Playing pool on curved surfaces and the wrong way to add fractions

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Mathematics: a *MOST* exact science

- Natural phenomena understood through quantitative measurements
  - Which are abstracted into mathematics.
  - These abstract ideas can be manipulated rigorously to make predictions.
  - Mathematical statements form a language in which measurements can be processed.
  - Mathematics represents an *ideal* situation which approximates the everyday world.
- For example:
  - Rates of change governed by laws of calculus.
  - Force = Mass · Acceleration.

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Billiards on a square

- A billiard ball starts moving once it is subjected to the initial force, and changes direction when it bounces off the side of a billiard table.
    - Here is an example of a billiard ball on a square billiard table, which follows a *periodic* path.
    - Here is a longer periodic path. When the slope is rational (a fraction of two whole numbers), the path is periodic.
    - When the slope is irrational, the path never closes up, and eventually fills the whole square.
- Example of the inter-relationship between seemingly *different* subjects of mathematics: arithmetic (number theory), and differential equations (mechanics).

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Looking for universal patterns

- The same kind of differential equations that govern the motion of a moving ball can govern population growth, financial markets, chemical reactions...
  - Because they exhibit *similar patterns*.
- Mathematics is *scalable:*
  - What's true in the small is true in the large.
- Mathematics is *reproducible:*
  - Governed only by abstract logic,
  - And does not need special equipment, just working conditions conducive for clear thinking.

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Language: striving for intellectual conciseness

- Promote recurring patterns into primitive concepts.
  - Break complicated relationships into simpler ones.
  - Consolidating definitions creates new concepts.
- Sometimes finding the right *question* is just as important as finding the right *answer!*
- Asking and answering questions about the simpler concepts *creates* new mathematics.
  - And it keeps on going...
  - And *growing*.
- More mathematics created in the last 50 years than before.
- Challenge: How can you learn enough of what has already been done to create new mathematics?

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Art: beauty in the simplicity of ideas

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The Golden Ratio

- The Parthenon is in the proportion of the *Golden Ratio:*

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803398874989484820458683436563811772030917980$$

- which also appears in the geometry of a seashell.

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A fraction which continues...

- $\phi \approx 1.618\ldots$ satisfies the algebraic equation

$$\phi = 1 + \frac{1}{\phi}$$

- Replacing ϕ by $1 + \frac{1}{\phi}$ in this expression:

$$\phi = 1 + \frac{1}{\phi} = 1 + \frac{1}{1 + \frac{1}{\phi}} = 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{\phi}}}$$

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What does this infinite fraction mean?

- This infinite expression is meaningless until we give it meaning!
  - Mathematicians change the questions to fit the answers!
- For example, define it to be the *limit* of the sequence $1,\ 1+\frac{1}{1}=2,\ 1+\frac{1}{2}=\frac{3}{2},\ 1+\frac{1}{3/2}=\frac{5}{3},\ 1+\frac{1}{5/3}=\frac{8}{5},\ 1+\frac{1}{8/5}=\frac{13}{8},\dots$
- Numerators and denominators are *Fibonacci numbers:*

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- Notice a pattern in the sequence of fractions approximating ϕ:

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}, \dots$$

- Each fraction is obtained from the preceding pair by *adding numerators and denominators:*

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$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}, \dots$$

- Each fraction is obtained from the preceding pair by *adding numerators and denominators:*

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a + c}{b + d}.$$

$$\frac{1}{1}, \frac{2}{1} \oplus \frac{3}{2} = \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}, \dots$$

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- $n = 6$:

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- $n = 1$:

$$\frac{0}{1}, \frac{1}{1}, \frac{2}{1}$$

- $n = 6$:

$$\frac{0}{1}, \frac{1}{5}, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1}, \frac{7}{6}, \frac{6}{5}, \frac{5}{4}, \frac{4}{3}, \frac{7}{5}, \frac{3}{2}, \frac{8}{5}, \frac{5}{3}, \frac{7}{4}, \frac{9}{5}, \frac{11}{6}, \frac{2}{1}$$

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$$\frac{0}{1}, \frac{1}{2}, \frac{1}{1}, \frac{3}{2}, \frac{2}{1}$$

- $n = 6$:

$$\frac{0}{1}, \frac{1}{5}, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1}, \frac{7}{6}, \frac{6}{5}, \frac{5}{4}, \frac{4}{3}, \frac{7}{5}, \frac{3}{2}, \frac{8}{5}, \frac{5}{3}, \frac{7}{4}, \frac{9}{5}, \frac{11}{6}, \frac{2}{1}$$

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- $n = 3$:

$$\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, \frac{2}{1}$$

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- $n = 4$:

$$\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}, \frac{5}{4}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \frac{2}{1}$$

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- $n = 5$:

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How a mathematical concept is created

- A pattern is isolated.
  - Focus on its essential qualitites.
- Promote it to a new concept
  - Give it a *definition.*
- Relate it to already defined concepts through *theorems,*
  - which must be *rigorously* proved!
    - The *right* definitions may make the theorems much easier to prove
- Similar to art: a human representation of an abstract pattern.

