DYNAMICS OF THE AUTOMORPHISM GROUP OF THE $GL(2,\mathbb{R})$-CHARACTERS OF A ONCE-PUNCUTRED TORUS

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Abstract. Let π be a free group of rank 2. Its outer automorphism group $\mathrm{Out}(\pi)$ acts on the space of equivalence classes of representations $\rho \in \mathrm{Hom}(\pi, \mathrm{SL}(2,\mathbb{C}))$. Let $\mathrm{SL}_-(2,\mathbb{R})$ denote the subset of $\mathrm{GL}(2,\mathbb{R})$ consisting of matrices of determinant -1 and let $\mathrm{ISL}(2,\mathbb{R})$ denote the subgroup $\mathrm{SL}(2,\mathbb{R}) \amalg i\,\mathrm{SL}_-(2,\mathbb{R}) \subset \mathrm{SL}(2,\mathbb{C})$. The representation space $\mathrm{Hom}(\pi, \mathrm{ISL}(2,\mathbb{R}))$ has four connected components, three of which consist of representations that send at least one generator of π to $i\,\mathrm{SL}_-(2,\mathbb{R})$. We investigate the dynamics of the $\mathrm{Out}(\pi)$-action on these components.

The group $\mathrm{Out}(\pi)$ is commensurable with the group Γ of automorphisms of the polynomial

$$\kappa(x,y,z) = -x^2 - y^2 + z^2 + xyz - 2$$

We show that for $-14 < c < 2$, the action of Γ is ergodic on $\kappa^{-1}(c)$. For $c < -14$, the group Γ acts properly and freely on an open subset $\Omega_c^M \subset \kappa^{-1}(c)$ and acts ergodically on the complement of Ω_c^M. We construct an algorithm which determines, in polynomial time, if a point $(x,y,z) \in \mathbb{R}^3$ is Γ-equivalent to a point in Ω_c^M or in its complement.

Conjugacy classes of $\mathrm{ISL}(2,\mathbb{R})$-representations identify with $\mathbb{R}^3$ via the appropriate restriction of the character map

$$\chi : \mathrm{Hom}(\pi, \mathrm{SL}(2,\mathbb{C})) \longrightarrow \mathbb{C}^3$$
$$\rho \longmapsto \begin{bmatrix} \xi(\rho) \\ \eta(\rho) \\ \zeta(\rho) \end{bmatrix} = \begin{bmatrix} \mathrm{tr}(\rho(X)) \\ \mathrm{tr}(\rho(Y)) \\ \mathrm{tr}(\rho(XY)) \end{bmatrix}$$

where X and Y are the generators of π. Corresponding to the Fricke spaces of the once-punctures Klein bottle and the once-punctured Möbius band are Γ-invariant open subsets Ω^K and Ω^M respectively. We give an explicit parametrization of Ω^K and Ω^M as subsets of $\mathbb{R}^3$ and we show that $\Omega^M \cap \kappa^{-1}(c) \neq \varnothing$ if and only if $c < -14$, while $\Omega^K \cap \kappa^{-1}(c) \neq \varnothing$ if and only if $c > 6$.

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1. Introduction

Let π be a free group of rank 2. Its outer automorphism group $\mathrm{Out}(\pi)$ acts on the space of equivalence classes of representations $\rho \in \mathrm{Hom}(\pi, \mathrm{SL}(2,\mathbb{C}))$. Let $\mathrm{SL}_-(2,\mathbb{R})$ denote the subset of $\mathrm{GL}(2,\mathbb{R})$ consisting of matrices of determinant -1, and let

$$\mathrm{SL}_\pm(2,\mathbb{R}) = \{A \in \mathrm{GL}(2,\mathbb{R}) \mid \det(A) = \pm 1\}$$

The group $\mathrm{SL}_\pm(2,\mathbb{R})$ is isomorphic to

$$\mathrm{ISL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R}) \amalg i\,\mathrm{SL}_-(2,\mathbb{R})$$

and in this context we identify the two as subgroups of $\mathrm{SL}(2,\mathbb{C})$. The representation space $\mathcal{R} = \mathrm{Hom}(\pi, \mathrm{ISL}(2,\mathbb{R}))$ has four connected components indexed by the elements of $\mathrm{H}^1(\pi, \mathbb{Z}_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. The three non-zero elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ correspond to the components of $\mathcal{R}$ consisting of representations that send at least one generator of π to $i\,\mathrm{SL}_-(2,\mathbb{R})$. We investigate the dynamics of the $\mathrm{Out}(\pi)$-action on these components. The action of $\mathrm{Out}(\pi)$ on the component of $\mathrm{SL}(2,\mathbb{R})$-representations has been recently studied by Goldman [8]

By a theorem of Fricke [4], the moduli space of $\mathrm{SL}(2,\mathbb{C})$-representations naturally identifies with affine 3-space $\mathbb{C}^3$ via the character map

$$\begin{aligned} \chi :\ \mathrm{Hom}(\pi, \mathrm{SL}(2,\mathbb{C})) &\longrightarrow \mathbb{C}^3 \\ \rho &\longmapsto \begin{bmatrix} \xi(\rho) \\ \eta(\rho) \\ \zeta(\rho) \end{bmatrix} = \begin{bmatrix} \mathrm{tr}(\rho(X)) \\ \mathrm{tr}(\rho(Y)) \\ \mathrm{tr}(\rho(XY)) \end{bmatrix} \end{aligned}$$

where X and Y are the generators of π. Let $[X,Y]$ be the commutator of X and Y. In terms of the coordinate functions ξ, η and ζ, the trace $\mathrm{tr}([X,Y])$ is given by the polynomial

$$\kappa(\xi,\eta,\zeta) := \xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta - 2$$

which is preserved under the action of $\mathrm{Out}(\pi)$. Moreover, the action of $\mathrm{Out}(\pi)$ on $\mathbb{C}^3$ is commensurable with the action of the group Γ of polynomial automorphisms of $\mathbb{C}^3$ which preserve κ (Horowitz [9]). Note that Γ is a finite extension of the modular group and is isomorphic to

$$\mathrm{PGL}(2,\mathbb{Z}) \ltimes (\mathbb{Z}/2 \oplus \mathbb{Z}/2)$$

Let $\mathcal{R}_{1,1}$ be the component of $\mathrm{Hom}(\pi, \mathrm{ISL}(2,\mathbb{R}))$ consisting of representations that send both X and Y to $i\,\mathrm{SL}_-(2,\mathbb{R})$. The restriction χ_{11} of the character map to $\mathcal{R}_{1,1}$ is a surjection onto $\mathfrak{X}_{1,1} = i\mathbb{R} \times i\mathbb{R} \times \mathbb{R}$. The latter is isomorphic to $\mathbb{R}^3$ and the restriction of κ to $\mathfrak{X}_{1,1}$ induces a polynomial

$$\kappa_{11}(x,y,z) := \kappa(ix, iy, z) = -x^2 - y^2 + z^2 + xyz - 2$$

where x, y and z are the standard coordinate functions on $\mathbb{R}^3$. If $u \in \mathfrak{X}_{1,1}$ is such that $k(u) \neq 2$, the fiber $\chi_{11}^{-1}(u)$ is an $\mathrm{SL}_\pm(2,\mathbb{R})$-conjugacy class of irreducible representations in $\mathcal{R}_{1,1}$. In this context, $\mathcal{X}_{1,1}$ identifies with a component of the $\mathrm{SL}_{\pm}(2,\mathbb{R})$-character variety of π.

