AN EXPOSITION OF RESULTS OF FRICKE AND VOGT

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Contents

The purpose of this paper is to give an elementary and relatively self-contained proof of the following basic theorem, apparently due to Fricke [6, 7] and, earlier by Vogt [17] Let $G = \mathsf{SL}(2,\mathbb{C})$ and let G act on $H = \mathsf{SL}(2,\mathbb{C}) \times \mathsf{SL}(2,\mathbb{C})$ by

$$g \cdot (\xi, \eta) \longmapsto (g\xi g^{-1}, g\eta g^{-1}).$$

Let $\kappa(x, y, z) = x^2 + y^2 + z^2 - xyz - 2$.

Theorem (Vogt [17],Fricke [6])**.** *Let $f : H \longrightarrow \mathbb{C}$ be a regular function which is invariant under the above G-action. Then there exists a polynomial function $F(x, y, z) \in \mathbb{C}[x, y, z]$ such that*

$$f(\xi, \eta) = F(\mathsf{tr}(\xi), \mathsf{tr}(\eta), \mathsf{tr}(\xi\eta)).$$

For every $(x, y, z) \in \mathbb{C}^3$, there exists $(\xi, \eta) \in H$ such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \mathsf{tr}(\xi) \\ \mathsf{tr}(\eta) \\ \mathsf{tr}(\xi\eta) \end{bmatrix}.$$

Conversely, if $\kappa(x, y, z) \neq 2$ and $(\xi, \eta), (\xi', \eta') \in H$ satisfy

$$\begin{bmatrix} \mathsf{tr}(\xi) \\ \mathsf{tr}(\eta) \\ \mathsf{tr}(\xi\eta) \end{bmatrix} = \begin{bmatrix} \mathsf{tr}(\xi') \\ \mathsf{tr}(\eta') \\ \mathsf{tr}(\xi'\eta') \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{1}$$

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then $(\xi', \eta') = g.(\xi, \eta)$ *for some* $g \in G$.

We regard H as a *variety of representations* of a free group. Let π be the group freely generated by elements X, Y, which we call *letters.* We call X, Y, X^{-1}, Y^{-1} *symbols.* An element of π is a *reduced word* $w(X, Y)$, that is a sequence of symbols such that no symbol follows its inverse. The length of a reduced word $w(X, Y)$ is denoted $\ell(w)$. The empty reduced word has length 0 and is the identity element $\mathbb{I} \in \pi$. Reduced words are multiplied by concatenation subject to cancellation relations of the form

$$XX^{-1} = X^{-1}X = YY^{-1} = Y^{-1}Y = \mathbb{I}.$$

The Cartesian product H identifies with the set $\mathsf{Hom}(\pi, G)$ of all representations $\pi \longrightarrow G$ under the correspondence

$$\begin{aligned} \mathsf{Hom}(\pi, G) &\longrightarrow H = G \times G \\ \rho &\longmapsto (\rho(X), \rho(Y)). \end{aligned}$$

Since π is generated by X, Y subject to no relations, this map is an isomorphism. Then the Vogt-Fricke theorem is the statement that the categorical quotient of $\mathsf{Hom}(\pi, G)$ by G is affine 3-space $\mathbb{C}^3$.

This result applies to non-Euclidean geometry. Let M be a compact surface homeomorphic to a three-holed sphere. The space of hyperbolic- or elliptic- geometric structures on M is then parametrized by the lengths of the boundary curves, which relate to the traces of the corresponding representation of the fundamental group. See Buser [3], Fenchel [5], Keen [10, 11, 12], and Harvey [8]. In [12], Keen uses Fricke moduli to parametrize the Teichmüller space of a one-holed torus and other surfaces.

1. Traces in $\mathsf{SL}(2, \mathbb{C})$

1.1. **Cyclic groups.** The analogous result for one generator is the following.

Theorem 1. *Let* $f : G \longrightarrow \mathbb{C}$ *be a polynomial function invariant under inner automorphisms of* G. *Then there exists a polynomial* $F(t) \in \mathbb{C}[t]$ *such that* $f(g) = F(\mathsf{tr}(g))$. *Conversely, if* $g, g' \in G$ *satisfy* $\mathsf{tr}(g) = \mathsf{tr}(g') \neq \pm 2$, *then there exists* $h \in G$ *such that* $g' = hgh^{-1}$.

Proof. Suppose f is an invariant function. For $t \in \mathbb{C}$, define

$$\xi_t := \begin{bmatrix} t & -1 \\ 1 & 0 \end{bmatrix}$$

and define $F(t)$ by

$$F(t) = f(\xi_t).$$

Suppose that $t \neq \pm 2$ and $\mathsf{tr}(g) = t$. Then both g and ξ_t have distinct eigenvalues

$$\lambda_\pm = \frac{1}{2}\left(t \pm (t^2-4)^{1/2}\right)$$

and $hgh^{-1} = \xi_t$ for some $h \in G$. Thus

$$f(g) = f(h^{-1}\xi_t h) = f(\xi_t) = F(t)$$

as desired. If $t = \pm 2$, then by taking Jordan normal form, either $g = \pm\mathbb{I}$ or g is conjugate to ξ_t. In the latter case, $f(g) = F(t)$ follows from invariance. Otherwise g lies in the closure of the G-orbit of ξ_t and $f(g) = f(\xi_t) = F(t)$ follows by continuity of f.

