euler characteristic of an orbifold by counting cells divided by the order of the group associated with the cells and summing with alternating signs. A compact orbifold with negative orbifold Euler characteristic admits hyperbolic structures, i.e., Riemannian metrics of constant negative curvature, in a suitable sense, and they are very good always by Selberg's lemma. (They form in fact a majority of 2-orbifolds.) Thurston showed that the deformation spaces of hyperbolic structures on these orbifolds, i.e., Teichmüller space, are homeomorphic to cells of certain dimensions depending on the Euler characteristic of the underlying space and the number of certain singular points. Since the Teichmüller space can be interpreted also as the space

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of conjugacy classes of discrete-faithful representations of the fundamental groups of the orbifolds to $\mathrm{PO}(1,2)$, his theorem gives us a classification of discrete $\mathrm{PO}(1,2)$-representations of orbifold-groups.

Let G be a Lie group acting on a space X transitively and effectively. A (G, X)-*structure* on an orbifold M is given by a maximal atlas of charts to orbit spaces of finite subgroups of G acting on open subsets of X. A (G, X)-structure on an orbifold implies that the orbifold is good as first observed by Thurston. This generalizes the notion of (G, X)-structures on manifolds introduced by Ehresmann.

We will define in this paper, the space $\mathcal{S}(\Sigma)$ of isotopy-equivalence classes of (G, X)-structures on a given orbifold Σ. Given a (G, X)-structure on Σ, we can define an immersion D from its universal cover to X and a homomorphism $h : \pi_1(\Sigma) \to G$ for the fundamental group $\pi_1(\Sigma)$ of Σ. (D, h) are essentially defined by analytic continuation on charts as in the manifold cases. (See Goldman [9] for more details on (G, X)-structures on manifolds.) The space $\mathcal{S}(\Sigma)$ can be considered as the space of equivalence classes of development of pairs (G, X)-structures on $\tilde{\Sigma}$ under the isotopy action of $\tilde{\Sigma}$ commuting with the deck-transformation group. The deformation space $\mathcal{D}(\Sigma)$ is a quotient space of $\mathcal{S}(\Sigma)$ by forgetting about D, and is obtained from $\mathcal{S}(\Sigma)$ as a quotient by an action of G. One can define a map

$$\mathcal{PH} : \mathcal{S}(\Sigma) \to \mathrm{Hom}(\pi_1(\Sigma), G)$$

by assigning a (G, X)-structure with D to its holonomy homomorphism associated with D. $\mathrm{Hom}(\pi_1(\Sigma), G)$ is naturally a real algebraic variety and hence is a topological space. We will show that this map is a local homeomorphism generalizing the proof in the manifold case by J. Lok [15] following J. Morgan's lectures and A. Weil [21] and also Canary-Epstein-Green [2]. (We mention that this can be also done using Goldman's idea in [9].) This result will be used in Choi-Goldman [6]. (There are related works by Kapovich [13] and Gallo-Kapovich-Marden [8] where these were proved for orbifolds partially.)

From now on, we look at the case when $X = \mathbf{R}P^2$ and $G = \mathrm{PGL}(3, \mathbf{R})$ acting on X. An $(\mathbf{R}P^2, \mathrm{PGL}(3, \mathbf{R}))$-structure on an orbifold is called a *real projective* structure, or *projectively flat* structure. They correspond to projectively flat torsion-free affine connections. ($\mathrm{PGL}(3, \mathbf{R})$ is a group isomorphic to $\mathrm{SL}(3, \mathbf{R})$, a group of linear maps with determinant 1.)

An orbifold with a real projective structure is called *convex* if it is projectively diffeomorphic to a convex domain in an affine patch quotient out by a properly discontinuous group of automorphisms. Among these special ones are *hyperbolic* ones. The domains are the standard disks in affine patches. They correspond naturally to hyperbolic metrics since the interior of the standard disk is the Klein model of hyperbolic plane and the group of isometries are the projective automorphism of the disk, an isomorphic copy of $\mathrm{PO}(1,2)$ in $\mathrm{PGL}(3, \mathbf{R})$.

Let Σ be a compact orbifold with negative orbifold Euler characteristic. The subspace of the deformation space $\mathcal{D}(\Sigma)$ of real projective structures on Σ corresponding to convex ones is denoted by $\mathcal{CP}(\Sigma)$ and the subspace corresponding to hyperbolic ones is denoted by $T(\Sigma)$, which is the same space as the ordinary Teichmüller spaces of Σ

as defined by Thurston. Then we see that $T(\Sigma)$ is a subspace of $\mathcal{CP}(\Sigma)$, and $\mathcal{CP}(\Sigma)$ is an open subset of $\mathcal{D}(\Sigma)$.

The map $\mathcal{PH}$ induces the following map

$$\mathcal{H} : \mathcal{D}(\Sigma) \to \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PGL}(3, \mathbf{R}))^{st}/\mathrm{PGL}(3, \mathbf{R})$$

where $\mathrm{PGL}(3, \mathbf{R})$ acts on

$$\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PGL}(3, \mathbf{R}))$$

by conjugation, and

$$\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PGL}(3, \mathbf{R}))^{st}$$

is the subset where this action is proper.

Let us denote by C_T the unique component of

$$\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PGL}(3, \mathbf{R}))$$

containing the holonomy homomorphisms of hyperbolic real projective structures on Σ. Then $C_T(\Sigma)$ is a component of

$$\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PGL}(3, \mathbf{R}))^{st},$$

$C_T/\mathrm{PGL}(3, \mathbf{R})$ is said to be a *Teichmüller component* along Hitchin [12], and we prove:

Theorem 0.1. *Let Σ be a compact 2-orbifold with negative Euler characteristic and empty boundary. Then*

$$\mathcal{H} : \mathcal{CP}(\Sigma) \to C_T(\Sigma)/\mathrm{PGL}(3, \mathbf{R})$$

is a homeomorphism, and $C_T(\Sigma)$ consists of discrete faithful representations of $\pi_1(\Sigma)$.

Since Choi-Goldman [6] show that $\mathcal{CP}(\Sigma)$ is a cell of certain dimension, this space is homeomorphic to cells of dimension $-8\chi(X_\Sigma) + 6k_c - 2k_b + 3l_c - l_b$ where X_Σ is the underlying space of Σ, k_c is the number of cone-points, l_c the number of corner-reflectors, k_b the number of cone-points of order two, and l_b the number of corner-reflectors of order two.

Corollary 0.1. *The Teichmüller component quotient space $C_T(\Sigma)/\mathrm{PGL}(3, \mathbf{R})$ is homeomorphic to cells of dimensions as above.*

This gives us some classification of discrete representations of orbifold-fundamental groups. We certain don't know if these are all the discrete representations.

Benoist [1] characterized the group of projective transformations acting on a convex domain for general dimensions. We only consider $n = 3$ case here. An element of $\mathrm{GL}(3, \mathbf{R})$ is *proximal* if it has an attracting fixed point in $\mathbf{R}P^2$. An element is *positive proximal* if the eigenvalue corresponding to the fixed point is positive. A subgroup Γ of $\mathrm{GL}(3, \mathbf{R})$ is *positive proximal* if all proximal elements of Γ are positive proximal. Proposition 1.1 of [1] shows that if Γ is an irreducible subgroup of $\mathrm{GL}(3, \mathbf{R})$, then Γ preserves a saillant cone in $\mathbf{R}^3$ if and only if Γ is positive proximal. Such a subgroup of $\mathrm{GL}(3, \mathbf{R})$ if discrete, acts on a convex domain Ω in an affine patch so that Ω/Γ is an orbifold. Suppose that Ω/Γ is compact, and Γ contains a free subgroup of two generators, then Ω/Γ is an orbifold of negative Euler-characteristic. Thus, Theorem 0.1

classifies such groups by cells. More precisely, one needs to act by surface automorphism groups of the deformation spaces to obtain the moduli spaces, and they classify such groups. (In fact, we are attempting to find a universal classifying spaces of such discrete groups.)

Essentially the purpose of this paper is on real projective orbifolds; however, due to lack of appropriate background expositions, we will include proofs of mostly known facts written by other authors in unpublished materials.

In Section one, we discuss the topology of orbifolds. We introduce orbifolds, orbifold-maps, isotopy of orbifold-maps, covering orbifolds, universal covering orbifolds, the deck transformation group of orbifolds, and so on. We faithfully follow and give some missing details of Chapter 5 of Thurston [19], in particular, the construction of the universal covering orbifolds. Also, we give a classification of singularities of 2-orbifolds, and the Euler characteristics of 2-orbifolds.

In Section two, we discuss the geometric structures on orbifolds. We show that an orbifold with geometric structures are good, and find the developing maps and the holonomy homomorphisms for orbifolds. We define the deformation spaces of (G, X)-structures on orbifolds, which is the space of equivalence classes of (G, X)-structures under isotopy. The so-called isotopy-equivalence spaces of (G, X)-structures on an orbifold Σ is defined to be the space of equivalence classes of a pair (D, f) where D is a developing map for a (G, X)-orbifold M, and f is a lift of an orbifold diffeomorphism defined on the universal cover of Σ. The equivalence relations is given by an isotopy action on f. We define a pre-hol map from the isotopy-equivalence spaces of (G, X)-structures to the space of representations $\mathrm{Hom}(\pi_1(\Sigma), G)$ given by sending (D, f) to the holonomy homomorphism composed with the homomorphism $\pi_1(\Sigma) \to \pi_1(M)$ induced by f. Here $\pi_1(\Sigma)$ denote the deck transformation group of Σ.

In Section three, we prove that the isotopy classes of (G, X)-structures on an orbifold is locally homeomorphic to the space of representations of the fundamental group to G by the pre-hol map. This generalizes the same result for manifolds written by Lok [15], along J. Morgan's lectures. The proofs are essentially the same but we modify slightly for clarity. Essentially, the idea is to deform first on small neighborhoods first and patch them together using "bump" functions as we change the representation by a small amount in a cone-neighborhood of representation variety. The finite group action complicates the proof somewhat but not greatly where use old ideas of Palais-Stewart [16].

We begin Section three by stating three lemmas on conjugating finite group action deformations. We introduce Riemannian metric on orbifolds. We choose a set of generators of $\pi_1(M)$. We show that there is a local section of the pre-hol map: as we deform holonomy, we deform the model neighborhoods by conjugating with respect to finite group action deformations. We patch the deformations together to form a deformation of M. We finally show that the pre-hol map is injective on a small neighborhood of the isotopy-equivalence space.

In Section four, we discuss real projective structures on manifolds. Define the Teichmüller component of the conjugacy classes of the representations of the fundamental groups to $\mathrm{PGL}(3, \mathbf{R})$. We prove that the deformation space of convex real projective structures on an orbifold maps into a closed subset of the Teichmüller component. For the purpose of the section, we prove that if two 2-orbifolds are homotopy-equivalent, then they are homeomorphic using harmonic diffeomorphisms.

A good reference on orbifolds is the Chapter 5 of Thurston's note [19] or Scott's survey paper [18]. Also, Ratcliffe [17] devotes a chapter to orbifolds with geometric structures but for one with invariant Riemannian metrics.

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1. Topology of orbifolds

An n-dimensional *orbifold* is a Hausdorff, second-countable space X so that each point has a neighborhood homeomorphic to a quotient of an open-ball U in $\mathbf{R}^n$ by an action of a finite group Γ. Moreover, if such a neighborhood V of y, modeled on a pair $(\tilde{V}, G_1)$ is a subset of another such neighborhood U, modeled on a pair $(\tilde{U}, G_2)$, then the inclusion map $\phi_{V,U} : V \rightarrow U$ lifts to an imbedding $\tilde{\phi}_{V,U} : \tilde{V} \rightarrow \tilde{U}$ equivariant with respect to a homomorphism $\psi_{V,U} : G_1 \rightarrow G_2$ so that the following diagram is commutative.

$$\begin{array}{ccc}
\tilde{V} & \stackrel{\tilde{\phi}_{V,U}}{\longrightarrow} & \tilde{U} \\
\downarrow & & \downarrow \\
\tilde{V}/G_1 & \longrightarrow & \tilde{U}/\psi_{V,U}(G_1) \\
& & \downarrow \\
\downarrow & & \tilde{U}/G_2 \\
& & \downarrow \\
V & \stackrel{\phi_{V,U}}{\longrightarrow} & U
\end{array} \tag{1}$$

Note that the pair $(\tilde{\phi}_{V,U}, \psi_{V,U})$ can be chosen differently; i.e., the pair $\vartheta \circ \tilde{\phi}_{V,U}$ and $\vartheta \circ \psi_{V,U}(\cdot)\vartheta^{-1}$ for $\vartheta \in G_2$ satisfies the above equation as well. Thus, an equivalence class of $(\tilde{\phi}_{V,U}, \psi_{V,U})$ is associated to the pair $\phi_{V,U}$ instead. When $\phi_{V,U} : V \hookrightarrow U$ and $\phi_{U,W} : U \hookrightarrow W$ are inclusion maps, then

$$\begin{aligned}
\tilde{\phi}_{V,W} &= \vartheta \circ \tilde{\phi}_{U,W} \circ \tilde{\phi}_{V,U} \text{ and} \\
\psi_{V,W}(\cdot) &= \vartheta \circ \phi_{U,W} \circ \phi_{V,U}(\cdot) \circ \vartheta^{-1}, \text{ for } \vartheta \in G_3
\end{aligned} \tag{2}$$

where G_3 is the finite group associated with W. (V is said to be a *model neighborhood* and $(\tilde{V}, \Gamma)$ the *model pair*.)

A maximal family of coverings with models satisfying the above conditions are said to be an orbifold structure on X is said to be an *orbifold structure* on X. X is said to be the *underlying space* of X with an orbifold structure.

Given two orbifolds M and N, an *orbifold-map* is a map $f : X_M \rightarrow X_N$ so that for each point x of X_N, a neighborhood of x modeled on (U, G), and an inverse image of y, there is a neighborhood of y modeled on (V, G') and a map $\tilde{f} : U \rightarrow V$ inducing f equivariant with respect to a homomorphism $\psi : G \rightarrow G'$. (That is, we record the lifting but $\tilde{f}$ is determined only up to G, G', i.e., the map $g \circ \tilde{f} \circ g''$, $g \in G'$, $g'' \in G$ and homomorphism $\psi(\cdot)$ changed to $g \circ \psi(g''(\cdot)g''^{-1}) \circ g^{-1}$ will do just as well.)

An *orbifold* with boundary is a Hausdorff, second-countable space so that each point has a neighborhood modeled on open-balls intersected with upper-half spaces and a finite group acting on them. The *interior* is a set of points with neighborhoods modeled on open-balls. The *boundary* is the complement of the interior. (The boundary is a boundaryless orbifold of codimension one.)