Theorem A. *Let $\kappa_{11}(x,y,z) = -x^2 - y^2 + z^2 + xyz - 2$ and let $c \in \mathbb{R}$. Let Γ be the automorphism group of k_{11}. Then*

- ▷ *For $-14 \le c < 2$, the group Γ acts ergodically on $\kappa_{11}^{-1}(c)$.*
- ▷ *For $c < -14$, the group Γ acts properly and freely on an open subset $\Omega_c^M \subset \kappa_{11}^{-1}(c)$ and acts ergodically on the complement of Ω_c^M.*

Since π is a free group, representations in $\mathrm{Hom}(\pi, \mathrm{SL}_{\pm}(2,\mathbb{R}))$, or equivalently in $\mathrm{Hom}(\pi, \mathrm{ISL}(2,\mathbb{R}))$, can be realized as lifts of representations in $\mathrm{Hom}(\pi, \mathrm{PGL}(2,\mathbb{R}))$ and thus can be interpreted geometrically via the identification of $\mathrm{PGL}(2,\mathbb{R})$ with the full isometry group of hyperbolic 2-space $\mathbf{H}^2$. More precisely, let S be a surface with fundamental group $\pi_1(S)$. Let G be a semisimple Lie group. Then $\mathrm{Hom}(\pi_1(S), G)$ is an analytic variety upon which G acts by conjugation. Let $\mathrm{Hom}(\pi_1(S), G)/G$ be the orbit space. The G-orbits parametrize equivalence classes of flat principal G-bundles over S. If X is a space upon which G acts, $\mathrm{Hom}(\pi_1(S), G)/G$ is the deformation space of flat (G, X)-bundles over S. In this context, $\mathrm{Hom}(\pi, \mathrm{PGL}(2,\mathbb{R}))/\mathrm{PGL}(2,\mathbb{R})$ identifies with the deformation space of flat $\mathbf{H}^2$-bundles over a surface S whose fundamental group is free of rank 2.

When $\rho \in \mathrm{Hom}(\pi, \mathrm{PGL}(2,\mathbb{R}))$ is a discrete embedding, the holonomy group $\rho(\pi)$ acts properly discontinuously on the fiber $\mathbf{H}^2$. The quotient $\mathbf{H}^2/\rho(\pi)$ is homotopy-equivalent to S and affords a hyperbolic structure induced by that of $\mathbf{H}^2$. If the quotient is also diffeomorphic to S we call ρ a *discrete S-embedding*. The set Ω^S of conjugacy classes of discrete S-embeddings is open in $\mathrm{Hom}(\pi, \mathrm{PGL}(2,\mathbb{R}))/\mathrm{PGL}(2,\mathbb{R})$ and parametrizes complete hyperbolic structures on S marked with respect to a fixed set of generators of π. We call Ω^S the *Fricke space* of S. In a certain sense Ω^S is a generalization of the Teichmüller space of an orientable closed surface.

Discrete embeddings inside $\mathcal{R}_{1,1}$ give rise to non-orientable surfaces and since π is free of rank 2 the only possibilities are the once-punctured[1] Möbius band M (equivalently, the twice-punctured projective plane), and the once-punctured Klein bottle K. Their respective Fricke spaces Ω^M and Ω^K can be parametrized as subsets of $\mathbb{R}^3$.

Theorem B. *Let Γ be the group of automorphisms of the polynomial*

$$\kappa_{11}(x,y,z) = -x^2 - y^2 + z^2 - xyz - 2$$

(1) *Let Ω_0^M be the region in $\mathbb{R}^3$ defined by the inequalities*

$$\begin{aligned} xy + z &> 2 \\ z &< -2 \end{aligned}$$

[1]By "punctured" in this context we mean a surface with the interiors of one or more disjoint disks removed, i.e. a surface with boundary

Then the Fricke space of the once-punctured Möbius band M identifies with

$$\Omega^M = \coprod_{\gamma\in\Gamma/\Gamma_M} \gamma\Omega_0^M$$

where $\Gamma_M \subset \Gamma$ is the stabilizer of Ω_0^M, and Γ/Γ_M denotes the coset space of Γ_M.

(2) *Let Ω_0^K be the region in $\mathbb{R}^3$ defined by the inequality*

$$x^2 + y^2 - xyz + 4 < 0$$

Then the Fricke space of the once-punctured Klein bottle K identifies with

$$\Omega^K = \coprod_{\gamma\in\Gamma/\Gamma_K} \gamma\Omega_0^K$$

where $\Gamma_K \subset \Gamma$ is the stabilizer of Ω_0^K, and Γ/Γ_K denotes the coset space of Γ_K.

Remark. The subgroups Γ_M and Γ_K correspond to the mapping class groups of M and K respectively.

The level sets $\kappa_{11}^{-1}(c)$ intersect Ω^M if and only if $c < -14$, and they intersect Ω^K if and only if $c > 6$. Assume $c < -14$, and let $\Omega_c^M = \kappa_{11}^{-1}(c) \cap \Omega^M$. This is precisely the region mentioned in the second part of Theorem A. In other words, on the Fricke space of the once-punctured Möbius band M the action of Γ is wandering; on the outside, the action is ergodic along the level sets $\kappa_{11}^{-1}(c)$ for each $c < 2$. Similar statements hold for the Fricke spaces of M sitting inside the other $\mathrm{SL}_{\pm}(2,\mathbb{R})$-moduli-space components that correspond to non-zero classes in $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Conjecture. *Let Ω^K be the Fricke space of the once-punctured Klein bottle. Let $\Omega_c^K = \kappa_{11}^{-1}(c) \cap \Omega^K$*

- ▷ *The action of Γ on Ω^K is wandering.*
- ▷ *For each $c > 2$, the action of Γ on the set $\kappa_{11}^{-1}(c) - \Omega_c^K$ is ergodic.*

2. Background and Motivation

In this part we provide some relevant background material and interpret it in our context. We also introduce new notation and terminology that will be used in the subsequent exposition.