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Challenges to doing mathematics

Its unique nature leads to basic challenges in its teaching, communication, and dissemination, unlike any other intellectual discipline.

A remarkably *successful* discipline

- Mathematics goes back thousands of years, and ...
  - continues to grow.
- Old mathematics is *not* discarded ...
  - but *condensed*.
- Leading to challenges in disseminating, organizing, teaching ...
- As more common relationships are discovered, ideas *generalize* ...
  - and the subject becomes more and more abstract ...
    - *And specialized.*

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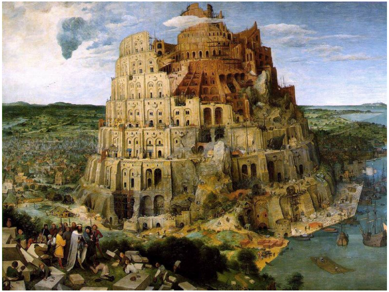
Going out of control?

- Too many subdivisions...
  - Despite basic unity, a natural tendency to splinter.
- Specialization must be controlled and resisted as the subject develops.
- Last 30 years: remarkable confluence of mathematical ideas.
  - Making it *even harder* to learn!



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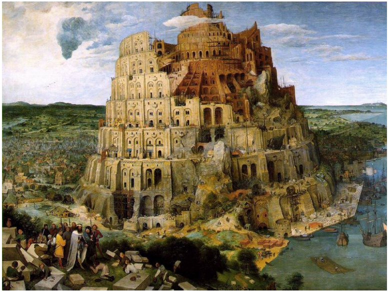

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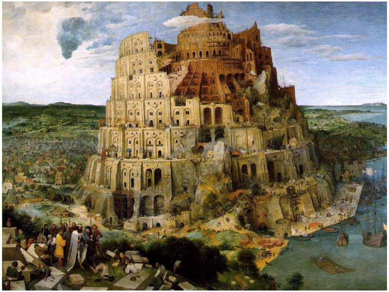

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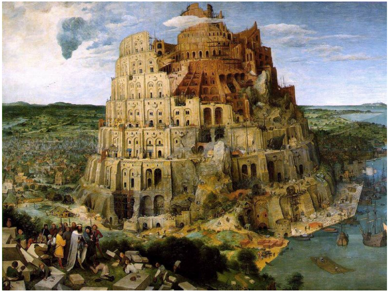

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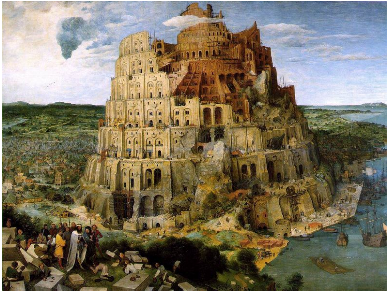

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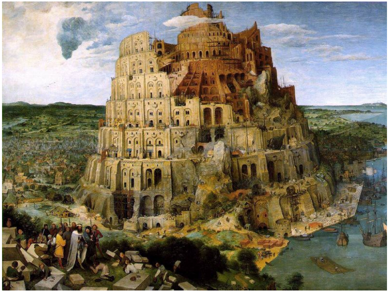

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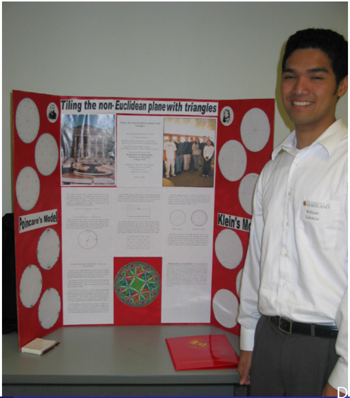
Investing in mathematics is investing in *people!*

- The speakers of a specialized language...
  - Are the audience ...
  - And the practitioners...
  - And the developers ...
  - And the first users.
- Build a community of technically literate and creative *people.*