A *singular* point x of an orbifold is a point of the underlying space which has a neighborhood with a model ball with a finite group fixing a point corresponding to x. A nonsingular point of an orbifold always has a neighborhood homeomorphic to a ball. The set of regular points is an open dense subset of the underlying space.

A *suborbifold* of an orbifold N is an imbedded subset Y of X_N with an orbifold structure so that for each point x of Y, and a neighborhood V modeled on (V', G), the neighborhood $V \cap Y$ is modeled on $(V' \cap P, G|P)$ where P is a submanifold of $\mathbf{R}^n$ where G acts, and $G|P$ denotes the image subgroup of the restriction to groups acting on P.

The boundary of an orbifold is a suborbifold clearly.

A class of examples are given as follows: Let M be a manifold and Γ a discrete group acting on M properly but not necessarily freely. Then M/Γ has an orbifold structure: Let x be a point of M/Γ and $\tilde{x}$ a point of M corresponding to x. Then a subgroup $I_{\tilde{x}}$ of Γ fixes $\tilde{x}$. There is a ball-neighborhood U of $\tilde{x}$ where $I_{\tilde{x}}$ acts on and for any $g \in \Gamma - I_{\tilde{x}}$, $g(U) \cap U$ is empty. Then $U/I_{\tilde{x}}$ is a neighborhood of x modeled on $(U, I_{\tilde{x}})$. If V is another such neighborhood in $U/I_{\tilde{x}}$ containing a point y. Then a component V' of its inverse image in U is acted upon by a subgroup I' of $I_{\tilde{x}}$. Also, for any $g \in \Gamma - I'$, $g(V') \cap V'$ is empty. Therefore, the inclusion $V \to U/I_{\tilde{x}}$ satisfies the conditions for equations 1.

Given two orbifolds M and N, the product space $X_M \times X_N$ obviously has an orbifold structure; i.e., we model on $(U \times V, \Gamma_U \times \Gamma_V)$ if (U, Γ_U) and (V, Γ_V) are model pairs for neighborhoods of M and N respectively. The product space with this orbifold structure is denote by $M \times N$.

A *homotopy* of two orbifold maps $f_1, f_2 : M \to N$ from an orbifold M to another one N is an orbifold-map $F : M \times I \to N$ where I is an interval and $F(x, 0) = f_1(x)$ and $F(x, 1) = f_2(x)$ for every $x \in M$. We define an orbifold map $F_t : M \to N$ to be given by $F_t(x) = F(x, t)$ with appropriate liftings in model pairs of M and N.

Given an orbifold M, an *isotopy* $f : M \to M$ is a self-orbifold-diffeomorphism so that there is a homotopy $F : M \times I \to M$ so that F_0 is the identity map and $F_1 = f$, and F_t is an orbifold-diffeomorphism for each t.

Two orbifold-diffeomorphisms $f_1, f_2 : M \to M'$ are *isotopic* if there is a homotopy $F : M \times I \to M'$ so that $F_0 = f_1$ and $F_1 = f_2$ and F_t are orbifold-diffeomorphisms.

Given a sequence of coverings map $p_i : X_i \to X$, in the ordinary sense, one can form a fiber-product $p^f : X^f \to X$ by setting X^f to be the subset of $\prod X_i$ where

$$p_i \circ \pi_i(x_1, x_2, \dots) = p_j \circ \pi_j(x_1, x_2, \dots)$$

for all i, j and $\pi_i : \prod X_i \to X_i$ a projection to the i-th factor. The covering map $p^f : X^f \to X$ is given by $p^f(x_1, x_2, \dots) = p_1(x_1)$, and X^f covers X_i by a map p'_i so that $p_i \circ p_i' = p^f$. It has a universal property that if $q_i : X'' \to X_i$ is a covering map for each i so that $p_1 \circ q_i$ is a fixed covering map $X'' \to X$, then there exists a covering map $q' : X'' \to X^f$ so that $p_i' \circ q' = q_i$.

Also, the universal property characterizes X^f up to covering isomorphisms. That is, if $p_Y : Y \to X$ is a covering map so that there are covering maps $q_{Y,i} : Y \to X_i$ and $p_i \circ q_{Y,i}$ is a fixed covering map $Y \to X$, and Y satisfies the universal properties of X^f above, then there exists a *unique* covering isomorphism $L : Y \to X^f$ such that $q_{Y,i} = p_i' \circ L$–(*).

A *covering orbifold* of an orbifold M is an orbifold $\tilde{M}$ with an onto-map $p : X_{\tilde{M}} \to X_M$ such that each point $x \in X_Q$ has a neighborhood U, so-called an elementary neighborhood, with a homeomorphism $\phi : \tilde{U}/\Gamma \to U$, with an open subset of $\tilde{U}$ in $\mathbf{R}^n$ or $\mathbf{R}^{n,+}$ with a group Γ acting on it, so that each component V_i of $p^{-1}(U)$ has a homeomorphism $\tilde{\phi}_i : \tilde{U}/\Gamma_i \to V_i$ (in the orbifold structure) where Γ_i is a subgroup of Γ. We require the quotient map $\tilde{U} \to V_i$ induced by $\tilde{\phi}_i$ composed with p is the quotient map $\tilde{U} \to U$ induced by ϕ. (We don't assume $X_{\tilde{M}}$ is connected.)

A *fiber* of a point of M is the inverse image $p^{-1}(x)$.

Given an orbifold map $f : X \to Y$ and a covering (Y_1, p_1) of Y, then if an orbifold map $\tilde{f} : X \to Y_1$ so that $p_1 \circ \tilde{f} = f$ and $\tilde{f}$ lifts for every model pairs of Y_1 in the consistent way for Y, $\tilde{f}$ is said to be a *lifting* of f.

Two covering orbifolds (X_1, p_1) and (X_2, p_2) of an orbifold X is *isomorphic* if there is an orbifold-diffeomorphism $f : X_1 \to X_2$ so that $p_2 \circ f = p_1$. A *covering automorphism* $X_1 \to X_1$ is a covering isomorphism of X_1 itself. More generally, a *covering morphism* $(X_1, p_1) \to (X_2, p_2)$ is an orbifold map $f : X_1 \to X_2$ so that $p_2 \circ f = p_1$. Thus, f is a lifting of p_1. A covering (X_1, p_1) is *regular* if the automorphism group acts transitively on fibers over regular points. Given coverings (X_1, p_1) over X and (X_2, p_2) over Y, a map $f : X_1 \to X_2$ *covers* a map $g : X \to Y$ if the following diagram is commutative:

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ \downarrow p_1 & & \downarrow p_2 \\ X & \xrightarrow{g} & Y \end{array}$$

Lemma 1.1. *A model neighborhood of a point of X is elementary for any orbifold covering maps.*

Proof. Let V be a model neighborhood of $x \in X$, and $(\tilde{V}, \Gamma)$ the model pair, and $p : X' \to X$ an orbifold-covering map. Then each component V' of $p^{-1}(V)$ orbifold-covers V. Let $q : \tilde{V} \to V$ be the quotient map, and V be covered by elementary neighborhoods of $p'|V' : V' \to V$. Choose elementary neighborhoods in V and components of its inverse image in V'. We cover $\tilde{V}$ by $\{O_i\}$ the components of the inverse images of the elementary neighborhoods. Let $\tilde{x}$ be a point of O_i so that $\tilde{q}(\tilde{x})$ is regular in V. For a path-class f in $\tilde{V}$ with base point $\tilde{x}$, we can lift $q \circ f$ to a path in V' easily by using the elementary neighborhoods. Two homotopic path-classes f and f' lift to homotopic path-classes again using elementary neighborhoods. Thus, we can lift q to a map $q' : \tilde{V} \to V'$ so that $p \circ q' = q$. q' is obviously an orbifold map. (This works in the same manner as in the covering space theory.)

Moreover, there is a transitive action by a subgroup of Γ on the fibers of $q'^{-1}(x)$ for a regular point $y \in V'$. This follows since we can lift from any point $\tilde{x}$ of $q'^{-1}(y)$ and any two lifts must differ by an automorphism in Γ. The subgroup has to be finite, and we see that $q' : \tilde{V} \to V'$ is equivalent to a quotient map of $\tilde{V}$ by a finite group action. This shows that V is elementary. $\square$

Lemma 1.2. *Let V be an n-orbifold which is a quotient of an n-ball $\tilde{V}$ by a finite group Γ acting on it. Then the following statements hold:*

- (i) *A covering orbifold V_1 of V is isomorphic to $\tilde{V}/\Gamma'$ for a subgroup $\Gamma' \subset \Gamma$ with a covering map $p : \tilde{V}/\Gamma' \to V = \tilde{V}/\Gamma$ induced the identity map $\tilde{V} \to \tilde{V}$; i.e., the covering space up isomorphism is in one-to-one correspondence with the conjugacy classes of subgroups of Γ.*
- (ii) *The covering automorphism group of a covering orbifold V' is given by $N(\Gamma')/\Gamma'$ where Γ' is a subgroup corresponding to V_1 and $N(\Gamma')$ is the normalizer of Γ' in Γ.*
- (iii) *Given two covering orbifolds $\tilde{V}/\Gamma_1$ and $\tilde{V}/\Gamma_2$, a covering morphism $\tilde{V}/\Gamma_1 \to \tilde{V}/\Gamma_2$ is induced by an element $g \in \Gamma : \tilde{V} \to \tilde{V}$ so that $g\Gamma_1 g^{-1} \subset \Gamma_2$. The covering maps are in one-to-one correspondence with double cosets of form $\Gamma_2 g \Gamma_1$ with g satisfying $g\Gamma_1 g^{-1} \subset \Gamma_2$.*

Proof. (i) follows from Lemma 1.1.

(ii) A covering automorphism $V' = \tilde{V}/\Gamma'$ lifts to Γ'-orbit preserving map $f : \tilde{V} \to \tilde{V}$. Since f covers the identity map of V, f is an element g of Γ. The orbit-preservation implies that g is in the normalizer of Γ'.

(iii) The morphism lifts to an orbit-preserving map $f : \tilde{V} \to \tilde{V}$, which sends Γ_1-orbit to Γ_2-orbit. Again f is an element g of Γ covering the identity map of V. Thus $g\Gamma_1(x) \subset \Gamma_2(g(x))$ for each point x of $\tilde{V}$. Thus, $g\Gamma_1 g^{-1} \subset \Gamma_2$.

The elements g and $g' \in \Gamma$ induce a same map $\tilde{V}/\Gamma_1 \to \tilde{V}/\Gamma_2$ if and only if $g' = g_2 g g_1$ for $g_1 \in \Gamma_1$ and $g_2 \in \Gamma_2$. $\square$

Lemma 1.3. *Let (X_1, p_1) and (X_2, p_2) be coverings over an orbifold X. Let $f : X_1 \to X_2$ be a covering morphism so that $f : X_1^n \to X_2^n$ is a covering isomorphism where X_1^n and X_2^n are inverse images of the nonsingular part X^n of X. Then f itself is a covering automorphism.*

Proof. Straightforward. $\square$

In the note of Thurston [19], he proved that each orbifold X has a so-called universal covering orbifold $\tilde{X}$ with an orbifold map $p_X : \tilde{X} \to X$ so that given a orbifold covering map $p : Y \to X$ where Y is connected, there is an orbifold map $q : \tilde{X} \to Y$ so that $p \circ q$ equals p_X. $\tilde{X}$ is required to have a connected underlying space.

Proposition 1.1 (Thurston)**.** *Let X be a connected orbifold. Then there exists a universal covering orbifold $\tilde{X}$ unique up to covering isomorphism.*

Proof. It is in Chapter 5 of Thurston [19]: We will repeat it here for reader's convenience and the difficulty of the writing there and some omissions.

Let us list connected covering orbifolds (X_i, p_i), $i \in I$, of X for some an index set I. We will define "orbifold fiber product" of these spaces which will serve as the universal covering space.

Let V be a model neighborhood with a model pair $(\tilde{V}, \Gamma)$. Take a component V_i^j of $p_i^{-1}(V)$ for each X_i. Let Γ_i^j denote the subgroup of Γ so that $\tilde{V} \to V_i^j$ is the covering map quotient along Γ_i^j acting on it. For each choice of j for i, we define a map J defined on the natural numbers $\mathbf{N}$ to the components of $p_i^{-1}(V)$. We will form a fiber product V^J of $V_1^{j_1}, V_2^{j_2}, \dots$. We define

$$V^J = (\tilde{V} \times \prod_{i \in I} \Gamma_i^{j(i)} \backslash \Gamma)/\Gamma$$

where Γ acts by

$$\gamma(v, \Gamma_1^{j(1)}\gamma_1, \Gamma_2^{j(2)}\gamma_2, \dots) = (\gamma v, \Gamma_1^{j(1)}\gamma_1\gamma^{-1}, \Gamma_2^{j(2)}\gamma^{-1}, \dots).$$

From V^J, we can define a covering map $q_i : V^J \to V_i^{j(i)}$ by sending $(u, (\Gamma_i^{j(i)}\gamma_i)_{i \in I})$ to $\gamma_i u$ in the equivalence class of $\tilde{V}/\Gamma_i^{j(i)}$. This is clearly a well-defined orbifold covering map.

We define $q_J : V^J \to V$ to be the obvious covering map sending $(u, (\Gamma_i^{j(i)}\gamma_i)_{i \in I})$ to the class of u in V, i.e., $q(u)$. Then $p_i \circ q_i = q_J$.

If V^n denote the nonsingular part of V, and $\tilde{V}^n$ the inverse image of it in $\tilde{V}$, then if we replace $\tilde{V}$ by $\tilde{V}^n$, we obtain an ordinary fiber product $V^{n,J}$ of the maps $p_1|V_1^{n,j(1)}, p_2|V_2^{n,j(2)}, \dots$, where $V_i^{n,j(i)}$ denote the inverse image of V^n in $V_i^{j(i)}$. We can easily see that $V^{n,J}$ is identifiable with the inverse image of $q_J^{-1}(V^n)$ by construction of V^J. (This is nicely explained in Chapter 5 of Thurston [19].)

Since $\tilde{V} \to V_i^{j(i)}$ is a covering with automorphism group $\Gamma_i^{j(i)}$, we can easily verify that $\tilde{V}^n \to V^{n,J}$ is a regular covering with automorphism group $\bigcap_i \Gamma_i^{j(i)}$ acting on $\tilde{V}$. Thus, $q_J|V^{n,J} : V^{n,J} \to V^n$ is a covering map corresponding to $\bigcap_i \Gamma_i^{j(i)}$.