2.1. **Algebraic Generalities.** If A,B are groups, then $\mathrm{Hom}(A, B)$ denotes the set of homomorphisms (representations) $A \longrightarrow B$. Let Aut(A) denote the group of all automorphisms of A. There is an action of the group $\mathrm{Aut}(A) \times \mathrm{Aut}(B)$ on $\mathrm{Hom}(A, B)$ defined by

$$\begin{aligned} (\mathrm{Aut}(A) \times \mathrm{Aut}(B)) \times \mathrm{Hom}(A, B) &\longrightarrow \mathrm{Hom}(A, B) \\ ((\alpha, \beta), \phi) &\longmapsto \beta \circ \phi \circ \alpha^{-1} \end{aligned} \tag{2.1}$$

In particular, the action of B on itself by conjugation embeds B in its automorphism group and thus induces an action:

$$\begin{aligned} B \times \mathrm{Hom}(A, B) &\longrightarrow \mathrm{Hom}(A, B) \\ (b, \phi) &\longmapsto \iota_b \circ \phi \end{aligned}$$

where $\iota_b : h \mapsto bhb^{-1}$ is the inner automorphism of B defined by conjugation by $b \in B$. The orbit space, $\mathrm{Hom}(A, B)/B$ is the set of conjugacy classes of representations in $\mathrm{Hom}(A, B)$ and the action (2.1) descends to an action of $\mathrm{Aut}(A)$ on $\mathrm{Hom}(A, B)/B$. Let $\mathrm{Inn}(A)$ denote the (normal) subgroup of $\mathrm{Aut}(A)$ consisting of *inner automorphisms.* Since $\mathrm{Inn}(A)$ preserves the conjugacy class of a representation, it acts trivially on $\mathrm{Hom}(A, B)/B$ and thus the action of $\mathrm{Aut}(A)$ factors through the action of the *outer automorphism group*

$$\mathrm{Out}(A) := \mathrm{Aut}(A)/\,\mathrm{Inn}(A)$$

2.2. **Geometric Motivation.** When A is a discrete group and B a Lie group, the representation space $\mathrm{Hom}(A, B)$ can have special geometric significance. In particular, it can be interpreted as the moduli space of flat bundles over manifolds. Consider, for example, a manifold M with fundamental group $\pi = \pi_1(M, x_0)$ and a Lie group G which acts on a space X. Then the space of *flat* (G, X)*-bundles over* M can be identified with $\mathrm{Hom}(\pi, G)/G$. In the case when M is a closed surface, and G is the group of orientation preserving isometries of the hyperbolic plane, the subset of $\mathrm{Hom}(\pi, G)/G$ consisting of equivalence classes of discrete embeddings of π, identifies with the Teichmüller space $\mathfrak{T}$ of M. Moreover, $\mathrm{Out}(\pi)$ is isomorphic to the mapping class group $\pi_0 \,\mathrm{Diff}(M)$ of M whose action on $\mathfrak{T}$ is well known to be *properly discontinuous.*

2.3. **The Structure of** $\mathrm{Hom}(\pi, G)$**.** Whenever π is a finitely generated group, the space $\mathrm{Hom}(\pi, G)$ inherits a natural (Hausdorff) topology from that of G. Namely, if $\{\gamma_1, \dots, \gamma_n\}$ is a set of generators of π then the evaluation map

$$\begin{aligned} \mathrm{Hom}(\pi, G) &\longrightarrow G^n \\ \rho &\longmapsto (\rho(\gamma_1), \dots, \rho(\gamma_n)) \end{aligned}$$

is an embedding which induces a topology on $\mathrm{Hom}(\pi, G)$ that is independent on the choice of generators. Furthermore, if G is an algebraic group, then there is an induced *algebraic structure* on $\mathrm{Hom}(\pi, G)$, and we refer to the resulting object as the *variety of representations*.

However, the topology of the space $\mathrm{Hom}(\pi, G)/G$ inherited from $\mathrm{Hom}(\pi, G)$ could be rather pathological (e.g. may not be even Hausdorff). When G is a reductive linear algebraic group, i.e a subgroup of $\mathrm{GL}(n, \mathbb{R})$, however, one can consider the algebraic-geometric quotient $\mathrm{Hom}(\pi, G)/\!\!/G$, which is defined as the variety whose coordinate ring is precisely the ring of G-invariant functions on $\mathrm{Hom}(\pi, G)$. Equivalently, it is the space of *semi-stable orbits*. In this case, the orbit of a representation ρ is semi-stable if ρ is a completely reducible representation. We call $\mathrm{Hom}(\pi, G)/\!\!/G$ the *character variety* of π.

2.4. **Obstruction Classes and Components of** $\mathrm{Hom}(\pi, G)$**.** Let S be a connected surface and let π denote its fundamental group. Let G be a real algebraic Lie group. An element ρ in the space $\mathrm{Hom}(\pi, G)$ determines a flat prinipal G-bundle over S. This bundle gives rise to the obstruction classes in $H^q(S, \pi_{q-1}(G))$ and thus induces an *obstruction class map*

$$o_q : \mathrm{Hom}(\pi, G) \longrightarrow H^q(S, \pi_{q-1}(G))$$

In particular, let G_0 be the identity component for G and $G/G_0 = \pi_0(G)$ be the group of components of G. Then $o_1(\rho) \in H^1(S, \pi_0(G))$, which via the Hurewicz isomorphism is $\mathrm{Hom}(\pi, G/G_0)$. Thus, $o_1(\rho)$ is just the composite

$$\pi \xrightarrow{\rho} G \longrightarrow G/G_0.$$

The map o_1 is continuous, hence, constant on each connected component. For example if $G = \mathrm{PGL}(2, \mathbb{R}) \simeq \mathrm{SO}(2, 1)$, then $\pi_0(G) = \mathbb{Z}_2$ and $o_1(\rho) \in H^1(S, \mathbb{Z}_2)$. In that case the action of $\mathrm{PGL}(2, \mathbb{R})$ on hyperbolic 2-space $\mathbf{H}^2$ gives rise to an associated circle bundle (with fiber $\partial\mathbf{H}^2 = \mathbb{RP}^1$), whose first Stiefel-Whitney class corresponds to $o_1(\rho)$ (see Steenrod [19], §38; Goldman [5], §2). If $n > 0$ then π is a free group of rank $r = 1 - \chi(S)$, where $\chi(S)$ is the Euler characteristic of S. Thus $H^1(S, \mathbb{Z}_2) \simeq \mathbb{Z}_2^r$, which is a free $\mathbb{Z}_2$-module. In this case $\mathrm{Hom}(\pi, G) \cong G^r$ and the fibers of o_1 identify with the connected components of $\mathrm{Hom}(\pi, G)$. In this context we shall refer to o_1 as the *first Stiefel-Whitney class map*.