The converse direction follows from Jordan normal form as already used above. □

1.2. **The Cayley-Hamilton theorem.** A principal tool in this investigation is the *Cayley-Hamilton theorem.* If ξ is a 2×2-matrix,

$$\xi^2 - \mathsf{tr}(\xi)\xi + \mathsf{det}(\xi)\mathbb{I} = 0. \tag{2}$$

Suppose $\xi, \eta \in \mathsf{SL}(2,\mathbb{C})$. Multiplying (2) by ξ^{-1} and rearranging,

$$\xi + \xi^{-1} = \mathsf{tr}(\xi)\mathbb{I} \tag{3}$$

from which follows

$$\mathsf{tr}(\xi) = \mathsf{tr}(\xi^{-1}) \tag{4}$$

Multiplying (3) by η and taking traces, we obtain:

Theorem 2 (The Basic Identity)**.** *Let* $\xi, \eta \in \mathsf{SL}(2)$*. Then*

$$\mathsf{tr}(\xi\eta) + \mathsf{tr}(\xi\eta^{-1}) = \mathsf{tr}(\xi)\mathsf{tr}(\eta). \tag{5}$$

1.3. **Traces of reduced words.** Here is an important special case of the Vogt-Fricke theorem. Namely, let $w(X,Y) \in \pi$ be a reduced word. Then

$$\begin{aligned} H &\longrightarrow \mathbb{C} \\ (\xi,\eta) &\longmapsto \mathsf{tr}w(\xi,\eta) \end{aligned}$$

is a G-invariant function on H and by the Vogt-Fricke theorem there exists a polynomial $f_w(x,y,z) \in \mathbb{C}[x,y,z]$ such that

$$\mathsf{tr}w(\xi,\eta) = f_w(\mathsf{tr}\xi, \mathsf{tr}\eta, \mathsf{tr}(\xi\eta)) \tag{6}$$

for all $\xi, \eta \in G$. We describe an algorithm for computing $f_w(x,y,z)$. For notational convenience we shall write $t(w(X,Y))$ for

$$f_{w(X,Y)}(x,y,z).$$

For example, $t(\mathbb{I}) = 2$ and

$$\begin{aligned} t(X^{-1}) &= t(X) = x, \\ t(Y^{-1}) &= t(Y) = y, \\ t(Z^{-1}) &= t(Z) = z, \end{aligned}$$

verifying assertion (6) for words w of length $\ell(w) \leq 1$. The reduced words of length two are

$$X^2, Y^2, XY, XY^{-1}, YX, YX^{-1}$$

and applications of the trace identities imply:

$$\begin{aligned} t(X^2) &= x^2 - 2 \\ t(Y^2) &= y^2 - 2 \\ t(Z^2) &= z^2 - 2 \\ t(XY^{-1}) &= xy - z \\ t(YZ^{-1}) &= yz - x \\ t(ZX^{-1}) &= zx - y \end{aligned}$$

For example, for X^2,

$$\mathsf{tr}(\xi^2) = \mathsf{tr}(\xi\xi) = \mathsf{tr}(\xi)\mathsf{tr}(\xi) - \mathsf{tr}(\xi\xi^{-1}) = \mathsf{tr}(\xi)^2 - 2.$$

and, for XY^{-1},

$$\mathsf{tr}(\xi\eta^{-1}) = \mathsf{tr}(\xi)\mathsf{tr}(\eta) - \mathsf{tr}(\xi\eta),$$

etc. For example

$$\begin{aligned} t(XYX^{-1}Y) &= t(XY)t(X^{-1}Y) - t(X^2) \\ &= z(xy - z) - (x^2 - 2) \\ &= 2 - x^2 - z^2 + xyz. \end{aligned} \tag{7}$$

Assume inductively that for all reduced words $w(X, Y) \in \pi$ with $\ell(w) < l$, there exists a polynomial $f_w(x, y, z) = t(w(X, Y))$ satisfying (6). Suppose that $u(X, Y) \in \pi$ is a reduced word of length $\ell(u) = l$.

The explicit calculations above begin the induction for $l \leq 2$. Thus we assume $l > 2$.

Furthermore, we can assume that u is *cyclically reduced,* that is the initial symbol of u is not inverse to the terminal symbol of u. For otherwise

$$u(X, Y) = Su'(X, Y)S^{-1},$$

where S is one of the four symbols

$$X, Y, X^{-1}, Y^{-1}$$

and $\ell(u') = l - 2$. Then $u(X,Y)$ and $u'(X,Y)$ are conjugate and

$$t(u(X,Y)) = t(u'(X,Y)).$$

If $l > 2$, and u is cyclically reduced, then $u(X,Y)$ has a repeated letter, which we may assume to equal X. That is, we may write

$$u(X,Y) = u_1(X,Y)u_2(X,Y)$$

where u_1 and u_2 are reduced words each ending in $X^{\pm 1}$. Furthermore we may assume that

$$\ell(u_1) + \ell(u_2) = \ell(u) = l,$$

so that $\ell(u_1) < l$ and $\ell(u_l) < l$. Suppose first that u_1 and u_2 both end in X. Then

$$u(X,Y) = \big(u_1(X,Y)X^{-1}\big)\, X \;\; \big(u_2(X,Y)X^{-1}\big)\, X$$

and each

$$u_1(X,Y)X^{-1}, u_2(X,Y)X^{-1}$$

is reduced. Then

$$(u_1(X,Y)X^{-1})(u_2(X,Y)X^{-1})^{-1} = u_1(X,Y)u_2(X,Y)^{-1}$$

is represented by a reduced word $u_3(X,Y)$ satisfying $\ell(u_3) < l$. By the induction hypothesis, there exist polynomials

$$t(u_1(X,Y)), t(u_2(X,Y)), t(u_3(X,Y)) \in \mathbb{C}[x,y,z]$$

such that, for all $\xi, \eta \in G$, $i = 1,2,3$,

$$\mathsf{tr}\big(u_i(\xi,\eta)\big) = t\big(u_i(X,Y))(\mathsf{tr}(\xi), \mathsf{tr}(\eta), \mathsf{tr}(\eta)\big).$$