We note the universal property of V^J that given a sequence of covering maps $q_i'' : V'' \to V_i^{j(i)}$ for each i so that $p_i \circ q_i''$ is a fixed covering map $V'' \to V$, there is a covering map $q' : V'' \to V^J$ so that $q_i \circ q' = q_i''$: We regard V'' as $\tilde{V}/\Gamma''$ for a subgroup Γ'' of Γ. We can lift q_i'' to an orbifold covering $\hat{q}_i : \tilde{V} \to V_i^{j(i)}$ for each i, and hence to a covering map $\tilde{q}_i : \tilde{V} \to \tilde{V}$, which is an element $g_i^{j(i)}$ of Γ by Lemma 1.2. $g_i^{j(i)}$ satisfies

$$g_i^{j(i)}\Gamma'' g_i^{j(i),-1} \subset \Gamma_i^{j(i)}. \tag{3}$$

We map $\tilde{V}$ to

$$V^J = (\tilde{V} \times \prod_{i \in I} \Gamma_i^{j(i)} \backslash \Gamma)/\Gamma$$

by sending u to the class of $(u, (\Gamma_i^{j(i)} g_i^{j(i)})_{i\in I})$. This map induces a well-defined (diagonal) map q' from V'' to V^J by equation 3.

$$\begin{array}{lcl} & \tilde V \longrightarrow & V^J = (\tilde V \times \prod \Gamma/\Gamma^{j(i)})/\Gamma \\ & \downarrow & \downarrow q_i \\ & V'' = \tilde V/\Gamma'' \xrightarrow{q_i''} & \tilde V/\Gamma_i^{j(i)} \\ & & \downarrow p_i \\ (4) & & V = \tilde V/\Gamma \end{array}$$

Now, we define $\hat V$ as the disjoint union $\coprod_J V^J$ for all functions J. It has an obvious covering map $\hat p : \hat V \to V$. We can define $q_i : \hat V \to \bigcup_{i,j} V_i^j$ by defining q_i as above for each of V^J.

Then $\hat V$ has a following universal property: given a sequence of covering maps $q_i'' : V'' \to \coprod_j V_i^j$ for each i so that $p_i \circ q_i''$ is a fixed covering map $V'' \to V$, there exists a covering map $q' : V'' \to \hat V$ so that $q_i \circ q' = q_i''$. This follows from considering each component of V'' and where it maps to.

The covering $\hat V$ is said to be a *fiber product* of $p_i^{-1}(V), i \in I$.

Let U be a connected open subset of V, such as $U = V \cap V'$ for another neighborhood V'. We assume that U is modeled on a pair $(\tilde U, \Gamma_U)$. Then components of $q^{-1}(U)$ in $\tilde V$ are homeomorphic to $\tilde U$ and the subgroup of Γ acting on a component is isomorphic to Γ_U.

Let $U_i^{j,k}$ denote the component of $p^{-1}(U)$ in V_i^j for each i, j. Let $\Gamma_{U,i}^{j,k}$ denote a subgroup of Γ_U so that $\tilde U/\Gamma_{U,i}^{j,k}$ equals the covering $U_i^{j,k}$.

Let K be a function defined from $\mathbf{N}$ by sending i to a component of $U_i^{j(i),k}$. Let the fiber product

$$U^{J,K} = (\tilde U \times \prod_{i\in I} \Gamma_{U,i}^{j(i),k(i)} \backslash \Gamma_U)/\Gamma_U$$

be defined where Γ_U act by sending $(u, (\Gamma_{U,i}^{j(i),k(i)}\gamma_i)_{i\in I})$ to $(\gamma u, (\Gamma_{U,i}^{j(i),k(i)}\gamma_i\gamma^{-1})_{i\in I})$ for $\gamma \in \Gamma_U$. Let $q_{U,i}^{J,K} : U^{J,K} \to U_{U,i}^{j(i),k(i)}$ be defined by sending $(u, (\Gamma_i^{j(i),k(i)}\gamma_i)_i)$ to $\gamma_i u$. Let $p_U^{J,K} : U^{J,K} \to U$ denote the covering map.

Define U^J by taking the disjoint union $\coprod_K U^{J,K}$, and $\hat U$ by $\bigcup_J U^J$. We define $q_{U,i}^J : U^J \to \coprod_k U_i^{j(i),k}$ by restricting to to be $q_{U,i}^{J,K}$ for appropriate components, and define $q_{U,i} : \hat U \to \coprod_{j,k} U_i^{j,k}$ similarly. We let $\hat p_U^J : U^J \to U$ and $\hat p_U : \hat U \to U$ denote the covering maps. We note that $\hat U$ has the appropriate universal property also: i.e., if $q_i'' : U'' \to \coprod_{j,k} U_i^{j,k}$ is a covering map for each i so that $p_i \circ q_i'' : U'' \to U$ is a fixed covering map, then there exists a covering map $q_U'' : U'' \to \hat U$ so that $q_{U,i} \circ q_U'' = q_i''$.–(**)

We will now identify $\hat p^{-1}(U)$ with $\hat U$: Since $q_i : \hat V \to p_i^{-1}(V)$ is a covering map, $\hat p^{-1}(U)$ covers $p_i^{-1}(U)$ by q_i. Thus, there is a covering map $f : \hat p^{-1}(U) \to \hat U$ by the universal property of $\hat{U}$. We obtain a covering map

$$f|\hat{p}^{-1}(U^n) : \hat{p}^{-1}(U^n) \to \hat{p}_U^{-1}(U^n).$$

Since $\hat{p}^{-1}(U^n)$ and $\hat{p}_U^{-1}(U^n)$ are same fiber-products, f is a covering isomorphism by the ordinary covering space theory. Moreover, since $q_{U,i} \circ f = q_i$ for each i by (**), the restricted f is a unique isomorphism $\hat{p}^{-1}(U^n) \to \hat{p}_U^{-1}(U^n)$ satisfying this equation; the restricted f is the identification of the fiber-product by (*). By Lemma 1.3, we can identify $\hat{U}$ as a subset of $\hat{V}$.

This type of identifications are consistent over a collection of finitely many open sets and their intersections since over the nonsingular parts the identifications are consistent. By taking a locally finite cover of X, and using identification as above for intersections of the covering open sets and induction, we can form an orbifold $\hat{X}$ with covering maps $q_i : \hat{X} \to X_i$, and $\hat{p} : \hat{X} \to X$. We call $\hat{X}$ a *fiber product* of X_i.

Given any covering $p : Y \to X$, since in the list of $X_1, X_2, \ldots$, we obviously have our Y, say X_i, we see that $q_Y : \hat{X} \to Y$ is a covering map so that $p \circ q_Y = \hat{q}$.

If Y' is another universal covering orbifold with covering map $p' : Y' \to X$, then since Y' has to be in the list also, we have a covering map $p'' : \hat{X} \to Y'$ so that $p' \circ p'' = \hat{p}$. Since Y' is also universal, there is a covering map $p''' : Y' \to \hat{X}$ so that $\hat{p} \circ p''' = p'$.

We obtain a covering map $p''' \circ p'' : Y' \to Y'$. Since over the nonsingular part $Y'^{,n} \to Y'^{,n}$, $p''' \circ p''$ is a covering map in the ordinary sense, it must be the covering automorphism. Thus, $p''' \circ p''$ is a covering isomorphism. So is $p'' \circ p''' : \hat{X} \to \hat{X}$. Therefore, $\hat{X}$ and Y' are isomorphic.

Actually, $\hat{X}$ is connected. If not, we may take a component of it, and it is still universal. □

The group of self-isomorphisms of a universal cover of X is said to be a *deck transformation group*, and we denote it by $\pi_1(X)$. The deck transformation groups act transitively on fibers, i.e, $\hat{p}^{-1}(x)$ for a regular point x of X: Actually, we do above construction by covering spaces with base points. One has to consider the base point x^0 of X, where x_0 is regular, and consider coverings (X_i, x_i^0) with a covering map $p_i : X_i \to X$ with $p_i(x_i^0) = x^0$. Define $\hat{X}$ with a base point $\hat{x}$ the point of the fiber product corresponding to x_i^0 in the nonsingular part, which is well-defined. Thus, we obtain a covering map $\hat{p} : \hat{X} \to X$ so that for any covering $p' : Y \to X$ with $p'(y^0) = x^0$, one has a covering map $p'' : \hat{X} \to Y$ so that $p''(\hat{x}) = y^0$. Thus, if one takes any other base point $x \in \hat{p}^{-1}(x)$, there is a covering map $\gamma : \hat{X} \to \hat{X}$ with $\gamma(\hat{x}) = x$. Since γ must be a covering automorphism, we have a transitivity.

Given two-orbifolds M and N, and an orbifold diffeomorphism $f : M \to N$ which lifts to a diffeomorphism $\tilde{f} : \tilde{M} \to \tilde{N}$, we obtain an induced homomorphism $\tilde{f}_* : \pi_1(M) \to \pi_1(N)$: For each deck-transformation ϑ of $\tilde{M}$, let $\tilde{f}_*(\vartheta)$ be the deck-transformation $\tilde{f} \circ \vartheta \circ \tilde{f}^{-1}$.

Lemma 1.4. *If $\tilde{f}_1 : \tilde{M} \to \tilde{N}$ is another diffeomorphism homotopic to $\tilde{f}$ by a homotopy $\tilde{M} \times I \to \tilde{N}$ equivariant with respect to $\tilde{f}_* : \pi_1(M) \to \pi_1(N)$, then $\tilde{f}_{1*} = \tilde{f}_*$.*

Proof. We have that $\gamma' = \tilde{f}_1 \circ \gamma \tilde{f}_1^{-1}$ is homotopic to $\gamma'' = \tilde{f} \circ \gamma \circ \tilde{f}^{-1}$, and let H be the homotopy between them. Then $H_t : \tilde{N} \to \tilde{N}$ is a deck transformation for each t. Since the group of deck transformations is discrete in C^r-topology, γ' and γ'' are equal. □

Lemma 1.5. *If $p : X \to Y$ be a regular covering map, the the following hold:*

- *X/Γ for a covering automorphism group Γ is an orbifold diffeomorphic to Y.*
- *Given an orbifold map $g : Z \to Y$, two lifts $g_1, g_2 : Z \to X$ differ by a unique automorphism; i.e., $g_2 = \gamma \circ g_1$ for $\gamma \in \Gamma$.*

Proof. The covering map $p : X \to Y$ induces an orbifold map $\hat{p} : X/\Gamma \to Y$. Over the regular point, $\hat{p}$ is a orbifold-diffeomorphism, and so $\hat{p}$ is an orbifold-diffeomorphism.

Let $x_1 = g_1(x)$ and $x_2 = g_2(x)$ for a regular point x of X. There exists a unique deck transformation $\gamma : X \to X$ so that $\gamma \circ g_1(x) = g_2(x)$. Over the regular points, we have $\gamma \circ g_1 | g^{-1}(Y^n) = g_2 | g^{-1}(Y^n)$. Again, an open and closedness argument as in the ordinary covering space theory shows that $\gamma \circ g_1 = g_2$. □

By above lemmas, we see that given a diffeomorphism $g : X_1 \to X_2$ of two orbifolds X_1 and X_2, there exists a diffeomorphism $\tilde{g} : \tilde{X}_1 \to \tilde{X}_2$ so that $\tilde{g}(x)$ is any point of $p_2^{-1}(g(x))$ where x is a regular point and $p_2 : \tilde{X}_2 \to X_2$ is the covering map. Any two different lifts of a diffeomorphism differ by a deck transformation of $\tilde{X}_2$.

Remark 1.1. The authors are not aware of the full theory of liftings for orbifold-covering spaces. But it might be desirable to have one for other purposes than required in this paper.

A *good* orbifold is an orbifold with a universal cover or a covering orbifold that is a manifold. A *very good* orbifold is an orbifold with a finite cover that is a manifold. A good orbifold X is always orbifold-diffeomorphic to M/Γ where M is a manifold and Γ is a discrete group acting on M properly. If M is simply-connected, then $\pi_1(X)$ is isomorphic to Γ.

A good orbifold M has a covering that is a simply-connected manifold $\tilde{M}$. Then it is a universal covering orbifold: there is an orbifold covering map from a universal covering orbifold $X \to \tilde{M}$ respecting covering maps. Since $\tilde{M}$ is a manifold, X has to be a manifold. Since X is connected, it is a simply-connected manifold, and by uniqueness, X is isomorphic to $\tilde{M}$ as covering spaces.

A singular point of a one-dimensional orbifold is always modeled on an open interval with $\mathbf{Z}_2$ acting as a reflection. Thus a compact one-dimensional orbifold is an orbifold diffeomorphic to one of the following: a circle, a closed interval without singular point, a closed interval with one singular point as an endpoint, and a closed interval with two singular points as endpoints. They are called a *circle*, a *segment*, a *half* 1-*orbifold*, and a *full* 1-*orbifold*.

They are all good orbifolds, and their orbifold-fundamental groups are respectively isomorphic to $\mathbf{Z}$, 1, $\mathbf{Z}_2$, and the extension $\mathbf{Z} \cdot \mathbf{Z}_2$ of $\mathbf{Z}$ by $\mathbf{Z}_2$.

A singular point x of a two-dimensional orbifolds has a neighborhoods modeled on the following pairs:

- an open ball in $\mathbf{R}^2$ with a Euclidean reflection acting on it fixing a point corresponding to x,: a *mirror point*.

- an open ball in $\mathbf{R}^2$ with a Euclidean rotation of order n acting on it fixing a point corresponding to x: a *cone-point of order* n.
- an open ball in $\mathbf{R}^2$ with a dihedral group generated by two Euclidean reflections with lines of fixed points meeting in an angle π/n, and the point corresponding to x a common fixed point of the group action: a *corner-reflector of order* n.

(This follows from an old result that a finite group action on $\mathbf{R}^n$ with fixed point is always conjugate to a linear action.)

From this, we see that the underlying space of a two-dimensional orbifold is homeomorphic to a surface with boundary or corners. The boundary of a two-dimensional orbifold is either a circle or a full 1-orbifold.

In dimension 2, the underlying space X_Q of an orbifold Q has a cellular decomposition such that each point of an open cell has the same model open set and the same finite group action. We define the Euler characteristic to be

$$\chi(Q) = \sum_{c_i} (-1)^{\dim(c_i)}(1/|\Gamma(c_i)|),$$

where c_i ranges over the open cells and $|\Gamma(c_i)|$ is the order of the group Γ_i associated with c_i.

Suppose that the orbifold Σ without boundary has the underlying space X_Σ and m cone-points of order q_i and n corner-reflectors of order r_j, then the following generalized Riemann-Hurwitz formula is very useful also:

$$\chi(\Sigma) = \chi(X_\Sigma) - \sum_{i=1}^{m}\left(1 - \frac{1}{q_i}\right) - \frac{1}{2}\sum_{j=1}^{n}\left(1 - \frac{1}{r_j}\right). \tag{5}$$

2. (G, X)-structures on orbifolds

Let G be a smooth Lie group acting on a smooth manifold X transitively. We give a restriction on action that if an element g of G acts as an identity map on an open subset of X, then g is the identity element.