2.5. $\mathrm{SL}(2, \mathbb{C})$**-character Varieties.** The example in the last paragraph is of particular geometric interest. Consider a finitely generated group π and the representation space $\mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))$. Since $\mathrm{SL}(2, \mathbb{C})$ acts as a group of symmetries of hyperbolic 3-space $\mathbf{H}^3$, the orbit space $\mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))/\mathrm{SL}(2, \mathbb{C})$ contains a subset that parametrizes equivalence classes of hyperbolic structures on 3-manifolds with fundamental group isomorphic to π.

The character variety

$$\mathfrak{X} = \mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))/\!\!/\mathrm{SL}(2, \mathbb{C})$$

admits an embedding of $\mathcal{X}$ as an algebraic subset of affine space. In particular, traces of a finite generating set of π. define coordinates on (X) (Procesi [17]). We consider the case when π is free of rank 2:

Theorem 2.1 (Fricke-Klein)**.** *Let X, and Y be the generators of π and let $G = \mathrm{SL}(2,\mathbb{C})$. Then the character map:*

$$\chi : \mathrm{Hom}(\pi, G)/\!\!/G \longrightarrow \mathbb{C}^3$$
$$[\rho] \longmapsto \begin{bmatrix} \xi(\rho) \\ \eta(\rho) \\ \zeta(\rho) \end{bmatrix} = \begin{bmatrix} \mathrm{tr}(\rho(X)) \\ \mathrm{tr}(\rho(Y)) \\ \mathrm{tr}(\rho(XY)) \end{bmatrix}$$

is an isomorphism. (Compare the discussion in Goldman [5], 4.1, [6],§§4–5 and [7].) Thus the traces of X, Y, XY parametrize $\mathrm{Hom}(\pi, G)/\!\!/G$ as the affine space $\mathbb{C}^3$. In particular, if $w(X,Y)$ is any word in X and Y, then $\mathrm{tr}(\rho(w(X,Y)))$ is expressed as a polynomial f_w in ξ, η, ζ.

Let $\mathcal{H}$ be a copy of the hyperbolic plane $\mathbf{H}^2$ sitting inside $\mathbf{H}^3$. The stabilizer of $\mathcal{H}$ in $\mathrm{SL}(2,\mathbb{C})$ is isomorphic to $\mathrm{PGL}(2,\mathbb{R})$ and in this way $\mathrm{PGL}(2,\mathbb{R})$ identifies with the full group of isometries of $\mathbf{H}^2$. Naturally, we are interested in the orbits of the action of $\mathrm{PGL}(2,\mathbb{R})$ by conjugation on $\mathrm{Hom}(\pi, \mathrm{PGL}(2,\mathbb{R}))$. However, in order to use the trace parametrization of the orbit space provided by the theorem of Fricke-Klein, we need to work with the relevant representation space, which in this case is $\mathrm{Hom}(\pi, \mathrm{SL}_{\pm}(2,\mathbb{R}))$. The subgroup

$$\mathrm{SL}_{\pm}(2,\mathbb{R}) := \{A \in \mathrm{GL}(2,\mathbb{R}) \mid \det(A) = \pm 1\}$$

of $\mathrm{GL}(2,\mathbb{R})$ is a double cover of $\mathrm{PGL}(2,\mathbb{R})$. Since π is a free group, any representation $\rho \in \mathrm{Hom}(\pi, \mathrm{PGL}(2,\mathbb{R}))$ lifts to a representation in $\mathrm{Hom}(\pi, \mathrm{SL}_{\pm}(2,\mathbb{R}))$. Thus a conjugacy class of representations in $\mathrm{Hom}(\pi, \mathrm{PGL}(2,\mathbb{R}))$ corresponds to a conjugacy class of representations in $\mathrm{Hom}(\pi, \mathrm{SL}_{\pm}(2,\mathbb{R}))$ together with a choice of a lift.

2.6. $\mathrm{SL}_{\pm}(2,\mathbb{R})$**-character Varieties.** Let $\mathrm{SL}_{-}(2,\mathbb{R})$ denote the subset of $\mathrm{GL}(2,\mathbb{R})$ consisting of matrices of determinant -1. The group $\mathrm{SL}_{\pm}(2,\mathbb{R})$ is isomorphic to

$$\mathrm{ISL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R}) \amalg i\,\mathrm{SL}_{-}(2,\mathbb{R})$$

and in this context we identify the two as subgroups of $G = \mathrm{SL}(2,\mathbb{C})$. In view of the discussion in section 2.4, the representation space

$$\mathcal{R} = \mathrm{Hom}(\pi, \mathrm{SL}_{\pm}(2,\mathbb{R})) \cong \mathrm{Hom}(\pi, \mathrm{ISL}(2,\mathbb{R})) \cong \mathrm{ISL}(2,\mathbb{R}) \times \mathrm{ISL}(2,\mathbb{R})$$

has four connected components indexed by the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Let $G_0 = \mathrm{SL}(2,\mathbb{R})$ and $G_1 = i\,\mathrm{SL}_{-}(2,\mathbb{R})$. Then the correspondence defined by the Stiefel-Whitney class map is

$$\mathcal{R}_{j,k} = G_j \times G_k \longmapsto (j,k) \in \mathbb{Z}_2 \times \mathbb{Z}_2$$

where $j, k \in \{0, 1\}$. The following Proposition shows that the the restriction of the character map χ to $\mathcal{R}$ can be used to parametrize conjugacy classes of $\mathrm{ISL}(2,\mathbb{R})$ representations (compare Xia [22], pp. 10-13).