By (5),

$$t\big(u(X,Y)\big) = t(u_1(X,Y))t(u_2(X,Y)) - t(u_3(X,Y))$$

is a polynomial in $\mathbb{C}[x,y,z]$. The cases when u_1 and u_2 both end in the symbols X^{-1}, Y, Y^{-1} are completely analogous. Since there are only four symbols, the only cyclically reduced words without repeated symbols are commutators of the symbols, for example $XYX^{-1}Y^{-1}$. Repeated applications of the trace identities give:

$$\begin{aligned} t(XYX^{-1}Y^{-1}) &= t(XYX^{-1})t(Y^{-1}) - t(XYX^{-1}Y) \\ &= y^2 - (2 - x^2 - z^2 + xyz) = \kappa(x,y,z). \end{aligned} \tag{8}$$

The other commutators of distinct symbols also have trace $\kappa(x,y,z)$ by identical arguments.

Thus every $w(X,Y)$ determines a polynomial $f_w(x,y,z)$ such that
$$\mathsf{tr} w(\xi,\eta) = f_w(\mathsf{tr}(\xi), \mathsf{tr}(\eta), \mathsf{tr}(\xi\eta))$$
for $\xi, \eta \in \mathsf{SL}(2,\mathbb{C})$.

2. Surjectivity of characters of pairs

We first show that
$$\tau : H \longrightarrow \mathbb{C}^3$$
$$(\xi,\eta) \longmapsto \begin{bmatrix} \mathsf{tr}\xi \\ \mathsf{tr}\eta \\ \mathsf{tr}\xi\eta \end{bmatrix}.$$
is surjective. Choose $\zeta \in \mathbb{C}$ so that
$$\zeta + \zeta^{-1} = z,$$
that is, $\zeta = \frac{1}{2}(z \pm \sqrt{z^2-4})$. Let
$$\xi_x = \begin{bmatrix} x & -1 \\ 1 & 0 \end{bmatrix}, \ \eta_{(y,\zeta)} = \begin{bmatrix} 0 & \zeta^{-1} \\ \zeta & y \end{bmatrix}. \tag{9}$$
Then $\tau(\xi_x, \eta_{(y,\zeta)}) = (x,y,z)$.

Next we show that every G-invariant regular function $f : H \longrightarrow \mathbb{C}$ factors through τ. To this end we need the following elementary lemma on symmetric functions:

Lemma 3. *Let R be an integral domain where 2 is invertible, and let $R' = R[\zeta,\zeta^{-1}]$ be the ring of Laurent polynomials over R. Let $R' \xrightarrow{\sigma} R'$ be the involution which fixes R' and interchanges ζ and ζ^{-1}. Then the subring of σ-invariants is the polynomial ring $R[\zeta+\zeta^{-1}]$.*

Proof. Let $F(\zeta,\zeta^{-1}) \in R[\zeta,\zeta^{-1}]$ be a σ-invariant Laurent polynomial. Begin by rewriting R' as the quotient of the polynomial ring $R[x,y]$ by the ideal generated by $xy-1$. Then σ is induced by the involution $\tilde{\sigma}$ of $R[x,y]$ interchanging x and y. Let $f(x,y) \in R[x,y]$ be a polynomial whose image in R' is F. Then there exists a polynomial $g(x,y)$ such that
$$f(x,y) - f(y,x) = g(x,y)(xy-1).$$
Clearly $g(x,y) = -g(y,x)$. Let
$$\tilde{f}(x,y) = f(x,y) - \frac{1}{2}\, g(x,y)(xy-1)$$
so that $\tilde{f}(x,y) = \tilde{f}(y,x)$. By the theorem on elementary symmetric functions,
$$\tilde{f}(x,y) = h(x+y, xy)$$

for some polynomial $h(u,v)$. Therefore $F(\zeta,\zeta^{-1}) = h(\zeta+\zeta^{-1},1)$ as desired. □

By definition $f(\xi,\eta)$ is a polynomial in the matrix entries of ξ and η; two polynomials which differ by elements in the ideal generated by $\mathsf{det}(\xi)-1$ and $\mathsf{det}(\eta)-1$ are regarded as equal. Thus $f(\xi_x,\eta_{(y,\zeta)})$ equals a function $g(x,y,\zeta)$ which is a polynomial in $x,y\in\mathbb{C}$ and a Laurent polynomial in $\zeta\in\mathbb{C}^*$.

Lemma 4. *Let $\xi,\eta\in G$ such that $\kappa(\tau(\xi,\eta))\neq 2$. Then there exists $g\in G$ such that*

$$g\cdot(\xi,\eta) = (\xi^{-1},\eta^{-1}).$$

Proof. Let $(x,y,z)=\tau(\xi,\eta)$. By a simple calculation

$$\mathsf{tr}[\xi,\eta] = \kappa(x,y,z)$$

where $[\xi,\eta]=\xi\eta\xi^{-1}\eta^{-1}$.