An (G, X)-*structure* on an orbifold M is a collection of charts $\phi_U : U \to X$ for each model pair (U, Γ_U) so that ϕ_U conjugates the action of Γ_U with that of a finite subgroup G_U of G on $\phi(U)$ by an isomorphism $i_U : \Gamma_U \to G_U$, and the inclusion map induced map $U \to V$ is always realized by an element ϑ of G and the homomorphism $G_U \to G_V$ is given by a conjugation by ϑ; i.e., $g \mapsto \vartheta \circ g \circ \vartheta^{-1}$.

A maximal such family of collections (ϕ_U, i_U) is said to be a (G, X)-*structure* of M. A (G, X)-structure on M induces a (G, X)-structure on any of its covering orbifolds.

A (G, X)-*map* f between two (G, X)-orbifolds M and N is a map so that for each point x of N and a point y of N so that $x = f(y)$, and a neighborhood U of x modeled on a pair $(\tilde{U}, \Gamma_U)$ with a chart ϕ_U and a homomorphism $i_U : \Gamma_U \to G_U \subset G$, there is a neighborhood V of y modeled on a pair $(\tilde{V}, \Gamma_V)$ with a chart ϕ_V and a homomorphism $i_V : \Gamma_V \to G_V$ so that f lifts to a map $\tilde{f} : \tilde{V} \to \tilde{U}$ equivariant with respect to a homomorphism $\Gamma_V \to \Gamma_U$ induced by a homomorphism $G_V \to G_U$ given by a conjugation $g \mapsto \vartheta g \vartheta^{-1}$ by some $\vartheta \in G$.

Theorem 2.1 (Thurston)**.** *A (G, X)-orbifold M is a good orbifold. There exists an immersion D from the universal covering manifold $\tilde{M}$ to X so that*

$$D \circ \vartheta = h(\vartheta) \circ D, \vartheta \in \pi_1(M)$$

hold for a homomorphism $h : \pi_1(M) \to G$, where D is a local (G, X)-map. Moreover, any such immersion equals $g \circ D$ for $g \in G$, with the associated homomorphism $g \circ h() \circ g^{-1}$.

Proof. This is found in Chapter 5 of Thurston [19] based on (G, X)-germs. We rewrite it here for the reader's convenience: Let N be a neighborhood of $x \in \Sigma$, and $(\tilde{N}, \Gamma)$ be the model pair for $\tilde{N}$ an open set in X and Γ the associated finite group acting on $\tilde{N}$. We assume that Γ fixes the point $\tilde{x}$ corresponding to x by taking a small neighborhood N if necessary. We form $G \times \tilde{N}$ and give an action of Γ by $\gamma(g, y) = (\gamma g, gy)$. Then $G(N) = G \times \tilde{N}/\Gamma$ is a manifold and has a projection $p_N : G(N) \to N$ induced by the projection to the second factor.

Find a locally finite cover of Σ by such neighborhoods $\{N_1, N_2, \ldots ...\}$. If N_i and N_j meet, then $N_i \cap N_j$ has an inclusion map $i : N_i \cap N_j \to N_i$. Then there is an open subset A of $\tilde{N}_i$ and a subgroup Γ_A acting on it being a model for $N_i \cap N_j$. We form $G(N_i \cap N_j)_A$ where A denote the fact we used A as a model and find a map $\tilde{i} : G(N_i \cap N_j)_A \to G(N_i)$ induced by $G \times A \to G \times \tilde{N}_i$. This is an imbedding. We find an open subset B of $\tilde{N}_j$ corresponding to $N_i \cap N_j$, and form $G(N_i \cap N_j)_B$ similarly, and find an imbedding $G(N_i \cap N_j)_B \to G(N_j)$. Since $G(N_i \cap N_j)_A$ and $G(N_i \cap N_j)_B$ can be identified by the identification of the model pairs, we see that $G(N_i)$ and $G(N_j)$ can be pasted by this relation. We can easily show that such identification of $G(N_1), G(N_2), \ldots$ are possible, and obtain a manifold $G(M)$ from the identification.

The manifold $G(M)$ has a projection p to M clearly. The foliation of $G(N_i)$ with leaves that are images of $g \times \tilde{N}_i$ for $g \in G$ give rise to a foliation on $G(M)$ whose leaves meet the fibers of p at unique points. Take a leaf L in $G(\Sigma)$, and $p|L : L \to \Sigma$ is an orbifold covering map, and L is a manifold. Take a universal cover $\tilde{L}$ of L with covering map p_L. Then $p \circ p_L$ is a universal covering map of M. L has a (G, X)-structure since it covers M: one can induces charts. Then $\tilde{L}$ has a (G, X)-structure.

By above discussions, $\tilde{L}/\Gamma$ for the deck transformation group Γ is (G, X)-diffeomorphic to M by a map induced by $p \circ p_L$. As $\tilde{L}$ is a (G, X)-manifold, it has a developing map $D : \tilde{L} \to X$. For a deck transformation γ, $D \circ \gamma$ is also a (G, X)-map, and this means that $D \circ \gamma = h(\gamma) \circ D$ for $h(\gamma) \in G$. We can clearly verify that $h : \Gamma \to G$ is a homomorphism. □

A pair (D, h) of immersions $D : \tilde{M} \to X$ equivariant with respect to a homomorphism $h : \pi_1(M) \to G$ is said to be a development pair of M. D is called a *developing map* and h a *holonomy homomorphism*. Conversely, given such a pair (D, h), they give charts to $\tilde{M}$, and hence induces a (G, X)-structure on $\tilde{M}$. Since a deck-transformation is a (G, X)-map $\tilde{M} \to \tilde{M}$, we see that $M = \tilde{M}/\pi_1(M)$ has an induced (G, X)-structure from $\tilde{M}$.

We say that two such pairs (D, h) and (D', h') are *G-equivalent* if $D' = \vartheta \circ D$ and $h'(\cdot) = \vartheta \circ h(\cdot) \circ \vartheta^{-1}$ for $\vartheta \in G$.

Let us look at the set $\mathcal{M}(M)$ of all (G, X)-structures on M and introduce an equivalence relation that two (G, X)-structures μ_1 and μ_2 are equivalent if there is an isotopy $\phi : M \to M$ so that the induced (G, X)-structure $\phi^*(\mu_1)$ obtained by pulling back charts equals μ_2. The *deformation space of* (G, X)*-structures on* M *without topology* is defined to be this set $\mathcal{M}(M)/\sim$.

We can reinterpret this space as follows: consider the set of diffeomorphisms $f : M \to M'$ where M' is a (G, X)-manifold. We introduce an equivalence relation that f and $f' : M \to M''$ are equivalent if there is a (G, X)-diffeomorphism $\phi : M' \to M''$ so that $\phi \circ f$ is isotopic to f'. The set of equivalence classes corresponds in one-to-one manner with the above space by sending $f : M \to M'$ to $f_*(\mu)$ for the (G, X)-structure μ on M'.

We can think of this space as follows: consider the set of diffeomorphisms $\tilde{f} : \tilde{M} \to \tilde{M}'$ equivariant with respect to an isomorphism $\pi_1(M) \to \pi_1(M')$ where M' is a (G, X)-manifold. Introduce an equivalence relation that $\tilde{f} : \tilde{M} \to \tilde{M}'$ and $\tilde{f}' : \tilde{M} \to \tilde{M}''$ are equivalent if there are a (G, X)-diffeomorphism $\tilde{\phi} : \tilde{M}' \to \tilde{M}''$ and an isotopy $H : \tilde{M}' \times I \to \tilde{M}''$ both equivariant with respect to an isomorphism $\pi_1(M') \to \pi_1(M'')$ so that $H_0 = \phi \circ \tilde{f}$ and $H_1 = \tilde{f}'$. (The set of the equivalence class is certainly in one-to-one correspondence with the above set since two different choices of lifts of $f : M \to M'$ differ by a deck transformation of $\tilde{M}'$ which is a (G, X)-diffeomorphism.) Let us denote this space by $D_I(M)$

Following Lok's thesis [15], which is a note of J. Morgan lectures, we define the *isotopy-equivalence space* $\mathcal{S}(M_0)$ *of* (G, X)*-structures* for an orbifold M_0 of (G, X)-structure to be the space of equivalence class of pairs $(D, \tilde{f} : \tilde{M}_0 \to \tilde{M})$ where $\tilde{f}$ is a diffeomorphism equivariant with respect to an isomorphism $\pi_1(M_0) \to \pi_1(M)$ and $D : \tilde{M} \to X$ is a developing map of a (G, X)-structure on M. Two such pairs $(D, \tilde{f})$ and $(D', \tilde{f}')$ are *isotopy-equivalent* if and only if there is a diffeomorphism $\phi : \tilde{M} \to \tilde{M}'$ with $D' \circ \phi = D$ and an isotopy $H : \tilde{M} \times I \to \tilde{M}'$ equivariant with respect to an isomorphism $\pi_1(M) \to \pi_1(M')$ so that $H_0 = \phi \circ \tilde{f}$ and $H_1 = \tilde{f}'$.

The topology is given on the set of pairs by C^r-topology on $D \circ \tilde{f}$, i.e., a sequence of functions converge if it does on every compact subsets of $\tilde{M}$ uniformly in C^r-sense. ($r = 1$ is sufficient for all purposes.) We give the quotient topology on $\mathcal{S}(M_0)$.

There is a natural G action on $\mathcal{S}(M_0)$ given by

$$\gamma(D, \tilde{f}) = (\gamma \circ D, \tilde{f}), \gamma \in G.$$

Let $\mathcal{D}(M_0)$ be the quotient space under this action. Then $D_I(M_0)$ and $\mathcal{D}(M_0)$ is also in one-to-one correspondence given by sending $\tilde{f} : M_0 \to M'$ to the equivalence class of $(D, \tilde{f})$ where $D : \tilde{M}' \to X$ is a developing map of M'. Therefore, we call $\mathcal{D}(M_0)$ the *deformation space* of (G, X)-structures on M_0.

The set of all homomorphisms $h : \pi_1(M) \to G$ is denoted by $\mathrm{Hom}(\pi_1(M), G)$. We assume that $\pi_1(M)$ is finitely presented. For example, if the underlying space of M is compact, this is true. Let $g_1, \dots, g_n$ denote the generators of $\pi_1(M)$, and $R_1, \dots, R_m$ relations. Then $H = \mathrm{Hom}(\pi_1(M), G)$ can be injectively mapped into G^n by sending a homomorphism to the elements corresponding to generators. The relations give us the subset of G^n where H can lie. Actually, the subset defined by the relations gives us precisely the image. Thus, we identify H with this subset. The subset has a subspace topology of real algebraic variety, which we give to H.

There is an action by conjugation on H sending a homomorphism $h(\cdot)$ to $\vartheta \circ h(\cdot) \circ \vartheta^{-1}$ for $\vartheta \in G$. H/G may not be a Hausdorff space. There is a subset H^s of H, where G acts properly, consisting of points lying in stable orbits when G is the group of $\mathbf{R}$-points of an algebraic group $\bar{G}$ defined over $\mathbf{R}$. H^s/G is a Hausdorff real analytic space.

We define a *pre-hol* map

$$\mathcal{PH} : \mathcal{S}(M_0) \to \mathrm{Hom}(\pi_1(M_0), G)$$

by sending $(D, \tilde{f} : \tilde{M}_0 \to \tilde{M})$ to the holonomy representation $h \circ \tilde{f}_*$ where $\tilde{f}_*$ is the induced homomorphism $\pi_1(M_0) \to \pi_1(M)$.

First of all, this is well-defined: Let $(D', \tilde{f}' : \tilde{M}_0 \to \tilde{M})$ be an equivalent pair. Then $D = D' \circ \tilde{\phi}$ for a lift of homeomorphism $\phi : M \to M'$. For a deck transformation ϑ of $\tilde{M}_0$,

$$\begin{aligned}
h(\tilde{f} * (\vartheta)) \circ D &= D \circ \tilde{f} \circ \vartheta \circ \tilde{f}^{-1} \\
&= D' \circ \tilde{\phi} \circ \tilde{f} \circ \vartheta \circ \circ \tilde{f}^{-1} \circ \tilde{\phi}^{-1} \circ \tilde{\phi} \\
&= D' \circ \tilde{f}' \circ \vartheta \circ \tilde{f}'^{-1} \circ \tilde{\phi} \text{ by Lemma 1.4} \\
&= h'(\tilde{f}'_*(\vartheta)) \circ D' \circ \tilde{\phi} \\
&= h'(\tilde{f}'_*(\vartheta) \circ D \qquad (6)
\end{aligned}$$

Therefore, we obtain $h \circ \tilde{f}_* = h' \circ \tilde{f}'_*$ meaning well-definedness.

Also $\mathcal{PH}$ is continuous: Let $C(M_0)$ denote the space of pairs $(D, \tilde{f} : \tilde{M}_0 \to \tilde{M})$ with C^1-topology. Then sending $(D, \tilde{f} : M_0 \to M)$ to $h \circ \tilde{f}_*$ is a continuous map, which we denote by

$$\mathcal{PPH} : C(M_0) \to \mathrm{Hom}(\pi_1(M), G)$$

for later purposes. This can be seen by the fact that C^1-convergence of sequence of $D_i \circ \tilde{f}_i$ for pairs $(D_i, \tilde{f}_i)$ implies the uniform C^∞-convergence of $h_i(\tilde{f}_*(\vartheta))$ for each deck transformation ϑ. (The sequence of locally defined maps $D_i \circ \tilde{f}_i \circ \vartheta \circ \tilde{f}_i^{-1} \circ D_i^{-1}$ converges in C^1-topology and hence in C^∞-topology as G acts smoothly on X.)

The proof of the following theorem for manifolds was first given by Thurston (perhaps earlier by Ehresmann), again by Canary-Epstein-Green, and simultaneously by J. Morgan and W. Lok:

Proposition 2.1. *The map $\mathcal{PH}$ is a local homeomorphism; i.e., for each point of $\mathcal{S}(M)$ there is a neighborhood mapping homeomorphic to a neighborhood of its image.*

Corollary 2.1. *Let D be a subset of $\mathcal{S}(M)$ which maps into a union of stable G-orbits of* $\mathrm{Hom}(\pi_1(M), G)$. *Then $\mathcal{PH}$ induces a local homeomorphism*

$$\mathcal{H} : \mathcal{S}/G \to \mathrm{Hom}(\pi_1(M), G)^s/G.$$

We call the above map $\mathcal{H}$ the *hol* map.

3. The proof of Proposition 2.1

Let us now present three lemmas 3.1, 3.2, and 3.3 on the perturbation of the finite group actions and conjugation by diffeomorphisms.