Proposition 2.2. *Let χ_{jk} be the restriction of χ to $\mathcal{R}_{j,k}$. Assume $(j,k) \neq (0,0)$. Then*

(1) *χ_{jk} is surjective*
(2) *The image of χ_{jk} is*

$$\mathcal{X}_{j,k} := \chi(\mathcal{R}_{j,k}) = \begin{cases} \mathbb{R} \times i\mathbb{R} \times i\mathbb{R} & \text{if} \quad (j,k) = (0,1) \\ i\mathbb{R} \times \mathbb{R} \times i\mathbb{R} & \text{if} \quad (j,k) = (1,0) \\ i\mathbb{R} \times i\mathbb{R} \times \mathbb{R} & \text{if} \quad (j,k) = (1,1) \end{cases}$$

(3) *Let $\rho \in \mathcal{R}$. Then*

$$\kappa \circ \chi(\rho) = \begin{cases} x^2 - y^2 - z^2 + xyz - 2 & \text{if} \quad \rho \in \mathcal{R}_{0,1} \\ -x^2 + y^2 - z^2 + xyz - 2 & \text{if} \quad \rho \in \mathcal{R}_{1,0} \\ -x^2 - y^2 + z^2 + xyz - 2 & \text{if} \quad \rho \in \mathcal{R}_{1,1} \end{cases} \tag{2.2}$$

(4) *Let $u \in \mathcal{X}_{j,k}$ be such that $k(u) \neq 2$. Then* $\mathrm{ISL}(2,\mathbb{R})$ *acts transitively on $\chi_{jk}^{-1}(u)$.*

Proof. We refer the proof to Xia [22], Proposition 12 and 13. □

By a result of Culler and Shalen, a representation in $\rho \in \mathcal{R}$ is reducible if and only if $\kappa(\chi(\rho)) = 2$. Thus Proposition 2.2 implies that if $u \in \mathcal{X}_{j,k}$ is not in $\kappa^{-1}(2)$, then the fiber $\chi_{jk}^{-1}(u)$ is an $\mathrm{ISL}(2,\mathbb{R})$-conjugacy class of irreducible representations in $\mathcal{R}_{j,k}$. In that sense $\mathcal{X}_{j,k}$ identifies with a component of the $\mathrm{ISL}(2,\mathbb{R})$-character variety of π.

Let κ_{jk} be the restriction of κ to $\mathcal{X}_{j,k}$. Since $\mathcal{X}_{j,k}$ is naturally isomorphic to $\mathbb{R}^3$ we will in fact consider the fibers of κ_{jk} as subsets of $\mathbb{R}^3$. Furthermore, whenever it is clear from the context, we will drop the subscript and use κ to denote the relevant polynomial as prescribed in (2.2).

2.7. **Surfaces Whose Fundamental Group is Free of Rank 2.** Let S be a surface whose fundamental group π is free of rank 2. Then S is precisely one of the following: a punctured[2] torus $S_{1,1}$, a pair of pants $S_{0,3}$, a punctured Klein bottle $S^\sharp_{1,1}$, or a punctured Möbius band $S^\sharp_{0,2}$. These are all connected surfaces with Euler characteristic -1.

Remark. The notation used for referencing these surfaces is based on the genus and the number of boundary components of each surface. Thus $S_{g,n}$ (respectively $S^\sharp_{g,n}$) denotes an orientable (respectively non-orientable) surface of genus g and n boundary components. For non-orientable surfaces, however, the notion of genus is not very consistently used throughout the literature. Some authors define genus as the number of copies of the projective plane in the cross-cap decomposition of the surface. In that context, the genus of the Klein bottle would be 2, and the genus of the Möbius band would be 1.

[2]In this context a "punctured surface" shall mean a surface with a topological disk removed

We adhere to the definition that is consistent with the formula for the Euler characteristic of a non-orientable surface S

$$\chi(S) = 1 - \beta_1(S) - n$$

where $\beta_1(S)$ is the first Betti number and n is the number of boundary components. We take $\beta_1(S)$ as a definition for the genus of a non-orientable surface S (see Moise [14], § 22).

Next, we show how the geometry of each surface is related to the presentation of π.

Definition. *An E-piece is a right-angled hyperbolic hexagon whose boundary is a piecewise-geodesic closed curve without self-intersections. Each edge of an E-piece is given the orientation of the underlying geodesic segment.*

Each surface of Euler characteristic -1 can be realized by pasting two E-pieces subject to a certain gluing scheme. Figure 1 illustrates this idea for the non-orientable cases. Let