Let $L=\xi\eta-\eta\xi$. (Compare §4 of Jørgensen [9] or Fenchel [5].) Then $\mathsf{tr}(L)=\mathsf{tr}(\xi\eta)-\mathsf{tr}(\eta\xi)=0$. Furthermore for any 2×2 matrix M, the characteristic polynomial

$$\lambda_M(t) := \mathsf{det}(t\mathbb{I}-M) = t^2-\mathsf{tr}(M)t+\mathsf{det}(M).$$

Thus

$$\begin{aligned}\mathsf{det}(L) &= \mathsf{det}([\xi,\eta]-I)\mathsf{det}(\eta\xi)\\ &= \mathsf{det}([\xi,\eta]-I)\\ &= -\lambda_{[\xi,\eta]}(1)\\ &= -2+\mathsf{tr}[\xi,\eta]\\ &= -2+\kappa(x,y,z)\neq 0.\end{aligned}$$

Choose $\mu\in\mathbb{C}^*$ such that $\mu^2\mathsf{det}(L)=1$ and let $g=\mu L\in G$.

Since $\mathsf{tr}(g)=0$ and $\mathsf{det}(g)=1$, the Cayley-Hamilton Theorem $\lambda_M(M)=0$ implies that $g^2=-\mathbb{I}$. Similarly

$$\begin{aligned}\mathsf{det}(g\xi) &= \mathsf{det}(g)=1\\ \mathsf{det}(g\eta) &= \mathsf{det}(g)=1,\end{aligned}$$

and

$$\begin{aligned}\mathsf{tr}(g\xi) &= \mu(\mathsf{tr}((\xi\eta)\xi)-\mathsf{tr}((\eta\xi)\xi))\\ &= \mu(\mathsf{tr}(\xi(\eta\xi))-\mathsf{tr}((\eta\xi)\xi))=0\end{aligned}$$

and

$$\begin{aligned}\mathsf{tr}(g\eta) &= \mu(\mathsf{tr}((\xi\eta)\eta)-\mathsf{tr}((\eta\xi)\eta))\\ &= \mu(\mathsf{tr}((\xi\eta)\eta)-\mathsf{tr}(\eta(\xi\eta))=0\end{aligned}$$

so $(g\xi)^2 = (g\eta)^2 = -\mathbb{I}$. Thus $g\xi g^{-1}\xi = -g\xi g\xi = \mathbb{I}$ whence $g\xi g^{-1} = \xi^{-1}$. Similarly $g\eta g^{-1} = \eta^{-1}$, concluding the proof of the lemma. □

Apply Lemma 4 to $\xi = \xi_x$ and $\eta = \eta_{(y,\zeta)}$ as above to obtain g such that conjugation by g maps

$$\xi \longmapsto \xi^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & x \end{bmatrix}$$

and

$$\eta \longmapsto \eta^{-1} = \begin{bmatrix} y & -1/\zeta \\ \zeta & 0 \end{bmatrix}.$$

Let

$$h = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Then

$$hg\xi(hg)^{-1} = h\xi^{-1}h^{-1} = \begin{bmatrix} x & -1 \\ 1 & 0 \end{bmatrix} = \xi$$

and

$$hg\eta(hg)^{-1} = h\eta^{-1}h^{-1} = \begin{bmatrix} 0 & \zeta \\ -1/\zeta & y \end{bmatrix}.$$

Thus

$$\begin{aligned} g(x,y,\zeta) &= f(\xi,\eta) \\ &= f(hg\xi(hg)^{-1}, hg\eta(hg)^{-1}) \\ &= g(x,y,\zeta^{-1}). \end{aligned}$$

Lemma 3 implies that there exists a polynomial $F \in \mathbb{C}[x,y,z]$ such that

$$g(x,y,\zeta) = F(x,y,\zeta + 1/\zeta) \tag{10}$$

whenever $\kappa(x,y,\zeta+1/\zeta) \neq 2$. Since this condition defines a nonempty Zariski-dense open set, (10) holds on all of $\mathbb{C}^2 \times \mathbb{C}^*$ and $f(\xi,\eta) = F(\mathsf{tr}(\xi), \mathsf{tr}(\eta), \mathsf{tr}(\xi\eta))$ as claimed.

3. Injectivity of characters of pairs

Finally we show that if $(\xi,\eta), (\xi',\eta') \in H$ satisfy (1) and $\kappa(x,y,z) \neq 2$, then (ξ,η) and (ξ',η') are G-equivalent. By §1.3, the triple

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \mathsf{tr}\xi \\ \mathsf{tr}\eta \\ \mathsf{tr}(\xi\eta) \end{bmatrix}$$

determines the character function

$$\begin{aligned}\chi : \pi &\longrightarrow \mathbb{C}\\ w(X,Y) &\longmapsto \mathsf{tr} w(\xi,\eta) = f_w(x,y,z).\end{aligned}$$

Let ρ and ρ' denote the representations $\pi \longrightarrow G$ taking X, Y to ξ, η and ξ', η' respectively and let χ, χ' denote their respective characters. Then our hypothesis (1) implies that $\chi = \chi'$.

Lemma 5. *Let $\rho : \pi \longrightarrow \mathsf{SL}(2,\mathbb{C})$ be a representation and (x, y, z) be as above. Then ρ is irreducible if and only if $\kappa(x, y, z) \neq 2$.*

Proof. Suppose first that ρ is reducible. If X, Y generate a representation with an invariant subspace of $\mathbb{C}^2$ of dimension one, this representation is conjugate to one in which $\rho(X)$ and $\rho(Y)$ are upper-triangular. Denoting their diagonal entries by a, a^{-1} and b, b^{-1} respectively, the diagonal entries of $\rho(XY)$ are $ab^{\pm 1}, a^{-1}b^{\mp 1}$. Thus

$$\begin{aligned}x &= a + a^{-1},\\ y &= b + b^{-1},\\ z &= ab^{\pm 1} + a^{-1}b^{\mp 1}.\end{aligned}$$

By direct computation, $\kappa(x, y, z) = 2$.