Lemma 3.1. *Let Γ be a finite subgroup of G acting on an n-ball B in X. Let $h_t : \Gamma \to G$, $t \in [0, \epsilon]$, $\epsilon > 0$, be an analytic parameter of representations of Γ so that h_0 is the inclusion map. Then for $0 \leq t \leq \epsilon$, there exists a continuous family of diffeomorphism $f_t : B \to B_t$ to an open ball B_t in X so that f_t conjugates $h(G)$-action to $h_t(G)$-action; i.e., $f_t^{-1} h_t(g) f_t = h(g)$ for each $g \in G$.*

Proof. We take a product $X \times I$ and let v be a vector field in the positive I-direction in the product space. The G acts smoothly on $X \times I$ by sending (x, t) to $(g_t(x), t)$. We average $g^*(v)$ for $g \in G$ to obtain a smooth Γ-invariant vector field V. The integral curve l of V starting from $(x, 0)$ is mapped to an integral curve m of V starting from $(g(x), 0)$. Thus, the endpoint $l(1)$ is sent to $m(1)$, and so $g(l(1)) = m(1)$. Hence, let $f'_t(x)$ equal the point of the path from $(x, 0)$ at time t. Then $f_t = p \circ f'_t : X \to X$ for the projection $p : X \times I \to X$ is a desired diffeomorphism and $f_t(B)$ is the desired open ball. □

A point x of a real algebraic variety has a neighborhood with a semi-algebraic homeomorphism to a cone over a semi-algebraic set S in the boundary of a small ball with a cone-point at the origin corresponding to x.

Lemma 3.2. *Suppose that h is a point of an algebraic variety $V = \mathrm{Hom}(\Gamma, G)$ for a finite group, and let N be a cone-neighborhood of h. Then for each $h' \in N$, there is a corresponding diffeomorphism $f_{h'} : B \to B_t$ so that $f_{h'}$ conjugates $h(G)$-action to $h'(G)$-action; i.e., $f_t^{-1} h'(g) f_t = h(g)$ for each $g \in G$. Moreover, the map $h' \mapsto f_{h'}$ is continuous from N to the space $C^\infty(B, X)$ of smooth functions from B to X.*

Proof. Parameterize N by $[0, \epsilon] \times S$ for a semi-algebraic variety S with $\{0\} \times S$ corresponding to h and, for each $x \in S$, there is a map $[0, \epsilon] \times x \to C^\infty(B, X)$ from the above lemma 3.1. Again, we obtain a smooth Γ-invariant vector field V_x on $X \times [0, \epsilon]$ as above, and V_x depends continuously on x. From this we see that $f_{x,t}$ corresponding to a representation corresponding to (x, t) depends continuously on (x, t). □

We assume the premise and the conclusion of the above lemma 3.2

Let $l(h') : [0, \epsilon] \to V$ be a ray in N so that $l(h')(0) = h$ and $l(h')(t) = h'$ for $t \in [0, \epsilon]$. We fix the collection of such rays by fixing the parameterization by the cone.

Let F be a submanifold of B. Let Γ_F denote the subgroup of Γ acting on F. A *$l(h')$-equivariant isotopy* $H : F \times I \to X$ is a map so that F_t is an imbedding for each $t \in [0, \epsilon]$ conjugating Γ_F action on X to $l(h')(t)(\Gamma_F)$-action on X, and H_0 is an inclusion map $F \to X$.

Let $i : B \to X$ be the inclusion map, and F a submanifold of B with finitely many components so that a subgroup Γ_F of Γ acts on F. (For this paper, $\Gamma_F = \Gamma$ is sufficient.) Let $\mathcal{E}(N, B, X)$ be the subset of $N \times C^\infty(B, X)$ so that its element (h', f) is a pair with an imbedding $f : B \to X$ equivariant with respect to h' with a $l(h')$-equivariant isotopy. We give $\mathcal{E}(N, B, X)$ the subspace topology from the product topology of N and the C^r-topology of $C^\infty(B, X)$. (We assume $r \geq 1$.)

Let S be a submanifold of X with a subgroup Γ_S of Γ acts on. Let $\mathcal{E}(N, S, X)$ be the subset of $N \times C^\infty(S, X)$ so that its elements (h', f) is a pair with an imbedding $f : S \to X$ equivariant with a $l(h')|G_S$-equivariant isotopy. Note S could have more than one components.

Let

$$\Phi : \mathcal{E}(N, B, X) \to \mathcal{E}(N, F, X)$$

be given by sending (h', f) to $(h', f|F)$.

The following lemma shows that $h'|\Gamma_i \to G$ imbeddings of F can be extend to B if it was close to the inclusion map $i : B \to X$.

Lemma 3.3. *There is a neighborhood W of $(h, i|F)$ in $\mathcal{E}(N, F, X)$ with a section s of Φ so that $s(h, i|F) = (h, i)$. W is of form $N \times O \cap \mathcal{E}(N, F, X)$ for an open neighborhood O of $i|F$ in the space of imbeddings from F to X.*

Proof. A $l(h')|\Gamma_F$-equivariant isotopy H of $F \to X$ gives us a vector field on $H(F \times [0, \epsilon])$ in $X \times I$ with the vertical component equal to 1. Extend this vector field arbitrarily to $X \times I$ but with vertical component 1, and now average by the Γ-action on $X \times I$ defined by $g(x, t) = (l(h')(t)(g), t)$. As before, integrate the vector field to obtain an isotopy of B. The last statement follows from the product topology. □

Remark 3.1. We choose some arbitrary Riemannian metric on a neighborhood of B, and can assume that the image of $f_{h'}$ are all in this neighborhoods. By our construction, given any $\epsilon > 0$, we can make sure that the C^r-norm of $f_{h'}$, constructed in above lemmas, minus the inclusion map of B is less than ϵ in some coordinate systems if we choose the neighborhoods N sufficiently small near h. In particular, we can assume that for each $\epsilon > 0$, there is a neighborhood N of h so that $d(f_{h'}(x), x) \leq \epsilon$ for $x \in B$ and $h' \in N$ where $f_{h'}$ is obtained from above three lemmas.

If B was strictly convex with smooth boundary, we see that $f_{h'}(B)$ is also strictly convex with smooth boundary as the boundary convexity is given by a C^2-condition.

We can trivially generalize Lemma 3.3 so that B could be a union of disjoint collection of balls with some finite groups acting on each.

A *Riemannian* metric on an orbifold is a Riemannian metric on each model open set invariant under the associated finite group action and inclusion induced maps for model pairs are isometries. We can always put a Riemannian metric on a compact orbifold. Cover the orbifold by the modeled neighborhoods and choose a finite subcover $\{V_i\}$, and a partition of unity. Let (U_i, Γ_i) be the modeling pairs. Choose a Riemannian metric on U_i and by taking an average over the finite group action Γ_i, we obtain an invariant metric on each modeled neighborhood V_i. Next, we use a partition of unity on M to obtain a Riemannian metric over the orbifold. This also induces a Riemannian metric on U_i invariant under Γ_i.

We make that a quotient of the tangent bundle $T(U_i)$ over U_i by Γ_i to obtain $2n$-dimensional orbifold O_i. We can easily patch O_is together to obtain a $2n$-orbifold $T(M)$ with a map $p : T(M) \to M$ so that an inverse image of a point is a vector space modulo a finite group action. Let $T_{x_0}(M)$ denote the fiber over $x_0 \in M$.

If $x_0 \in M$ is a singular point in V_i, then we can choose an open ball U_{x_0} in U_i so that the subgroup Γ_{x_0} of Γ_i fixing the point $\tilde{x}_0$ corresponding to x_0 acts on it. Then there is a neighborhood of V_{x_0} of x_0 which is modeled on (U_{x_0}, Γ_{x_0}).

An exponential map from $T_{x_0}(M)$ to V_{x_0} is locally defined by the exponential map on the modeling open set U_{x_0} which is clearly invariant under the finite group action if x_0 is singular. If x_0 is nonsingular, we can use the ordinary exponential map. We can obviously patch these maps to obtain a global map $\exp_{x_0} : T_{x_0}(M) \to M$.

We can find $r > 0$ so that under $\exp_{\tilde{x}_0}$ imbeds the ball $B_r(0) \subset T_{\tilde{x}_0}U_{x_0}$ of radius r to a strictly convex ball in U_{x_0}. (They have smooth convex boundary.) Thus, the exponential map from each $x_0 \in M$ sends a quotient of a ball to a quotient of a strictly convex ball in M. Cover M by finitely many of them, say $V_1, \ldots, V_k$.

We may choose quotients of strictly convex balls $X_i \subset W_i \subset V_i$ for each i so that X_i is precompact in W_i and W_i is so in V_i, and we assume that they have smooth boundary in V_i.

Let $\tilde{M}$ be the universal cover of M. Since $\tilde{M}$ is a manifold, it has an induced Riemannian metric in the ordinary sense. The components of inverse images of the balls V_i above, are strictly convex balls which are images of exponential maps. By their strict convexity, any two of them meet in a strictly convex ball, i.e., in a contractible subset.

For each V_i, choose an arbitrary component L_i in $\tilde{M}$ of its inverse image. L_i is homeomorphic to an n-ball, and there exists a finite subgroup Γ_i of Γ acting on L_i, and (L_i, Γ_i) is a model pair for V_i. We choose M_i and N_i in L_i corresponding to X_i and W_i respectively.

Given i, j, if V_i and V_j meet, then there exists a deck-transformation γ_{ij} so that $L_i \cap \gamma_{ij} L_j \neq \emptyset$. The choice of γ_{ij} is not unique if Γ_i and Γ_j are not trivial since one can always multiply γ_{ij} in the left by an element of Γ_i in the right by an element of Γ_j. Let Γ_{ij} denote the all such possibilities for L_i and L_j. ($\Gamma_{ii} = \Gamma_i$.)

Clearly, every element of Γ_{ij} can be written $\gamma_1\gamma\gamma_2$ where $\gamma_1 \in \Gamma_i$, $\gamma_2 \in \Gamma_j$, and γ is a fixed element of Γ_{ij}. Thus, one can make sense of the coset space Γ_{ij}/Γ_j, one-to-one correspondence with Γ_i.

We note that $L_i \cap \gamma L_j$ for $\gamma \in \Gamma_{ij}$ is a convex ball, hence contractible. The same can be said for $M_i \cap \gamma M_j$ and $N_i \cap \gamma N_j$. We assume that $L_i \cap \gamma L_j \neq \emptyset$ if and only if $M_i \cap \gamma M_j \neq \emptyset$ if and only if $N_i \cap \gamma N_j \neq \emptyset$.

We claim that $\bigcup_{i,j} \Gamma_{ij}$ is a set of generators of $\pi_1(M)$: Let $\gamma \in \pi_1(M)$. Since $\tilde{M}$ is connected, there is a path from L_1 to $\gamma(L_1)$. There exists a collection $V_1, V_2, \ldots, V_n$ of open sets so that $V_1 = L_1$, $V_n = \gamma(L_1)$, and V_i is of form $\gamma_i(L_{k_i})$ for $i = 1, \ldots, n$. Since V_j and V_{j+1} meet, and so $\gamma_j(L_{k_j})$ and $\gamma_{j+1}(L_{k_{j+1}})$ meet, it follows that $\gamma_j^{-1}\gamma_{j+1}$ lies in $\Gamma_{k_j,k_{j+1}}$. We have

$$
\begin{aligned}
V_1 &= L_1 \\
V_2 &= \gamma_{1k_2} L_{k_2}, \text{ for } \gamma_{1k_2} \in \Gamma_{1k_2} \\
V_3 &= \gamma_{1k_2}\gamma_{k_2k_3} L_{k_3}, \text{ for } \gamma_{k_2k_3} \in \Gamma_{k_2k_3} \\
& \cdots \\
V_n = \gamma(L_1) &= \gamma_{1k_2}\gamma_{k_2k_3}\cdots\gamma_{k_{n-1}1} L_1, \text{ for } \gamma_{k_{n-1}1} \in \Gamma_{k_{n-1}1} \qquad (7)
\end{aligned}
$$

Thus, we see that

$$\gamma = \gamma_{1k_2}\gamma_{k_2k_3}\cdots\gamma_{k_{n-1}1}.$$

We can write any element of Γ as a product of elements in $\bigcup_{i,j}\Gamma_{ij}$.

Also, we see that

$$\begin{aligned} &\gamma\circ\gamma''\circ\gamma' \in \Gamma_{ik} \quad \text{for some choices of } \gamma'' \in \Gamma_j \\ &\qquad\qquad \text{if } \gamma \in \Gamma_{ij} \text{ and } \gamma' \in \Gamma_{jk} \text{ and } V_i \cap V_j \cap V_k \neq \emptyset, \\ (8) \qquad &\text{and} \qquad \gamma^{-1} \in \Gamma_{ji} \text{ if } \gamma \in \Gamma_{ij}. \end{aligned}$$

Let $(D, \tilde{f} : \tilde{M}_0 \to \tilde{M})$ be an element of $C(M_0)$. Let h be the associated holonomy homomorphism $\pi_1(M_0) \to G$.

We will find a neighborhood Ω of $h \circ \tilde{f}_*$ in Hom $(\pi_1\ (M_0),\ G)$ so that there is a continuous map $s : \Omega \to C(M_0)$ where $\mathcal{PPH} \circ s$ is the identity map and $s(h \circ f_*) = (D, \tilde{f})$. The map s induces a continuous map

$$\tilde{s} : \Omega \to \mathcal{C}(M_0),$$

which is a local inverse of $\mathcal{PH}$.

One can construct the underlying space of X_M from V_is. That is, we introduce an equivalence relation on the disjoint union $\coprod_{i=1}^n L_i$ given by letting $x \sim y$ if $x = \gamma_{ij}(y)$ for $x \in L_i, y \in L_j$. Obviously, the orbifold structure is encoded in this construction; thus, we can construct M back from this construction given Γ_{ij}s.

We can also construct M from $\coprod_{i=1}^n D(L_i)$ from the equivalence relation that $x \sim_M y$ if $x \in h(\gamma_{ij})(y)$ for $x \in D(L_i), y \in D(L_j)$. This is easily shown to be an equivalence relation (see equation 8). Let $Q : \coprod D(L_i) \to M$ denote the quotient map. The components of the inverse images of $Q(D(L_i))$ under the universal covering map $\tilde{M} \to M$ form a covering of $\tilde{M}$.

For a moment, we identify M with M_0 and $\pi_1(M)$ with $\pi_1(M_0)$ by $\tilde{f} : \tilde{M}_0 \to \tilde{M}$: We choose a cone-neighborhood Ω of h in $\mathrm{Hom}(\pi_1(M), G)$ so that for each finite Γ_i associated with L_i are in a neighborhood N of $\mathrm{Hom}(\Gamma_i, G)$ satisfying Remark 3.1 for B equal to $D(\mathrm{Cl}(N_i))$ or $D(\mathrm{Cl}(M_i))$ and Riemannian metrics from M pushed to X by $D|L_i$. That is, we assume that $f_{h'}(B)$ are subsets of $D(L_i)$ or $D(N_i)$ respectively for $f_{h'}$ obtained in the lemmas 3.1, 3.2, and 3.3.