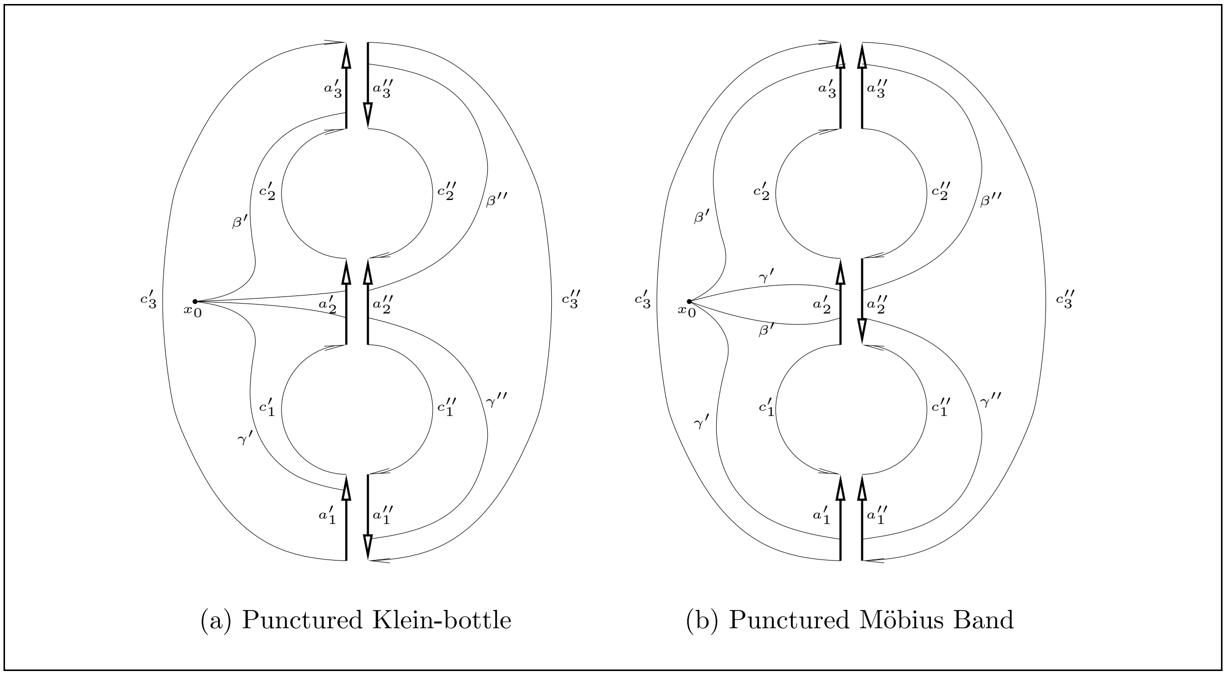


Figure 1: Constructing non-orientable surfaces from E-pieces

$\rho \in \mathrm{Hom}(\pi, \mathrm{PGL}(2,\mathbb{R}))$ be a discrete S-embedding. The union of two E-pieces E' and E'' is a fundamental region for $\rho(\pi)$ acting on $\mathbf{H}^2$ with quotient S. From the gluing diagrams we obtain a geometric presentation of π in each case of a surface of Euler characteristic -1. For example, suppose $S = S^{\sharp}_{1,1}$ is the punctured Klein-bottle. Then, $S = E' + E''$ modulo the gluing pattern shown on Fig. 1(a). In the quotient, $\beta' * \beta'' = \beta$, and $\gamma' * \gamma'' = \gamma$ become the two generators of π, and the boundary geodesic that represents the puncture is $\delta = c_2' * c_3'' * c_1' * c_1'' * c_3' * c_2''$. Here the "$*$" operator indicates the obvious left-to-right curve concatenation with respect to a suitable parametrization on each geodesic segment. Thus, for a fixed base point x_0, we obtain a presentation of $\pi \equiv \pi_1(S, x_0)$ as follows:

$$\pi = \langle \beta, \gamma, \delta \mid \gamma\beta^2\gamma = \delta \rangle \tag{2.3}$$

Similarly, from the diagram in Fig. 1(b) we obtain the following presentation of the fundamental group $\pi \equiv \pi_1(S^\sharp_{0,2}, x_0)$ of the punctured Möbius band:

$$\pi = \langle \beta, \gamma, \delta_1, \delta_2 \mid \delta_1 = \beta\gamma, \delta_2 = \beta\gamma^{-1} \rangle \tag{2.4}$$

where again $\beta = \beta' * \beta''$ and $\gamma = \gamma' * \gamma''$ are the two generators of π, while $\delta_1 = c'_3 * c''_3$ and $\delta_2 = c'_2 * c''_2 * c'_1 * c''_1$ represent the two boundary components of $S^\sharp_{0,2}$.

2.8. **Mapping Class Group, and the Structure of** $\mathrm{Out}(\pi)$**.** For a compact connected surface S, the Mapping Class Group of S is defined as the group $\pi_0(\mathrm{Homeo}(S, \partial S))$ of isotopy classes of homeomorphisms on S. There is a well-defined homomorphism:

$$N : \pi_0(\mathrm{Homeo}(M)) \longrightarrow \mathrm{Out}(\pi) \cong \mathrm{Aut}(\pi)/\,\mathrm{Inn}(\pi). \tag{2.5}$$

If S is a closed surface then by Dehn (unpublished) and Nielsen [16], N is an isomorphism. When $\partial S \neq \emptyset$, then each component $\partial_i S$ determines a conjugacy class C_i of elements of $\pi_1(S)$ and the image of N consists of elements of $\mathrm{Out}(\pi)$ represented by automorphisms which preserve each C_i. Another theorem of Nielsen [15], implies that when S is a punctured torus, N is also an isomorphism.

The action of $\mathrm{Out}(\pi)$ on homology $H_1(S; \mathbb{Z}) \cong \mathbb{Z}^2$ defines a homomorphism

$$h : \mathrm{Out}(\pi) \longrightarrow \mathrm{GL}(2, \mathbb{Z}). \tag{2.6}$$

which, again by Nielsen [15], is an isomorphism (see also Magnus-Karrass-Solitar [13], § 3.5, Corollary N4). Thus for a free group π of rank 2, $\mathrm{Out}(\pi)$ is isomorphic to $\mathrm{GL}(2, \mathbb{Z})$.

2.9. **The modular group.** In general the group $\mathrm{Aut}(\pi)$ acts on the $\mathrm{SL}(2, \mathbb{C})$ character variety $\mathfrak{X}$ by:

$$\phi_*([\rho]) = [\rho \circ \phi^{-1}]$$

Let $\gamma \in \pi$ and let ι_γ denote conjugation by γ. Since

$$\rho \circ \iota_\gamma = \iota_{\rho(\gamma)} \circ \rho$$

the subgroup $\mathrm{Inn}(\pi)$ acts trivially. Thus $\mathrm{Out}(\pi)$ acts on $\mathbb{C}^3$ and since an automorphism ϕ of π is determined by

$$(\phi(X), \phi(Y)) = (w_1(X, Y), w_2(X, Y)),$$

for some words w_1, w_2 in the generators, the action of ϕ on $\mathbb{C}^3$ is given by a triple of polynomials

$$(f_{w_1}(\xi, \eta, \zeta), f_{w_2}(\xi, \eta, \zeta), f_{w_1 w_2}(\xi, \eta, \zeta))$$

Hence $\mathrm{Out}(\pi)$ acts on $\mathbb{C}^3$ by *polynomial automorphisms.* Nielsen's theorem (see Magnus-Karras-Solitar [13], Theorem 3.9) implies that any such automorphism preserves $\kappa : \mathbb{C}^3 \longrightarrow \mathbb{C}$, that is

$$\kappa\left((f_{w_1}(\xi,\eta,\zeta), f_{w_2}(\xi,\eta,\zeta), f_{w_1w_2}(\xi,\eta,\zeta))\right) = \kappa(\xi,\eta,\zeta).$$

Horowitz[9] determined the group $\mathrm{Aut}(\mathbb{C}^3,\kappa)$ of polynomial mappings $\mathbb{C}^3 \longrightarrow \mathbb{C}^3$ preserving κ. Let $\mathfrak{S}_3$ be the symmetric group consisting of permutations of the coordinates ξ,η,ζ. Horowitz proved that the automorphism group of $(\mathbb{C}^3,\kappa)$ is generated by the linear automorphism group

$$\mathrm{Aut}(\mathbb{C}^3,\kappa) \cap \mathrm{GL}(3,\mathbb{C}) = (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \rtimes \mathfrak{S}_3$$

and the *quadratic reflection:*

$$\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \longrightarrow \begin{bmatrix} \xi \\ \eta \\ \xi\eta - \zeta \end{bmatrix},$$

This group is commensurable with $\mathrm{Out}(\pi)$. The factor $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ corresponds to *sign change automorphisms* of $\mathfrak{R}$. In particular, the three non-trivial elements $(0,1),(1,0),(1,1)$ act on representations by σ_{10}, σ_{01}, σ_{11} respectively:

$$\sigma_{01}\cdot\rho : \begin{cases} X & \longmapsto \rho(X) \\ Y & \longmapsto -\rho(Y) \end{cases}$$

$$\sigma_{10}\cdot\rho : \begin{cases} X & \longmapsto -\rho(X) \\ Y & \longmapsto \rho(Y) \end{cases}$$

$$\sigma_{11}\cdot\rho : \begin{cases} X & \longmapsto -\rho(X) \\ Y & \longmapsto -\rho(Y) \end{cases}$$

The corresponding action on characters is:

$$(\sigma_{01})_* : \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \longmapsto \begin{bmatrix} \xi \\ -\eta \\ -\zeta \end{bmatrix}$$

$$(\sigma_{10})_* : \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \longmapsto \begin{bmatrix} -\xi \\ \eta \\ -\zeta \end{bmatrix}$$

$$(\sigma_{11})_* : \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \longmapsto \begin{bmatrix} -\xi \\ -\eta \\ \zeta \end{bmatrix}.$$

Note that the sign change automorphisms are not induced by automorphisms of π.