Conversely, suppose that $\kappa(x, y, z) = 2$. Let $\mathfrak{A} \subset M_2(\mathbb{C})$ denote the linear span of $\mathbb{I}, \xi, \eta, \xi\eta$. Identities derived from the Cayley-Halmilton theorem (2) such as (3) imply that $\mathfrak{A}$ is a subalgebra of $M_2(\mathbb{C})$. For example, ξ^2 is the linear combination $-1 + x\xi$ and

$$\eta\xi = (z - xy)1 + y\xi - x\eta - \xi\eta.$$

In the basis of $M_2(\mathbb{C})$ by elementary matrices, the map

$$\begin{aligned}\mathbb{C}^4 &\longrightarrow M_2(\mathbb{C})\\ \begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} &\longmapsto x_1\mathbb{I} + x_2\xi + x_3\eta + x_4\xi\eta\end{aligned}$$

has determinant $2 - \kappa(x, y, z) = 0$ and is not surjective. Thus $\mathfrak{A}$ is a proper subalgebra of $M_2(\mathbb{C})$ and the representation is reducible. $\square$

Thus ρ and ρ' are irreducible representations on $\mathbb{C}^2$. Burnside's Theorem (see Lang [13], p.445) implies the corresponding representations (also denoted ρ, ρ' respectively) of the group algebra $\mathbb{C}\pi$ into $M_2(\mathbb{C})$

are surjective. Since the trace form

$$\begin{aligned} M_2(\mathbb{C}) \times M_2(\mathbb{C}) &\longrightarrow \mathbb{C} \\ (A,B) &\longmapsto \mathsf{tr}(AB) \end{aligned}$$

is nondegenerate, the kernel K of $\rho : \mathbb{C}\pi \longrightarrow M_2(\mathbb{C})$ consists of all

$$\sum_{\alpha\in\pi} a_\alpha \alpha \in \mathbb{C}\pi$$

such that

$$\begin{aligned} 0 &= \mathsf{tr}\Bigg(\Big(\sum_{\alpha\in\pi} a_\alpha \rho(\alpha) \Big) \rho(\beta) \Bigg) \\ &= \sum_{\alpha\in\pi} a_\alpha \mathsf{tr}\rho(\alpha\beta) \\ &= \sum_{\alpha\in\pi} a_\alpha \chi_{\alpha\beta} \end{aligned}$$

for all $\beta \in \pi$. Thus the kernels of both representations of $\mathbb{C}\pi$ are equal, and ρ and ρ' induce algebra isomorphisms

$$\tilde\rho, \tilde\rho' : \mathbb{C}\pi/K \longrightarrow M_2(\mathbb{C})$$

respectively. The composition $\tilde\rho' \circ \rho^{-1}$ is an automorphism of the algebra $M_2(\mathbb{C})$, which must be induced by conjugation by $g \in \mathsf{GL}(2,\mathbb{C})$. (See, for example, Corollary 9.122, p.734 of Rotman [16].) In particular $\rho'(\gamma) = g\rho(\gamma)g^{-1}$ as desired.

4. Trace relations for triples

Finally, we consider the case of triples $(\xi, \eta, \zeta) \in G$, or equivalently representations of the free group of rank three. Unlike the rank two case, the quotient is no longer an affine space. Rather, the coordinate ring of the quotient has dimension six, but generated by the eight functions

$$f_X, f_Y, f_Z, f_{XY}, f_{YZ}, f_{ZX}, f_{XYZ}, f_{XZY}$$

subject to the two relations expressing the sum and product of traces of the length 3 monomials in terms of traces of monomials of length 1 and 2:

$$\begin{aligned}
f_{XYZ} + f_{XZY} &= f_{XY} f_Z + f_{YZ} f_X + f_{ZX} f_Y + f_X f_Y f_Z \\
f_{XYZ}\, f_{XZY} &= (f_X)^2 + (f_Y)^2 + (f_Z)^2 \\
&\quad + f_{XY}^2 + f_{YZ}^2 + f_{ZX}^2 \\
&\quad\quad - (f_X f_Y f_{XY} + f_Y f_Z f_{YZ} + f_Z f_X f_{ZX}) \\
&\quad\quad\quad + f_{XY} f_{YZ} f_{ZX} - 4
\end{aligned}$$

We call these two identities Fricke's Sum and Product Relations respectively.