Given h' in Ω, we will construct a real projective manifold M' which is homeomorphic to M_0.

We define a topological space $\coprod_{j\in J} D(N_j)/\sim_{M'}$ where $\sim_{M'}$ is defined as follows: $x \in D(N_i)$ and $y \in D(N_j)$ are equivalent if $x = h'(\gamma)(y)$ for $\gamma \in \Gamma_{ij}$. This obviously reflexive and transitive by equation 8. Let $Q' : \coprod D(N_i)/\sim_{M'} \to M'$ be the quotient map. Note here that $Q'(D(M_i)) = W_i$ exactly.

It is clear that M' is a manifold, possibly non-Hausdorff. We show that M' is Hausdorff. Let $x \in D(N_i)$ and $y \in D(N_j)$, and suppose that they are not equivalent. If $i \neq j$ and $\Gamma_{ij} = \emptyset$, then $Q'(D(N_i))$ and $Q'(D(N_j))$ are disjoint neighborhoods of $Q'(x)$ and $Q'(y)$. If $i \neq j$ and $\Gamma_{ij} \neq \emptyset$, then define a map

$$D' : D(N_i) \coprod \coprod_{[\gamma]\in\Gamma_{ij}/\Gamma_j} D(N_j)^{[\gamma]} \to X$$

by letting $D'|D(N_i)$ be the inclusion map, and $D'|D(N_j)^{[\gamma]}$ be the map $h'(\gamma)|D(N_j)$ where γ is a representative of $[\gamma]$ and $D(N_j)^{[\gamma]}$ is a copy of $D(N_j)$ for each $[\gamma] \in \Gamma_{ij}/\Gamma_j$. Since x and y are not equivalent, $D'(x)$ and $D'(\gamma' y^{[\gamma]})$ for a copy $y^{[\gamma]}$ of y, each $\gamma' \in \Gamma_j$, and $[\gamma] \in \Gamma_{ij}/\Gamma_j$ are not equal. Thus, there exist disjoint neighborhoods of $D'(x)$ and the set $\{D'(\gamma' y^{[\gamma]})\}$ which are Γ_{ii}-invariant, and the later of which intersected with $D'|D(N_j)^{[\gamma]}$ is Γ_{ii}-invariant when taken an inverse image by D' in $D(N_j)^{[\gamma]}$. Then the component of the neighborhood containing y and that containing x have no equivalent points since every equivalence between $D(L_i)$ and $D(L_j)$ arises from Γ_{ij} (see equation 8). The disjoint neighborhoods clearly map to disjoint neighborhoods in M' by Q'. If $i = j$, a similar argument applies. Thus, M' is a Hausdorff space.

The second countability of M' is easy to show.

Since $D(\mathrm{Cl}(N_i))$ are compact, and we can easily define a map from $\coprod D(\mathrm{Cl}(N_i))$ to M' by extending the quotient map $\coprod D(N_i) \to M'$, we see that M' is compact.

Also, M' is obviously an (G, X)-orbifold since we obtained M' by patching together the finite subgroup orbits in an open subsets of X. Clearly, $Q'(D(N_j))$ are open cover of M' modeled on the pairs $(D(N_j), \Gamma_j)$.

First, we will construct an orbifold diffeomorphism $\phi : M \to M'$. The construction is generalized from Lok [15]: Define an imbedding $I_i : Q(D(\mathrm{Cl}(M_i))) \to Q'(D(N_i))$ by

$$I_i = Q' \circ f_{h'|\Gamma_i} \circ (Q|D((\mathrm{Cl}(N_i)))^{-1}$$

obtained by Lemma 3.2 if Γ_i is not trivial, or

$$I_i = Q' \circ (Q|D(\mathrm{Cl}(N_i)))^{-1}|D(\mathrm{Cl}(M_i))$$

if Γ_i is trivial. (We define $\tilde{I}_i : D(\mathrm{Cl}(M_i)) \to D(N_i)$ to be $f_{h'|\gamma_i}$, which covers the above map.) The problem is that I_is are not consistently defined over the overlaps of $Q(D(\mathrm{Cl}(M_i)))$ and hence, we need to modify the map. We have an ordering $M_1, M_2, \ldots, M_n$ for some n. We look at the sets of form

$$Q(D(\mathrm{Cl}(M_{i_1}))) \cap Q(D(\mathrm{Cl}(M_{i_2}))) \cap \cdots \cap Q(D(\mathrm{Cl}(M_{i_t}))), i_1 \leq i_2 \leq \cdots \leq i_t$$

for some t. There is an upper bound t_0 on t. Note that for given t_0, the collection of the sets of above forms is composed of disjoint contractible compact submanifolds. We define a map $\phi : M \to M$ by defining it to be I_{i_1} on each sets of above form for $t = t_0$ and the lowest index i_1. We note that I_{i_1} lifts to a smooth embedding $\tilde{I}_{i_1}$ restricted to the inverse image of the above set in $D(\mathrm{Cl}(M_{i_1}))$.

We begin an induction. Suppose that we defined an immersion ϕ from the union of sets of form

$$\begin{aligned} & Q(D(\mathrm{Cl}(M_{i_1}))) \cap Q(D(\mathrm{Cl}(M_{i_2}))) \cap \cdots \cap Q(D(\mathrm{Cl}(M_{i_t}))), i_1 \leq i_2 \leq \cdots \leq i_t \\ \text{to} \quad & Q'(D(N_{i_1})) \cap Q'(D(N_{i_2})) \cap \cdots \cap Q'(D(N_{i_t})) \subset M' \end{aligned} \tag{9}$$

to M' so that ϕ lifts to a smooth map on the inverse image under Q in $D(\mathrm{Cl}(M_{i_j}))$ to $D(N_{i_j})$ which is $h' \circ h^{-1}|h(\Gamma_{i_j})$-equivariant for $j = 1, \ldots, t$. (We should think of the situation of Lemma 3.3.)

Then we define a map from the union of sets of form

$$
\begin{aligned}
(10)\qquad & Q(D(\mathrm{Cl}(M_{i_1}))) \cap Q(D(\mathrm{Cl}(M_{i_2}))) \cap \cdots \cap Q(D(\mathrm{Cl}(M_{i_{t-1}}))), i_1 \leq i_2 \leq \cdots \leq i_{t-1} \\
\text{to} \quad & Q'(D(N_{i_1})) \cap Q'(D(N_{i_2})) \cap \cdots \cap Q'(D(M_{i_{t-1}})) \subset M'.
\end{aligned}
$$

Take one of them say A of form

$$Q(\mathrm{Cl}(D(M_{i_1}))) \cap Q(D(\mathrm{Cl}(M_{i_2}))) \cap \cdots \cap Q(D(\mathrm{Cl}(M_{i_{t-1}}))), i_1 \leq i_2 \leq \cdots \leq i_{t-1}.$$

The subset $\tilde{A} = Q^{-1}(A) \cap D(\mathrm{Cl}(M_{i_1}))$ is an imbedded submanifold where Γ_{i_1} acts on. Let A' be the subset

$$\bigcup_{i_t=1}^{n} \mathrm{Cl}(Q'(D(M_{i_1}))) \cap \mathrm{Cl}(Q'(D(M_{i_2}))) \cap \cdots \cap \mathrm{Cl}(Q'(D(M_{i_{t-1}}))) \cap \mathrm{Cl}(Q'(D(M_{i_t}))),$$

of A where $i_1 \leq i_2 \leq \cdots \leq i_{t-1}$, $i_t \neq i_1, \ldots, i_{t-1}$, and ϕ is already defined. The subset $\tilde{A}' = Q^{-1}(A') \cap D(\mathrm{Cl}(M_{i_1}))$ is an imbedded submanifold of $\tilde{A}$ where Γ_{i_1} acts on. ϕ lifts to $\tilde{\phi}$ on $\tilde{A}'$ and using Lemma 3.3, we obtain an embedding $\phi'|\tilde{A} : \tilde{A} \to X$ equivariant with respect to $h' \circ h|h(\Gamma_{i_1})$. (See Remark 3.1.) Therefore, the map ϕ on A' extends to a smooth map $\phi' : A \to M'$. We can do this for sets of form A consistently since they overlap in sets of form A' where ϕ is already defined. By induction, we obtain a map $\phi : M \to M'$.

We note that for each M_i, there is a map $\tilde{\phi}_i : D(M_i) \to D(L_i)$ lifting ϕ. This is a lifting of ϕ in the models of neighborhoods $Q(D(M_i))$ to $Q'(D(L_i))$. By taking a finite cover of M initially, so that there are some points which is covered by the open sets only once, we see that the degree of ϕ is one, and so ϕ is an orbifold-diffeomorphism. (Note ϕ is smooth almost everywhere, and M removed with singularity is smooth, and hence M can be considered a CW-complex and hence topological degree is defined. By continuation, we see that the degree is always one on nonsingular points, and hence ϕ is an orbifold-diffeomorphism.)

Since M' and M are orbifold-diffeomorphic, their universal covers $\tilde{M}'$ and $\tilde{M}$ are diffeomorphic also equivariant with respect to an isomorphism $\pi_1(M') \to \pi_1(M)$.

Actually, we can construct $\tilde{M}'$ explicitly from $\tilde{M}$ as follows: $\tilde{M}$ is covered by open sets of form γL_i for $\gamma \in \pi_1(M)$, $i = 1, \ldots, n$. $\tilde{M}$ can be considered a quotient space of $\coprod h(\gamma)D(M_i)$ under the equivalence relation that $x \in h(\gamma)D(M_i) \sim y \in h(\gamma')D(M_j)$ if $x = y$ and $\gamma^{-1}\gamma' \in \Gamma_{ij}$ (or γM_i and $\gamma' M_j$ meet). Let $\tilde{Q} : \coprod h(\gamma)D(M_i) \to \tilde{M}$ denote the quotient map. (We take distinct copies in the disjoint union of $h(\gamma)D(M_i)$ for each γ unless $\gamma^{-1}\gamma'$ belongs to Γ_i, in which case, we consider $h(\gamma)D(M_i)$, same as $h(\gamma')D(M_i)$.)

We define $\tilde{M}'$ as the quotient space of $\coprod h'(\gamma)D(N_i)$ again with the relation $x \in h'(\gamma)D(N_i) \sim y \in h'(\gamma')D(N_j)$ if $x = y$ and $\gamma^{-1}\gamma' \in \Gamma_{ij}$. (Again, we use the above copying rule.) $\tilde{M}'$ is clearly a manifold. Also, from nerve consideration, $\tilde{M}'$ has a same nerve of covering as $\tilde{M}$. Thus, $\tilde{M}'$ is a simply-connected manifold. We define a map $p_{M'} : \tilde{M}' \to M'$ by defining

$$p_{M'}|\tilde{Q}'(h'(\gamma)D(N_i)) : \tilde{Q}'(h'(\gamma)D(N_i)) \to Q'(D(N_i))$$

by sending a point corresponding to $h'(\gamma)(x)$ to $Q'(x)$ for $x \in \coprod D(N_i)$ $p_{M'}$ is clearly an orbifold-covering map. Moreover, $\pi_1(M)$ acts on $\tilde{M}'$ by letting $\vartheta \in \pi_1(M)$ act by sending $x \in \tilde{Q}'(h'(\gamma)D(N_i))$ to a point in $\tilde{Q}'(h'(\vartheta)h'(\gamma)D(N_i)$ by a map $\tilde{Q}' \circ h'(\vartheta) \circ \tilde{Q}'^{-1}$. This is a well-defined automorphism of $\tilde{M}'$, and $\tilde{M}'/\pi_1(M)$ is orbifold-diffeomorphic to M'. (Of course, the covering map $p_M : \tilde{M} \to M$ and the action of $\pi_1(M)$ on $\tilde{M}$ can be defined the same way.)

Let $\tilde{Q}' : \coprod h'(\gamma)D(N_i) \to \tilde{M}'$ denote the quotient map.

The above diffeomorphism ϕ lifts to a diffeomorphism $\tilde{\phi} : \tilde{M} \to \tilde{M}'$: We first define $\tilde{\phi} : D(M_i) \to D(N_i)$ by lifting $\phi : Q(D(M_i)) \to Q'(D(N_i))$, and for $h(\gamma)D(M_i)$, with $\gamma \in \pi_1(M)$, we define $\tilde{\phi} : h(\gamma)D(M_i) \to h'(\gamma)D(N_i)$ by letting $\tilde{\phi}(x)$ to be $h'(\gamma) \circ \tilde{\phi}_i \circ h(\gamma)^{-1}(x)$. This is well-defined: Let y be a point of $h(\gamma')D(M_j)$ for some j, $\gamma' \in \pi_1(M)$ so that $x = y$ and $\gamma^{-1}\gamma' \in \Gamma_{ij}$. Then

$$h'(\gamma') \circ \tilde{\phi}_j \circ h(\gamma')^{-1}(y) = h'(\gamma)h'(\gamma^{-1}\gamma') \circ \tilde{\phi}_j \circ h(\gamma'^{-1}\gamma)h(\gamma^{-1})(x).$$

For $\eta \in \Gamma_{ij}$,

$$h'(\eta) \circ \tilde{\phi}_j \circ h(\eta^{-1})|D(M_i) \cap \eta D(M_j) = \tilde{\phi}_i|D(M_i) \cap \eta D(M_j)$$

by the fact that $\tilde{\phi}_j$ descends to a well-defined function ϕ_j agreeing with ϕ_i on $Q(M_i) \cap Q(M_j)$. The right-hand side of the above equation is now $h'(\gamma) \circ \tilde{\phi}_i \circ h(\gamma^{-1})(x)$. This defines a smooth map $\tilde{\phi} : \tilde{M} \to \tilde{M}'$, which is an immersion.

We see that $p_{M'} \circ \tilde{\phi} = \phi \circ p_{M'}$ clearly, and $\tilde{\phi}$ is a lift of an orbifold-diffeomorphism ϕ. Hence, $\tilde{\phi}$ is a diffeomorphism $\tilde{M} \to \tilde{M}'$, which follows easily. The above map $\tilde{\phi}$ is equivariant, i.e., $\tilde{\phi} \circ \gamma = \gamma \circ \tilde{\phi}$ for $\gamma \in \pi_1(M)$. Thus, we see that $\tilde{M}'$ is the universal covering space of M' and $\pi_1(M)$ and $\pi_1(M')$ are isomorphic by $\tilde{\phi}_*$ induced from $\tilde{\phi}$.