Thus the group $\mathrm{Aut}(\mathbb{C}^3,\kappa)$ is isomorphic to a semidirect product

$$\mathrm{PGL}(2,\mathbb{Z}) \ltimes (\mathbb{Z}/2 \oplus \mathbb{Z}/2)$$

See Goldman [8], §1.3 and Appendix, for a detailed discussion.

The action of $\mathrm{Aut}(\mathbb{C}^3, \kappa)$ on $\mathbb{C}^3$ restricts to an action on the $\mathrm{ISL}(2, \mathbb{R})$-character variety. However the subgroup $\mathfrak{S}_3$ does not preserve the individual components $\mathfrak{X}_{j,k}$. For example, the automorphism induced by transposition of x and y maps characters in $\mathfrak{X}_{1,0}$ to characters in $\mathfrak{X}_{0,1}$, and stabilizes $\mathfrak{X}_{1,1}$. For each component $\mathfrak{X}_{j,k}$ there is a subgroup of finite index of $\mathrm{Aut}(\mathbb{C}^3, \kappa)$ that preserves $\mathfrak{X}_{j,k}$. We call this subgroup *the modular group of* $\mathfrak{X}_{j,k}$ and denote it by Γ_{jk}.

For instance, Γ_{11} is generated by the quadratic reflections

$$Q_x : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \longrightarrow \begin{bmatrix} yz - x \\ y \\ z \end{bmatrix}, \quad Q_y : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ xz - y \\ z \end{bmatrix}, \quad Q_z : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ y \\ -xy - z \end{bmatrix},$$

the sign-change automorphisms σ_{jk}, and the transposition

$$t_{xy} : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \longrightarrow \begin{bmatrix} y \\ x \\ z \end{bmatrix}$$

where $x, y, z \in \mathbb{R}$. Since $\Gamma_{01} \cong \Gamma_{10} \cong \Gamma_{11}$, whenever it is clear from the context, we will drop the subscript and use Γ to denote the relevant modular group for the given component.

2.10. **Goldman's Result on the Real $\mathrm{SL}(2)$-Characters of the Punctured Torus.** Recently, Goldman [8] studied the action of the modular group on the $\mathrm{SL}(2, \mathbb{R})$ component of $\mathfrak{X}$. In this case the modular group Γ is isomorphic to

$$\mathrm{PGL}(2, \mathbb{Z}) \ltimes (\mathbb{Z}/2 \oplus \mathbb{Z}/2)$$

We summarize Goldman's results in the theorem below.

Theorem 2.3 (Goldman)**.** *Let $\kappa(\xi, \eta, \zeta) = \xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta - 2$ and let $c \in \mathbb{R}$.*

- ▷ *For $c < -2$, the group Γ acts properly and freely on $\kappa^{-1}(c) \cap \mathbb{R}^3$;*
- ▷ *For $-2 \leq c < 2$, there is a compact connected component K_c of $\kappa^{-1}(c) \cap \mathbb{R}^3$ and Γ acts properly and freely on the complement $\kappa^{-1}(c) \cap \mathbb{R}^3 - K_c$;*
- ▷ *For $18 \geq c > 2$, the group Γ acts ergodically on $\kappa^{-1}(c) \cap \mathbb{R}^3$;*
- ▷ *For $c > 18$, the group Γ acts properly and freely on an open subset $\Omega_c \subset \kappa^{-1}(c) \cap \mathbb{R}^3$ and acts ergodically on the complement of Ω_c.*

3. Fricke Spaces Within the Character Variety

We now focus on the components $\mathcal{X}_{j,k}$ of the $\mathrm{ISL}(2,\mathbb{R})$-character variety of π that correspond to non-zero Steifel-Whitney classes in $\mathbb{Z}_2 \times \mathbb{Z}_2$. Our goal is to identify the regions in $\mathcal{X}_{j,k}$ that parametrize the Fricke spaces of the two non-orientable surfaces whose fundamental group is isomorphic to π, namely $S^{\sharp}_{1,1}$and $S^{\sharp}_{0,2}$(see section 2.7). For short, we call these surfaces the *PK-bottle* and the *PM-band* respectively.

3.1. **The Fricke Space of the Punctured Klein-Bottle.** We first derive a presentation of the fundamental group of the PK-bottle $S^{\sharp}_{1,1}$ from a presentation of $\pi_1(S_{0,3})$ via the Higman-Neumann-Neumann (HNN) extension construction. Let δ, α', and α'' be the homotopy classes in $\pi_1(S_{0,3})$ of the boundary loops of $S_{0,3}$. Then $\pi_1(S_{0,3})$ admits the following (redundant) presentation:

$$\pi_1(S_{0,3}) = \langle \delta, \alpha', \alpha'' \mid \delta\alpha'\alpha'' = I\rangle$$

The PK-bottle $S^{\sharp}_{1,1}$ is obtained from the pair of pants $S_{0,3}$ via identifying the boundary components α' and α'' by a (orientation reversing) diffeomorphism ϕ (see Fig 3.1). Let $H = \langle \alpha' \rangle$ be the cyclic subgroup of $\pi_1(S_{0,3})$ generated by α'. Then the induced homomorphism ϕ_* sends α' to α'' and thus its restriction to H is a monomorphism:

$$\phi_*|_H : H \longmapsto \pi_1(S_{0,3})$$

The corresponding HNN-extension of $\pi_1(S_{0,3})$ is given by:

$$\pi_1(S_{0,3}) *_\phi H = \langle \delta, \alpha', \alpha'', \beta \mid \delta\alpha'\alpha'' = I, \phi_*(\alpha') := \alpha'' = \beta\alpha'\beta^{-1}\rangle \tag{3.1}$$