Fricke's Sum Relation.

$$f_{XYZ} + f_{XZY} = f_{XY} f_Z + f_{YZ} f_X + f_{ZX} f_Y + f_X f_Y f_Z \tag{11}$$

To prove this formula, apply the Basic Identity three times:

$$t(XYZ) + t(XYZ^{-1}) = t(XY)t(Z) \tag{12}$$

$$\begin{aligned}
t(Z^{-1}XY) + t(Z^{-1}XY^{-1}) &= t(Z^{-1}X)t(Y) \\
&= (t(Z)t(X) - t(ZX))t(Y)
\end{aligned} \tag{13}$$

$$\begin{aligned}
t(Y^{-1}Z^{-1}X) + t(Y^{-1}Z^{-1}X^{-1}) &= t(Y^{-1}Z^{-1})t(X) \\
&= t(YZ)t(X)
\end{aligned} \tag{14}$$

Now subtract (13) from the sum of (12) and (14), using the facts

$$\begin{aligned}
t(XYZ^{-1}) &= t(Z^{-1}XY) \\
t(Z^{-1}XY^{-1}) &= t(Y^{-1}Z^{-1}X) \\
t(Z^{-1}X^{-1}Y^{-1}) &= t(Y^{-1}Z^{-1}X^{-1}) = t(XZY)
\end{aligned}$$

to obtain

$$\begin{aligned}
t(XYZ) + t(XZY) &= t(XY)t(Z) \\
&\quad - t(Z)t(X)t(Y) + t(ZX)t(Y) \\
&\quad\quad + t(YZ)t(X)
\end{aligned}$$

from which (11) follows.

Fricke's Product Relation.

$$\begin{aligned}
f_{XYZ}\, f_{XZY} &= (f_X)^2 + (f_Y)^2 + (f_Z)^2 \\
&\quad + f_{XY}^2 + f_{YZ}^2 + f_{ZX}^2 \\
&\quad\quad - (f_X f_Y f_{XY} + f_Y f_Z f_{YZ} + f_Z f_X f_{ZX}) \\
&\quad\quad\quad + f_{XY} f_{YZ} f_{ZX} - 4
\end{aligned} \tag{15}$$

We derive this formula in several steps. A direct application of the Basic Identity (5) yields:

$$
\begin{aligned}
t(ZXZY) &= t(ZX)t(ZY) - t(XY^{-1}) \\
&= t(ZX)t(ZY) - (t(X)t(Y) - t(XY)) \\
&= t(ZX)t(ZY) - t(X)t(Y) + t(XY)
\end{aligned} \tag{16}
$$

By (7) applied to X, Z^{-1}:

$$
t(XZ^{-1}X^{-1}Z^{-1}) = t(X)t(Z)t(ZX) - t(ZX)^2 - t(X)^2 + 2 \tag{17}
$$

$$
\begin{aligned}
t(XYZXZY) &= t(XY)t(ZXZY) \\
&\qquad - t(XZ^{-1}X^{-1}Z^{-1}) \\
&= t(XY)\left(t(ZX)t(ZY) - t(X)t(Y) + t(XY)\right) \\
&\qquad - \big(t(X)t(Z)t(ZX) - t(ZX)^2 - t(X)^2 + 2\big) \\
&\qquad\qquad \text{(by (16) and (17))} \\
&= t(XY)t(ZX)t(YZ) \\
&\qquad - t(X)t(Y)t(XY) - t(Z)t(X)t(ZX) \\
&\qquad + t(XY)^2 + t(X)^2 - 2
\end{aligned} \tag{18}
$$

Finally, appplying (18) and the Commutator Identity (8) to Y, Z:

$$
\begin{aligned}
t(XYZ)t(XZY) &= t(XYZXZY) + t(YZY^{-1}Z^{-1}) \\
&= \Big(t(XY)t(ZX)t(YZ) - t(X)t(Y)t(XY) \\
&\qquad - t(Z)t(X)t(ZX) + t(XY)^2 + t(ZX)^2 + t(X)^2 - 2\Big) \\
&\qquad + \Big(t(Y)^2 + t(Z)^2 + t(YZ)^2 - t(Y)t(Z)t(YZ) - 2\Big)
\end{aligned}
$$

from which (15) follows.

4.1. **The coordinate ring is a quadratic extension.** If π is freely generated by X, Y, Z, then the ring of G-invariant polynomials on $\mathsf{Hom}(\pi, G)$ is a quadratic extension of the polynomial ring

$$
\mathbb{C}[f_X, f_Y, f_Z, f_{XY}, f_{YZ}, f_{ZX}].
$$

The algebraic generator λ (which corresponds to f_{XYZ} or f_{XZY}) satisfies the quadratic equation

$$
\lambda^2 - f_\Sigma \lambda + f_\Pi = 0
$$

where

$$
f_\Sigma = f_{XY} f_Z + f_{YZ} f_X + f_{ZX} f_Y - f_X f_Y f_Z
$$

and

$$\begin{aligned} f_\Pi = &(f_X^2 + f_Y^2 + f_Z^2) \\ &+ (f_{XY}^2 + f_{YZ}^2 + f_{ZX}^2) \\ &- (f_X f_Y f_{XY} + f_Y f_Z f_{YZ} + f_Z f_X f_{ZX}) \\ &+ f_{XY} f_{YZ} f_{ZX} - 4 \end{aligned}$$

(Compare Magnus [15].)

5. Surjectivity in rank 3

Let V be the $\mathsf{SL}(2)$-character variety of the (rank three) free group $\pi = \langle A_1, A_2, A_3\rangle$. In trace coordinates

$$t_I([\rho]) := \mathsf{tr}\big(\rho(A_{i_1} A_{i_2} \dots A_{i_k})\big)$$

where

$$I = (i_1, i_2, \dots i_k),$$

$V \subset \mathbb{C}^8$ is the codimension two subvariety defined by the two equations:

$$t_{123} + t_{132} = t_{12}t_3 + t_{13}t_2 + t_{23}t_1 + t_1t_2t_3 \tag{19}$$

$$\begin{aligned} t_{123}\ t_{132} = &(t_1^2 + t_2^2 + t_3^2) + (t_{12}^2 + t_{23}^2 + t_{13}^2) + \\ &(t_1t_2t_{12} + t_2t_3t_{23} + t_3t_1t_{13}) + t_{12}t_{23}t_{13} - 4. \end{aligned} \tag{20}$$