We define a developing map $D' : \tilde{M}' \to X$ by defining $D'|\tilde{Q}'(h'(\gamma)D(N_i))$ to be given by $\tilde{Q}'^{-1}|h'(\gamma)D(N_i)$. This defines a smooth immersion over $\tilde{M}'$ in a consistent manner. We consider $D' \circ \vartheta$ for $\vartheta \in \pi_1(M)$. Then on $\tilde{Q}'(D(N_i))$, it equals

$$\tilde{Q}'^{-1} \circ \tilde{Q}' \circ h'(\vartheta) \circ \tilde{Q}'^{-1}$$

which equals $h'(\vartheta) \circ \tilde{Q}'^{-1}$. We obtain $D' \circ \vartheta = h'(\vartheta) \circ D'$. Therefore, the holonomy homomorphism is $h' : \pi_1(M) \to G$ under the identification $\pi_1(M') = \pi_1(M)$. Or equivalently, $h'' \circ \tilde{\phi}_* = h'$ where h'' is the holonomy homomorphism of M'.

So for each $h' \in \Omega$, we defined $M'(h')$ with development pair (D', h'') and a diffeomorphism $\tilde{\phi}_{h'}$ so that $h'' \circ \tilde{\phi}_{h'*} = h'$. In fact, we constructed a map $s' : \Omega \to C(M)$ where

$$s'(h') = (D', \tilde{\phi}_{h'} : \tilde{M} \to \tilde{M}'(h')).$$

We can show that this is a continuous function since we can make $\tilde{\phi}_{h'}$ used above depend continuously in C^r-topology on h' and $D' \circ \tilde{\phi}_{h'}$ restricted on $\tilde{M}_i$s is just $f_{h'}$s, and so $D' \circ \tilde{\phi}_{h'}$ restricted on an arbitrary compact subset of $\tilde{M}$ depends continuously on h' in C^r-topology. This proves the continuity of section s.

Now, we will show that

$$\mathcal{PH} : \mathcal{S}(M_0) \to \mathrm{Hom}(\pi_1(M_0), G)$$

is locally injective; i.e., for each $(D, \tilde{f} : \tilde{M}_0 \to \tilde{M})$ there is a neighborhood where $\mathcal{PH}$ is injective.

Again, we identify $\tilde{M}$ with $\tilde{M}_0$ by $\tilde{f}$. Let us give M a Riemannian metric as above with covering by neighborhoods modeled on (L_i, Γ_i), $i = 1, \dots, n$, in $\tilde{M}$ as above. We choose M_i, N_i as above in L_i.

We choose a neighborhood $\mathcal{O}$ of $(D, id : \tilde{M} \to \tilde{M})$ in $C(M)$ so that any two elements $(D_1, \tilde{f}_1 : \tilde{M} \to \tilde{X}_1)$ and $(D_2, \tilde{f}_2 : \tilde{M} \to \tilde{X}_2)$ satisfy that $D_1 \circ \tilde{f}_1$ is sufficiently C^r-close to $D_2 \circ \tilde{f}_2$ so that $D_1 \circ \tilde{f}_1(\mathrm{Cl}(M_i)) \subset D_2 \circ \tilde{f}_2(N_i) \subset D(L_i)$ and $D_2 \circ \tilde{f}_2(\mathrm{Cl}(M_i)) \subset D_1 \circ \tilde{f}_1(N_i) \subset D(L_i)$ and the corresponding holonomy homomorphisms $h_1, h_2 \in \Omega$ for Ω defined above. (We will add two more conditions on $\mathcal{O}$ making it smaller.)

Let $q : \mathcal{C}(M) \to \mathcal{S}(M)$ be the quotient map defined above. q is an open map since $\mathcal{S}(M)$ is the space of orbits in $\mathcal{C}(M)$ under the action of the group of isotopies of M.

We claim that on $q(\mathcal{O})$, which is a neighborhood of the equivalence class of $(D, id : M \to M)$ in $\mathcal{S}(M)$, $\mathcal{PH}|q(\mathcal{O})$ is injective.

This will prove Proposition 2.1 since $\mathcal{PH}|q(\mathcal{O})$ has an inverse map s restricted to the image in Ω. The image of $\mathcal{PH}|q(\mathcal{O})$ is open since that of $\mathcal{PPH}|\mathcal{O}$ is open. The latter image is open since for each point of its image, we can find a small neighborhood Ω' so that a section s' defined on Ω' has images in $\mathcal{O}$ as we can control the size of conjugating diffeomorphisms of model sets by the size of the holonomy perturbations (see Remark 3.1.) Thus, $\mathcal{PH}|q(\mathcal{O})$ is a homeomorphism to an open subset of $\mathrm{Hom}(\pi_1(M), G)$.

Given $(D_1, \tilde{f}_1 : \tilde{M} \to \tilde{X}_1)$ and $(D_2, \tilde{f}_2 : \tilde{M} \to \tilde{X}_2)$ in $\mathcal{O}$ with the holonomy homomorphisms $h_1 \circ \tilde{f}_1^*$ and $h_2 \circ \tilde{f}_2^*$ which are equal, we show that $(D_1, \tilde{f}_1)$ and $(D_2, \tilde{f}_2)$ are isotopy equivalent. We assumed that $D_1 \circ \tilde{f}_1(M_i) \subset D_2 \circ \tilde{f}_2(N_i)$ for each i. We start from $\tilde{f}_1(M_1)$, and lift the map $D_1|\tilde{f}_1(M_1)$ by D_2^{-1} to $\tilde{f}_2(N_1)$.

We identify $\pi_1(M)$ with $\pi_1(X_1)$ and $\pi_1(X_2)$ by $\tilde{f}_{1*}$ and $\tilde{f}_{2*}$. If $\gamma(M_j)$ meet M_1 for $\gamma \in \Gamma_{1j}$, then $D_1(\tilde{f}_1\gamma(M_j)) = h_1(\gamma)D_1(\tilde{f}_1(M_j)) \subset h_2(\gamma)D_2(\tilde{f}_2(L_j))$ since $h_1(\gamma) = h_2(\gamma)$. We lift $D_1|\tilde{f}_1\gamma(M_j)$ by $(D_2|\tilde{f}_2(\gamma(L_j)))^{-1}$ into $\tilde{f}_2(\gamma(N_j))$. By an induction in this manner, we see that we can lift an immersion $D_1 : \tilde{X}_1 \to X$ to an immersion $f_{12} : \tilde{X}_1 \to \tilde{X}_2$ by D_2 so that $D_2 \circ f_{12} = D_1$.

Since $h_1 = h_2$, considering $\tilde{X}_1$ and $\tilde{X}_2$ as quotient spaces of the sets of form $h_1(\gamma)D_1(\tilde{f}_1(M_j))$ and $h_2(\gamma)D_2(\tilde{f}_2(N_j))$, this map is also seen to be $\pi_1(M)$-equivariant, i.e., $f_{12} \circ \tilde{f}_1 \circ \gamma = \gamma \circ f_{12} \circ \tilde{f}_1$, $\gamma \in \pi_1(M)$; or in other words, $f_{12} \circ \tilde{f}_{1*}(\gamma) = \tilde{f}_{2*}(\gamma) \circ f_{12}$ for $\gamma \in \pi_1(M)$. Thus, $f_{12*} \circ f_{1*}(\gamma) = f_{2*}(\gamma)$.–(***)

We now show that $f_{12} \circ \tilde{f}_1$ is isotopic to $\tilde{f}_2$ by an isotopy $H : \tilde{M} \times I \to \tilde{X}_2$ equivariant with respect to the homomorphism $f_{2*} : \pi_1(M) \to \pi_1(X_2)$.

Let X_2 have the Riemannian metric pushed from f_2 with distance metric d_{X_2}. Then $\tilde{f}_2$ is an isometry. $f_{12} \circ \tilde{f}_1 : \tilde{M} \to \tilde{X}_2$ is a map so that

$$d_{X_2}(\tilde{f}_2(x), f_{12} \circ \tilde{f}_1(x)) \leq \epsilon \text{ for } x, y \in \mathrm{Cl}(M_i)$$

for some small $\epsilon > 0$.

We may choose our neighborhood $\mathcal{O}$ in the beginning so that ϵ may be chosen to be smaller that the radius of the normal neighborhoods chosen above. Thus, one can find a unique geodesic from $\tilde{f}_2(x)$ to $f_{12} \circ \tilde{f}_1(x)$ for each $x \in M$. For each point y of $\tilde{X}_2$, let v be a vector at $T_y\tilde{X}_2$ so that $\exp_y(v) = f_{12} \circ \tilde{f}_1 \circ \tilde{f}_2^{-1}(y)$. Since $f_{12} \circ \tilde{f}_1 \circ \tilde{f}_2^{-1}$ is a $\pi_1(X_2)$-equivariant diffeomorphism by (***), v is $\pi_1(X_2)$-invariant vector field.

If we choose $\mathcal{O}$ sufficiently near $(D, id : \tilde{M} \to \tilde{M})$, then v is very small so that the map $\tilde{f}_t : \tilde{X}_2 \to \tilde{X}_2$ defined by $f_t(x) = \exp_x(tv)$ are immersions for $t \in [0, 1]$. Moreover, since $\tilde{f}_2$ covers $f_2 : X_2 \to X_2$, which must be a diffeomorphism, $\tilde{f}_2$ is a diffeomorphism. Thus, we add this requirement to Ω.

Let us denote by $E : T(\tilde{X}_2) \to \tilde{X}_2 \times \tilde{X}_2$ the map given by sending (y, v) to $(y, \exp_y(v))$ for $y \in \tilde{X}_2$ and $v \in T_y(\tilde{X}_2)$. Then E is a differentiable map invertible near the diagonal $\triangle$ in $\tilde{X}_2 \times \tilde{X}_2$. Let us call E^{-1} the inverse in a neighborhood of $\triangle$. Since E^{-1} is smooth map, v is a smooth vector field on $\tilde{X}_2$.

Let us denote by $H(y, t)$ the point $\exp_y(tv)$ for $t \in [0, 1]$. Then H is a smooth function $\tilde{X}_2 \times I \to \tilde{X}_2$ so that $H(y, 0) = y$ and $H(y, 1) = f_{12} \circ \tilde{f}_1(y)$ for every $y \in \tilde{X}_2$. In fact $H(y, t)$, $t \in [0, 1]$, with y fixed is the flow line of a time-dependent vector field v_t defined by pushing v by a map $y \mapsto H(y, t)$. This implies that H is an isotopy. Moreover, H is $\pi_1(X_2)$-equivariant since so are v and v_t.

Clearly, $H(\tilde{f}_1(y), 0) = \tilde{f}_1(y)$ and $H(\tilde{f}_1(y), 1) = f_{12} \circ \tilde{f}_2(y)$ for $y \in \tilde{X}_2$. Thus, $\tilde{f}_1$ and $f_{12}\tilde{f}_2$ are isotopic, and $(D_1, \tilde{f}_1)$ and $(D_2, \tilde{f}_2)$ are isotopy-equivalent. □

4. Real projective structures on 2-orbifolds

We will now look at two-dimensional orbifolds with real projective structures, i.e., $(\mathbf{R}P^2, \mathrm{PGL}(3, \mathbf{R}))$-structure. The subspace $\mathcal{CP}(\Sigma)$ of the deformation space $\mathcal{D}(\Sigma)$ consists of the equivalence classes of convex real projective structures, i.e., Σ with the structure is projectively diffeomorphic to a quotient of a strictly convex domain in an affine patch by a properly discontinuous action of a group of projective automorphisms.

We recall that $\mathrm{PO}(1, 2)$ is a subgroup of $\mathrm{GL}(3, \mathbf{R})$ acting on the upper part of the hyperboloid given by $x_0^2 - x_1^2 - x_2^2 = 1$. Thus, we see that there is a natural isomorphic copy of it in $\mathrm{PGL}(3, \mathbf{R})$. $\mathrm{PO}(1, 2)$ acts on a standard circle in an affine patch of $\mathbf{R}P^2$; conversely, given a conic in $\mathbf{R}P^2$, the group of projective transformations acting on it is a conjugate of $\mathrm{PO}(1, 2)$.

The Teichmüller space $T(\Sigma)$ of real projective structures on Σ is a subset of $\mathcal{D}(\Sigma)$ of the equivalence classes of real projective structures on Σ so that Σ with the structure is projectively diffeomorphic to a quotient of the interior or a conic in $\mathbf{R}P^2$. One can show that this is homeomorphic to the Teichmüller space in the ordinary sense by a natural manner since the interior of a conic admits a hyperbolic metric where the projective automorphism acts as isometries. (Σ with these structures has a hyperbolic structure.) Clearly, $T(\Sigma)$ is a subset of $\mathcal{CP}(\Sigma)$.

To show the equivalence of topology of $T(\Sigma)$ to that of ordinary Teichmüller space as defined by Thurston, we simply note that the topology of the Teichmüller space can also be defined as a quotient space of the space of $(D, \tilde{f} : \tilde{\Sigma}_0 \to \tilde{\Sigma})$ where D is a developing map to the hyperbolic plane H^2 and f a diffeomorphism. The topology on the space of pairs is given by C^1-topology of $D \circ \tilde{f}$s. The quotient process is again exactly as in the projective case.

Since the holonomy group preserves a conic, we see that a holonomy homomorphism of a hyperbolic real projective structure can be conjugated so that $h : \pi_1(\Sigma) \to \mathrm{PO}(1,2)$ where $\mathrm{PO}(1,2)$ is a subgroup of linear automorphisms preserving the standard quadratic form of type $(-1,1,1)$. This follows since a conic can always be put to the boundary of a standard disk in an affine patch by a projective automorphism with holonomy homomorphism conjugated by the same automorphism.

The *pre-Teichmüller component* of

$$\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PGL}(3, \mathbf{R}))$$

is a component C_T of it which contains representations

$$\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PO}(1,2))$$

corresponding to holonomy homomorphisms of hyperbolic structures on Σ.

The group $\mathrm{PGL}(3, \mathbf{R})$ acts on

$$\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PGL}(3, \mathbf{R}))$$

by conjugation, i.e., $h(\cdot) \mapsto \vartheta h(\cdot)\vartheta^{-1}, \vartheta \in \mathrm{PGL}(3, \mathbf{R})$. Let

$$\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PGL}(3, \mathbf{R}))^{st}$$

be the subspace of representations r not fixing a point of $\mathbf{R}P^2$. Then Lemma 1.12 of Goldman [10] shows that $\mathrm{PGL}(3, \mathbf{R})$ acts properly on this subset.

Also, he shows (Lemma 2.5 of [10]):

Lemma 4.1. *Let Γ be a holonomy group of a compact real projective orbifold with negative Euler characteristic and empty boundary. Then Γ does not fix a point in $\mathbf{R}P^2$.*

Proof. Goldman proved for closed surfaces. But by taking a finite-index subgroup of the fundamental group of the orbifold, the lemma follows. □

Therefore, we see that

$$\mathcal{PH} : \mathcal{S}(\Sigma) \to \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PGL}(3, \mathbf{R}))^{st}$$

is defined, and is a local homeomorphism, an open map, and by conjugation action, we obtain a local homeomorphism

$$\mathcal{PH} : \mathcal{D}(\Sigma) \to \mathrm{Hom}(\pi_1(\Sigma, \mathrm{PGL}(3, \mathbf{R}))^{st}/\mathrm{PGL}(3, \mathbf{R}).$$

As a consequence $\mathcal{D}$ is a locally Hausdorff real analytic variety.