Investing in mathematics is investing in *people!*

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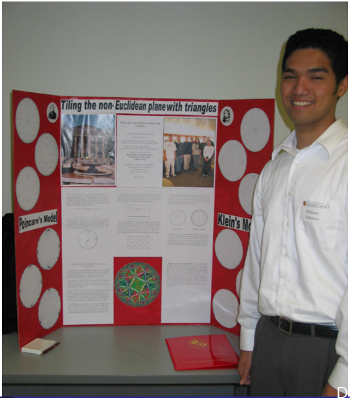

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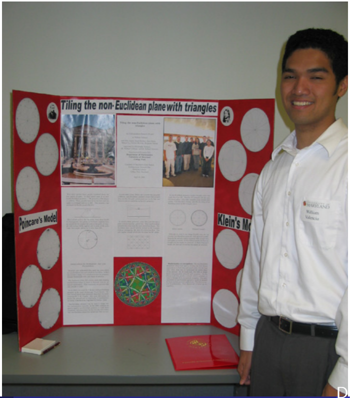

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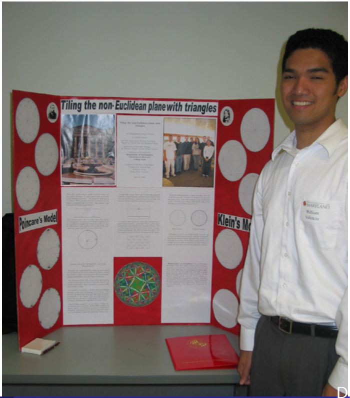

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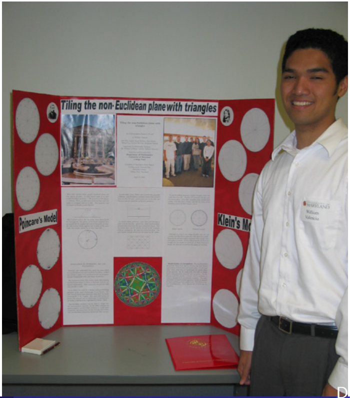

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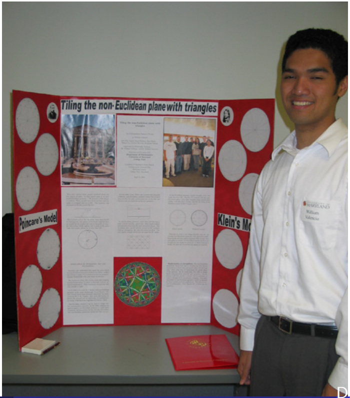

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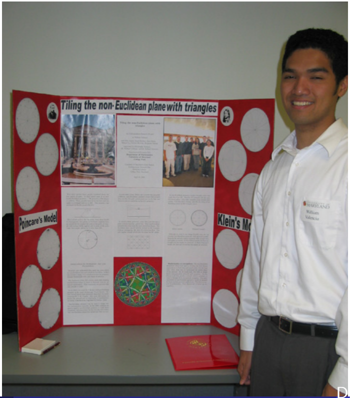
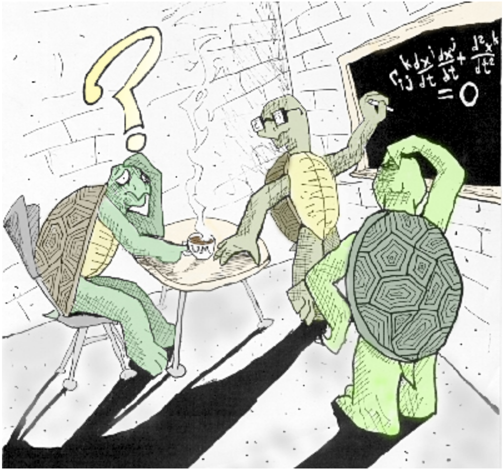


Mathematics:A fundamentally *human* activity.



Terrapins work out the equations of straight lines on curved surfaces.

Building communities to promote mathematics

Potomac High School students visit the Experimental Geometry Lab.

Why support mathematics?

- A rapidly changing society needs people who can:
  - Learn and work with abstract ideas,
  - *Communicate* them effectively
    - ... all in a short period of time...

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A community activity

Summary

Mathematics:

- A Science: a rigorous exact discipline which formulates statements modeling natural phenomena.
- A Language: a collection of ideas, represented symbolically and organized into units of communication.
- An art: an esthetic activity, characterized by elegance and simplicity, despite its innate complexity.

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- These three roles complement each other in a unique way.
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Playing pool on curved surfaces...