By eliminating t_{132} in (19) as

$$t_{132} = t_{12}t_3 + t_{13}t_2 + t_{23}t_1 + t_1t_2t_3 - t_{123},$$

the variety V may be expressed as the hypersurface in $\mathbb{C}^7$ consisting of all

$$\big(t_1, t_2, t_3, t_{12}, t_{23}, t_{13}\big) \in \mathbb{C}^7$$

satisfying

$$\begin{aligned} &t_{123}\ \big(t_{12}t_3 + t_{13}t_2 + t_{23}t_1 + t_1t_2t_3 - t_{123}\big) = \\ &\quad (t_1^2 + t_2^2 + t_3^2) + (t_{12}^2 + t_{23}^2 + t_{13}^2) + (t_1t_2t_{12} + t_2t_3t_{23} + t_3t_1t_{13}) + t_{12}t_{23}t_{13} - 4. \end{aligned}$$

Proposition 6. *The projection*

$$V \xrightarrow{\mathsf{t}} \mathbb{C}^6$$

$$[\rho] \longmapsto \begin{bmatrix} t_1(\rho) \\ t_2(\rho) \\ t_3(\rho) \\ t_{12}(\rho) \\ t_{23}(\rho) \\ t_{13}(\rho) \end{bmatrix}$$

is surjective.

The polynomial ring

$$\mathbb{C}[t] := \mathbb{C}[t_1, t_2, t_3, t_{12}, t_{23}, t_{13}]$$

is the coordinate ring of affine space $\mathbb{C}^6$. Projection $V \xrightarrow{\mathsf{t}} \mathbb{C}^6$ is a 2-1 map corresponding to the quadratic extension

$$\mathbb{C}[V] \;\cong\; \mathbb{C}[t][z] \Big/ \Big(z^2 - P(t)z + Q(t) \Big)$$

where P, Q are the polynomials appearing above:

$$\begin{aligned} P(t) &= t_{12}t_3 + t_{13}t_2 + t_{23}t_1 + t_1t_2t_3 \\ Q(t) &= (t_1^2 + t_2^2 + t_3^2) + (t_{12}^2 + t_{23}^2 + t_{13}^2) + \\ &\qquad (t_1t_2t_{12} + t_2t_3t_{23} + t_3t_1t_{13}) + t_{12}t_{23}t_{13} - 4. \end{aligned}$$

Proof. By the Vogt-Fricke theorem in rank two, there exist $A_1, A_2 \in \mathsf{SL}(2)$ such that

$$\begin{aligned} \mathsf{tr}(A_1) &= t_1, \\ \mathsf{tr}(A_2) &= t_2, \\ \mathsf{tr}(A_1A_2) &= t_{12}. \end{aligned} \tag{21}$$

We seek $A_3 \in \mathsf{SL}(2)$ such that

$$\begin{aligned} \mathsf{tr}(A_3) &= t_3, \\ \mathsf{tr}(A_2A_3) &= t_{23}, \\ \mathsf{tr}(A_1A_3) &= t_{13}. \end{aligned} \tag{22}$$

To this end, consider the affine subspace $\mathcal{W}$ of $\mathsf{M}_2(\mathbb{C})$ consisting of matrices W satisfying

$$\begin{aligned} \mathsf{tr}(W) &= t_3, \\ \mathsf{tr}(A_2W) &= t_{23}, \\ \mathsf{tr}(A_1W) &= t_{13}. \end{aligned} \tag{23}$$

Since the bilinear pairing

$$\begin{aligned} \mathsf{M}_2(\mathbb{C}) \times \mathsf{M}_2(\mathbb{C}) &\longrightarrow \mathbb{C} \\ (X, Y) &\longmapsto \mathsf{tr}(XY) \end{aligned}$$

is nondegenerate, each of the three equations in (23) describes an affine hyperplane in $\mathsf{M}_2(\mathbb{C})$. We first suppose that $\kappa(t_1, t_2, t_{12}) \neq 2$, that is,

$$4 - t_1^2 - t_2^2 - t_{12}^2 + t_1t_2t_{12} \neq 0. \tag{24}$$

Lemma 7. *There exists $A_1, A_2 \in \mathsf{SL}(2)$ satisfying (21) such that $\mathcal{W}$ is nonempty.*

Proof. Since $\kappa(t_1, t_2, t_{12}) \neq 2$. the pair A_1, A_2 generates an irreducible representation. In particular $\{\mathbb{I}, A_1, A_2\}$ is a linearly independent subset of the 4-dimensional vector space $\mathsf{M}_2(\mathbb{C})$. The three conditions of (23) are independent, so $\mathcal{W} \subset \mathsf{M}_2(\mathbb{C})$ is an affine line. □

Let $W_0, W_1 \in \mathcal{W}$ be distinct elements in this line. Then the function function

$$\begin{aligned} \mathbb{C} &\longrightarrow \mathbb{C} \\ s &\longmapsto \mathsf{det}\big(sW_1 + (1-s)W_0\big) \end{aligned}$$

is polynomial of degree ≤ 2, and is thus onto unless it is constant.