We will show that C_T is a component of the stable set above. PGL $(3, \mathbf{R})$ acts on C_T properly, and we call a component $C_T/\mathrm{PGL}(3, \mathbf{R})$ of

$$\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PGL}(3, \mathbf{R}))^{st}/\mathrm{PGL}(3, \mathbf{R})$$

a *Teichmüller component* of Σ following Hitchin.

Theorem 4.1. *Let Σ be a compact orbifold with negative Euler characteristic and empty boundary. Then the map*

$$\mathcal{H} : \mathcal{CP}(\Sigma) \to C_T/\mathrm{PGL}(3, \mathbf{R})$$

is a homeomorphism, where C_T is a subset of

$$\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PGL}(3, \mathbf{R}))^{st}.$$

Proposition 4.1. *Let Σ_1 and Σ_2 be compact 2-orbifolds of negative Euler characteristic without boundary where $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are homeomorphic to disks. If $k : \pi_1(\Sigma_1) \to \pi_1(\Sigma_2)$ is an isomorphism, we claim that there is a diffeomorphism $f : \Sigma_1 \to \Sigma_2$ so that $\tilde{f}_* = k$ for a lift $\tilde{f} : \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$.*

Proof. Since Σ_1 admits a hyperbolic structure, $\pi_1(\Sigma_1)$ is isomorphic to a discrete cocompact subgroup of $\mathrm{PSL}(2, \mathbf{R})$. There is a torsion-free finite-index normal subgroup Γ of $\pi_1(\Sigma)$ by Selberg's lemma. Let Γ' be $k(\Gamma)$ in $\pi_1(\Sigma_2)$. There is a finite covering surface Σ_1' of Σ_1 corresponding to Γ and Σ_2' of Σ_2 corresponding to Γ'.

The finite group $G_1 = \pi_1(\Sigma)/\Gamma$ maps into $\mathrm{Out}(\Gamma)/\mathrm{Inn}(\Gamma)$ where $\mathrm{Out}(\Gamma)$ is the group of outer-automorphisms of the surface group Γ and $\mathrm{Inn}(\Gamma)$ the group of inner-automorphisms of Γ. We note that k induces an isomorphism of $G_1 \to G_2$ and $\mathrm{Out}(\Gamma)/\mathrm{Inn}(\Gamma) \to \mathrm{Out}(\Gamma')/\mathrm{Inn}(\Gamma')$ so that the following diagram is commutative:

$$\begin{array}{ccc} G_1 & \longrightarrow & \mathrm{Out}(\Gamma)/\mathrm{Inn}(\Gamma) \\ \downarrow & & \downarrow \\ G_2 & \longrightarrow & \mathrm{Out}(\Gamma')/\mathrm{Inn}(\Gamma'). \end{array} \tag{11}$$

Since $\mathrm{Out}(\Gamma)/\mathrm{Inn}(\Gamma)$ is isomorphic to the mapping class group $MC(\Sigma_1')$ of Σ_1', and $\mathrm{Out}(\Gamma')/\mathrm{Inn}(\Gamma')$ to $MC(\Sigma_2')$ of Σ_2', G_1 acts on Σ_1 and G_2 on Σ_2. A homeomorphism $f' : \Sigma_1' \to \Sigma_2'$ realizes $k|\Gamma$ and satisfies $f' \circ g$ for $g \in G_1$ is homotopic to $k(g) \circ f'$ for $k(g)$ an element of G_2 corresponding to g under k.

Give Σ_1' and Σ_2' arbitrary hyperbolic metrics which are G_1- and G_2-invariant respectively (i.e., using Thurston's orbifold hyperbolization of Σ_1 and Σ_2). Then choose a unique harmonic diffeomorphism $\hat{f} : \Sigma_1' \to \Sigma_2'$ in the homotopy class of f' as proved by Jost-Schoen (see (5.10) of Eells-Lemaire [7]). Then $\hat{f} \circ g$ and $k(g) \circ \hat{f}$ are harmonic and in the same homotopy class. By uniqueness, we obtain $\hat{f} \circ g = k(g) \circ \hat{f}$. Thus $\hat{f}$ induces an orbifold diffeomorphism $f : \Sigma_1 \to \Sigma_2$. Clearly, a lift $\tilde{f} : \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$ induces k since $\tilde{f}$ induces $k|\Gamma$ and $\hat{f} \circ g = k(g) \circ \hat{f}$ for $g \in G$. ◻

Proof of Theorem 4.1. Let $\mathcal{C}'(\Sigma)$ the subset of $\mathcal{S}(\Sigma)$ consisting of convex structures on Σ, and let $\mathcal{T}'(\Sigma)$ the subset of $\mathcal{S}(\Sigma)$ consisting of hyperbolic real projective structures on Σ. Then $\mathcal{T}'(\Sigma)$ is obviously a connected subset, and we see that $\mathcal{C}'(\Sigma)$ is also connected as $\mathcal{CP}(\Sigma)$ is homeomorphic to cells (see [6]).

We have

$$\mathcal{PH} : \mathcal{T}'(\Sigma) \to \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PGL}(3, \mathbf{R}))$$

maps into C_T. By connectedness, $\mathcal{PH}$ sends $\mathcal{C}'(\Sigma)$ into C_T as well. By Proposition 3.3 of Goldman [10], $\mathcal{C}'(\Sigma)$ is an open subset of $\mathcal{S}(\Sigma)$ (using the result of Koszul [14]). To see this one needs to cover Σ be a surface and see the openness for the deformation space of surfaces implies the same for that of orbifolds, which is a subspace. Therefore the image $\mathcal{PH}(\mathcal{C}'(S))$ is an open subset of C_T.

We show that the image is a closed subset also. Clearly, C_T is a Hausdorff space being realized as a real analytic subvariety of $\mathrm{PGL}(3, \mathbf{R})^m$ for m the number of generators of $\pi_1(M)$. Let $g_1, \ldots, g_m$ denote the generators of $\pi_1(\Sigma)$.

Choose a sequence of representations

$$h_i : \pi_1(\Sigma) \rightarrow \mathrm{PGL}(3, \mathbf{R})$$

so that

$$(h_i(g_1), \ldots, h_i(g_m)) \rightarrow (h(g_1), \ldots, h(g_m))$$

for a representation $h : \pi_1(M) \rightarrow \mathrm{PGL}(3, \mathbf{R})$; i.e., h_i converges to h algebraically. Assume that h_i is in the image and we show that h is in the image proving the closedness.

Let D_i be a developing map of Σ_i associated with h_i. Then $D_i : \tilde{\Sigma}_i \rightarrow \mathbf{R}P^2$ maps $\tilde{\Sigma}_i$ to a strictly convex domain Ω_i in an affine patch of $\mathbf{R}P^2$. (This follows since $\tilde{\Sigma}_i$ is a universal cover of a convex real projective surface covering Σ_i finitely. (See [10].) $D_i : \tilde{\Sigma} \rightarrow \Omega_i$ induces a real projective diffeomorphism $\Sigma_i \rightarrow \Omega_i/h_i(\pi_1(\Sigma_i))$.

Since the sphere $\mathbf{S}^2$ covers $\mathbf{R}P^2$ so that $p_{\mathbf{R}P^2} : \mathbf{S}^2 \rightarrow \mathbf{R}P^2$ is a projective map, and the group of projective automorphisms $\mathrm{Aut}(\mathbf{S}^2)$ of $\mathbf{S}^2$ is isomorphic to $\mathrm{SL}_{\pm}(3, \mathbf{R})$.

We can show that $D_i : \tilde{\Sigma} \rightarrow \mathbf{R}P^2$ always lift to an imbedding $D'_i : \tilde{\Sigma}_i \rightarrow \mathbf{S}^2$ and h_i lifts to a homomorphism $h'_i : \pi_1(\Sigma) \rightarrow \mathrm{Aut}(\mathbf{S}^2)$. (See [3]): we can lift first, and for a deck-transformation ϑ of $\pi_1(\Sigma)$, $D'_i \circ \vartheta$ is another developing map, and hence it must equal $\varphi \circ D'_i$ for $\varphi \in \mathrm{Aut}(\mathbf{S}^2)$. Letting $h'_i(\vartheta) = \varphi$, we see that h'_i is a lift of h_i.

The image Ω'_i of $D'_i(\tilde{\Sigma})$ is a convex open subset of an open hemisphere in $\mathbf{S}^2$ with standard geodesic structure.

By choosing a subsequence, the sequence of the closures $\mathrm{Cl}(\Omega'_i)$ converges to a compact convex subset of $\mathbf{S}^2$ in a closed hemisphere (Choi-Goldman [5]). We claim that the limit Ω'_∞ is not a point, a line segment, a lune, or the closed hemisphere. If not, by taking a finite subcover Σ' of Σ, we see that $D_i(\tilde{\Sigma})$ are also images of a sequence of developing images of convex real projective structures on Σ'. We showed that such degeneration cannot happen in [5]. (See also [4].) Thus, Ω'_∞ is a compact convex subset of an open hemisphere in $\mathbf{S}^2$.

By choosing a subsequence, we can show that h'_i converges to a representation $h' : \pi_1(\Sigma) \rightarrow \mathrm{Aut}(\mathbf{S}^2)$ lifting h. We also see as in [5] that $h'(\pi_1(\Sigma))$ acts on Ω'_∞. As before, since h' is a map to $\mathrm{SL}_{\pm}(3, \mathbf{R})$, h' is discrete and faithful by Lemma 1.1 of Goldman-Millson [11]. ($\pi_1(\Sigma)$ has a finite index subgroup which is torsion-free. Apply Lemma 1.1 of [11] here and the finite index extension argument is trivial.)

Therefore, $h'(\pi_1(\Sigma))$ acts on an open disk Ω'^o_∞ to obtain an orbifold Σ'. By above Proposition 4.1, we see that Σ' is homeomorphic to Σ by a homeomorphism inducing h'. Since $p_{\mathbf{R}P^2}|\Omega'^o_\infty$ is an imbedding onto a convex open domain Ω in $\mathbf{R}P^2$ with $h(\pi_1(\Sigma))$ acting on it, we see that Σ' is realized also as the quotient space of Ω by $h(\pi_1(\Sigma))$. Thus, h is realized as a holonomy homomorphism of a convex real projective structure on Σ. Thus, the image of $\mathcal{PH}(\mathcal{CP}(\Sigma))$ is closed in C_T and hence the image equals C_T.

The holonomy group of a convex real projective orbifold is discrete since it acts on an open domain discontinuously. Thus, C_T consists of discrete faithful representations.

We also see that

$$C_T \subset \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PGL}(3, \mathbf{R})))^{st}$$

as they consist of holonomy representations of convex real projective structures on Σ.

Also, $\mathcal{PH}|\mathcal{C}'$ is injective: That is, we are given two holonomy representations h and h' for $(D, \tilde{f} : \tilde{\Sigma} \to \tilde{M})$ and $(D', \tilde{f}' : \tilde{\Sigma} \to \tilde{M}')$ for projective orbifolds M and M'. That is $h = h_1 \circ \tilde{f}_*$ and $h' = h'_1 \circ \tilde{f}'_*$ for holonomy homomorphisms of h_1 and h'_1 of M and M' respectively.

Let $\hat{\Sigma}$ be a closed surface covering Σ finitely. Let Ω be the image of D composed with $\tilde{f}$ and Ω' that of D' composed with $\tilde{f}'$. Since h and h' restricted to $\pi_1(\hat{\Sigma})$ are the same, Proposition 3.4 of [10] shows $\Omega = \Omega'$. The images of $D \circ \tilde{f}$ and $D' \circ \tilde{f}'$ are the same, and they are both equivariant under the homomorphism $h = h' : \pi_1(\Sigma) \to \mathrm{PGL}(3, \mathbf{R})$. The map $g = D'^{,-1} \circ D : \tilde{M}_1 \to \tilde{M}_2$ is so that $D' \circ g = D$.

Let Σ have a hyperbolic metric μ. Then $\tilde{M}_1$ and $\tilde{M}_2$ has induced ones by $\tilde{f}_1$ and $\tilde{f}_2$. Then we now show that $g \circ \tilde{f}$ and $\tilde{f}'$ are isotopic. Then M_2 have induced hyperbolic metrics μ_0 and μ_1 induced from $g \circ \tilde{f}$ and $\tilde{f}'$ respectively. There is a path of Riemannian metrics $\mu_t = t\mu_1 + (1-t)\mu_0$ for $t \in [0, 1]$ from μ_1 to μ_2. Let Σ' be the cover of Σ which is a surface and let M'_2 denote the corresponding ones. Let μ'_t denote the Riemannian metrics of M'_2 corresponding to μ_t. By Theorem B.26 of Tromba [20], there exists a smooth parameter of harmonic diffeomorphisms $S'(\mu_t) : (M'_2, \mu_t) \to (\Sigma', \mu)$. Clearly, $S'(\mu_t)$ descends to a parameter of diffeomorphisms $S(\mu_t) : (M_2, \mu_t) \to (\Sigma, \mu)$. By uniqueness of harmonic diffeomorphisms, we see that the inverse map $S^{-1}(\mu_0)$ lifts to $g \circ \tilde{f}_1 : \tilde{\Sigma}_1 \to \tilde{M}_2$, and $S^{-1}(\mu_1)$ lifts to $\tilde{f}_2 : \tilde{\Sigma}_2 \to \tilde{M}_2$. One can lift $S^{-1}(\mu_t)$ to a smooth parameter diffeomorphisms $\tilde{S}(\mu_t) : \tilde{\Sigma}_2 \to \tilde{M}_2$ using analytic continuations. This gives us an isotopy between $g \circ \tilde{f}$ and $\tilde{f}'$; $(D, \tilde{f})$ and $(D', \tilde{f}')$ are equivalent.

Therefore,

$$\mathcal{PH} : \mathcal{C}'(\Sigma) \to C_T$$

is a homeomorphism and its quotient

$$\mathcal{H} : \mathcal{CP}(\Sigma) \to C_T/\mathrm{PGL}(3, \mathbf{R})$$

is a homeomorphism. □

Remark 4.1. We use the harmonic map arguments since topological proofs of the facts that a homotopy equivalence of 2-orbifolds is homotopic to a diffeomorphism, and the homotopy of two diffeomorphisms of 2-orbifolds is homotopic to an isotopy are missing from the mathematical literature. We conjecture that there are topological proofs of these facts.

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