Lemma 8. *Let* $W_0, W_1 \in \mathsf{M}_2(\mathbb{C})$. *Then*

$$\begin{aligned} &\mathsf{det}\big(W_0 + s(W_1 - W_0)\big) = \\ &\qquad \mathsf{det}(W_0) \ + \ s\ \big(\mathsf{tr}(W_0)\mathsf{tr}(W_1 - W_0) - \mathsf{tr}\big(W_0(W_1 - W_0)\big)\big) \\ &\qquad\qquad + \ s^2\, \mathsf{det}(W_1 - W_0) \end{aligned}$$

Proof. Clearly

$$\mathsf{tr}\big(W_0 \ + \ s\,(W_1 - W_0)\big) = \mathsf{tr}(W_0) \ + \ s\,\mathsf{tr}(W_1 - W_0). \tag{25}$$

Now

$$\mathsf{det}(W) = \frac{\mathsf{tr}(W)^2 - \mathsf{tr}(W^2)}{2} \tag{26}$$

whenever $W \in \mathsf{M}_2(\mathbb{C})$. Now apply (26) to (25) taking

$$W = W_0 + s(W_1 - W_0)$$

□

Thus the restriction $\mathsf{det}|_{\mathcal{W}}$ is constant only if $\mathsf{det}(W_1 - W_0) = 0$.

Work in the slice

$$A_1 = \begin{bmatrix} t_1 & -1 \\ 1 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & \xi \\ -\xi^{-1} & t_2 \end{bmatrix},$$

where $\xi + \xi^{-1} = t_{12}$.

The matrix $W_0 \in \mathsf{SL}(2)$ defined by:

$$W_0 = \begin{bmatrix} t_3 & \big((t_{13} - t_1t_3)\xi + t_{23}\big)\xi/(\xi^2 - 1) \\ \big((t_{13} - t_1t_3) + t_{23}\xi\big)/(\xi^2 - 1) & 0 \end{bmatrix},$$

satisfies (23). Any other $W \in \mathcal{W}$ must satisfy

$$\begin{aligned} \mathsf{tr}\big(W - W_0\big) &= 0, \\ \mathsf{tr}\big(A_2(W - W_0)\big) &= 0, \\ \mathsf{tr}\big(A_1(W - W_0)\big) &= 0. \end{aligned} \tag{27}$$

Lemma 9. *Any solution $W - W_0$ of (27) is a multiple of*

$$[A_1, A_2] = \begin{bmatrix} \xi^{-1} - \xi & -t_2 + t_1\xi \\ -t_2 + \xi^{-1}t_1 & \xi - \xi^{-1} \end{bmatrix}.$$

Proof. The first equation in (27) asserts that $W - W_0$ lies in the subspace $\mathsf{SL}(2, \mathbb{C})$, upon which the trace form is nondegenerate. The second and third equations assert that $W - W_0$ is orthogonal to A_1 and A_2. By (24), A_1, A_2 and $\mathbb{I}$ are linearly independent in $\mathsf{M}_2(\mathbb{C})$, so the solutions of (27) form a one-dimensional linear subspace. The Lie product

$$[A_1, A_2] = A_1A_2 - A_2A_1$$

is nonzero, lies in $\mathsf{SL}(2, \mathbb{C})$, and since

$$\mathsf{tr}(A_iA_1A_2) = \mathsf{tr}(A_iA_2A_1)$$

for $i = 1, 2$, orthogonal to A_1 and A_2. The lemma follows. $\square$

Parametrize $\mathcal{W}$ explicitly as

$$W = W_0 + s[A_1, A_2].$$

Since

$$\mathsf{det}([A_1, A_2) = 4 - (t_1^2 + t_2^2 + t_{12}^2 - t_1t_2t_{12}) = 2 - \kappa(t_1, t_2, t_{12}) \neq 0,$$

Thus the polynomial

$$\mathcal{W} \xrightarrow{\mathsf{det}} \mathbb{C}$$

is nonconstant, and hence onto. Taking $W_1 \in \mathsf{det}_{\mathcal{W}}^{-1}(1)$, the proof is complete assuming (24).

Now consider the case when $4 - t_1^2 - t_2^2 - t_{12}^2 + t_1t_2t_{12} = 0$. In that case there exist $a_1, a_2 \in \mathbb{C}^*$ such that

$$t_i = a_i + (a_i)^{-1}$$

for $i = 1, 2$. Then either

$$t_{12} = a_1a_2 + (a_1a_2)^{-1} \tag{28}$$

or

$$t_{12} = a_1(a_2)^{-1} + (a_1)^{-1}a_2. \tag{29}$$

In the first case (28), set

$$A_1 := \begin{bmatrix} a_1 & t_{13} - a_1t_3 \\ 0 & (a_1)^{-1} \end{bmatrix},$$

$$A_2 := \begin{bmatrix} a_2 & t_{23} - a_2t_3 \\ 0 & (a_2)^{-1} \end{bmatrix},$$

$$A_3 := \begin{bmatrix} t_3 & -1 \\ 1 & 0 \end{bmatrix}$$

and in the second case (28), set

$$A_1 := \begin{bmatrix} (a_1)^{-1} & t_{13} - (a_1)^{-1} t_3 \\ 0 & a_1 \end{bmatrix},$$
$$A_2 := \begin{bmatrix} a_2 & t_{23} - a_2 t_3 \\ 0 & (a_2)^{-1} \end{bmatrix},$$
$$A_3 := \begin{bmatrix} t_3 & -1 \\ 1 & 0 \end{bmatrix}$$

obtaining $(A_1, A_2, A_3) \in \mathsf{SL}(2)$ explicitly solving (21) and (22). This completes the proof of Proposition 6.. $\square$

Writing

$$W = a_0 I + a_1 A_1 + a_2 A_2$$

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