This defines a presentation of $\pi_1(S^{\sharp}_{1,1})$ which reduces to the more familiar:

$$\pi_1(S^{\sharp}_{1,1}) = \langle \alpha, \beta, \delta \mid \alpha\beta\alpha\beta^{-1} = \delta^{-1}\rangle \tag{3.2}$$

Both $\pi_1(S_{0,3})$ and $\pi_1(S^{\sharp}_{1,1})$ are free of rank two. In particular, $\pi_1(S_{0,3})$ is freely generated by α' and α'', while $\pi_1(S^{\sharp}_{1,1})$ is freely generated by α and β. The quotient map

$$\pi_1(S_{0,3}) \longrightarrow \pi_1(S_{0,3}) *_\phi H$$

defines a monomorphism of the fundamental groups:

$$\begin{aligned} \phi_* : \pi_1(S_{0,3}) &\longrightarrow \pi_1(S^{\sharp}_{1,1}) \\ \alpha' &\longmapsto \alpha \\ \alpha'' &\longmapsto \beta\alpha\beta^{-1} \\ \delta &\longmapsto \beta\alpha^{-1}\beta^{-1}\alpha^{-1} \end{aligned}$$

Let $\rho \in \mathcal{R}_{0,1}$ and let $X \in G_0$, $Y \in G_1$ denote the images under ρ of α and β respectively. Then the images under $\rho \circ \phi_*$ of α' and α'' will be X and YXY^{-1} respectively. Thus the

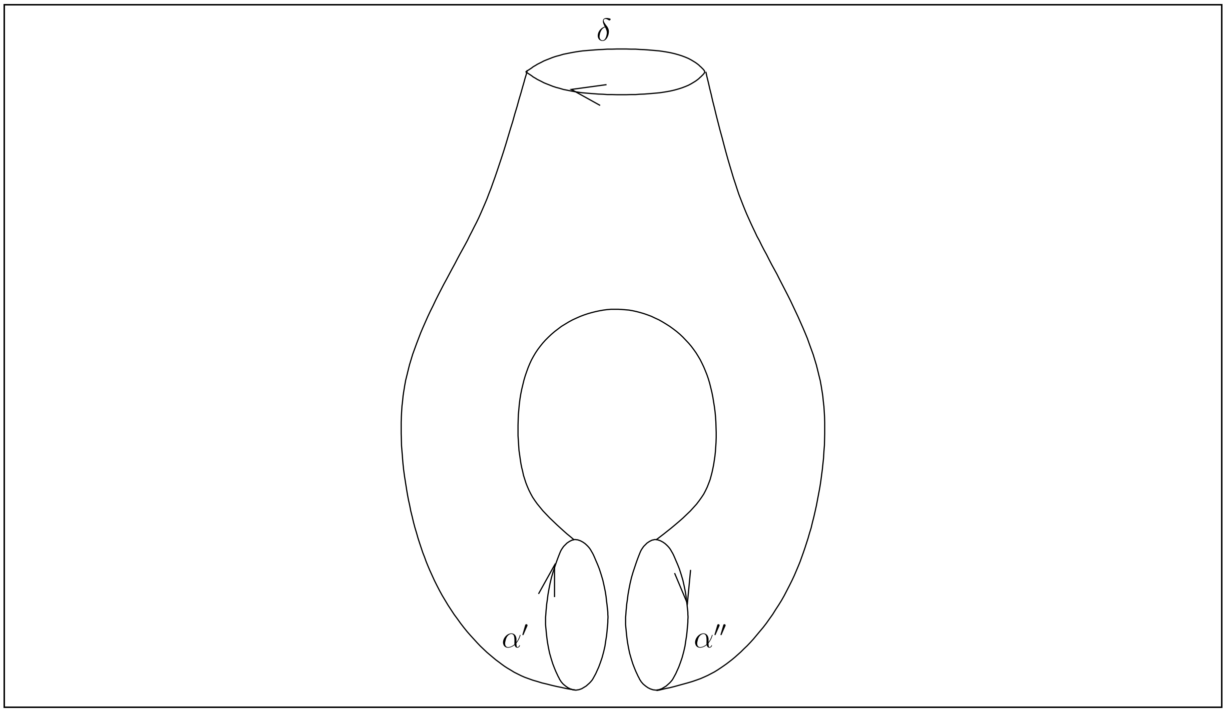


Figure 2: $S^{\sharp}_{1,1}$ as a quotient space of $S_{0,3}$

induced map ϕ^* of representation spaces is:

$$\begin{aligned}\phi^* : \mathcal{R}_{0,1} \cong G_0 \times G_1 &\longrightarrow G_0 \times G_0 \cong \mathcal{R}_{0,0}\\ (X,Y) &\longmapsto (X, YXY^{-1})\end{aligned}$$

A representation of $\pi_1(S_{0,3})$ defined by $(X,Z) \in G_0 \times G_0 \cong \mathcal{R}_{0,0}$ pulls back to a representation of $\pi_1(S^{\sharp}_{1,1})$ in $\mathcal{R}_{0,1}$ if and only if Z is in the centralizer of X in G_1, that is, there exists $Y \in G_1$, such that $Z = YXY^{-1}$. The map ϕ^* descends to a map (denoted again, by abuse of notation, as ϕ^*) between the corresponding components of the character varieties:

$$\begin{aligned}\phi^* : \mathfrak{X}_{0,1} \cong \mathbb{R} \times i\mathbb{R} \times i\mathbb{R} &\longrightarrow \mathbb{R}^3 \cong \mathfrak{X}_{0,0}\\ \begin{bmatrix}\xi\\ \eta\\ \zeta\end{bmatrix} &\longmapsto \begin{bmatrix}\xi\\ \xi\\ -\eta^2 - \zeta^2 + \xi\eta\zeta + 2\end{bmatrix}\end{aligned}$$

Setting $\xi = x$, $\eta = iy$, and $\zeta = iz$ with $x, y, z \in \mathbb{R}$ establishes an isomorphism $\mathbb{R} \times i\mathbb{R} \times i\mathbb{R} \cong \mathbb{R}^3$ such that the map ϕ^* can be expressed as follows:

$$\phi^* : \mathcal{X}_{0,1} \cong \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \cong \mathcal{X}_{0,0}$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} x \\ x \\ y^2 + z^2 - xyz + 2 \end{bmatrix}$$

A representation $\rho \in \mathcal{R}_{0,0}$ gives rise to a hyperbolilc structure on $int(S_{0,3})$ if and only if $\rho(\alpha')$, $\rho(\alpha'')$ and $\rho(\delta)$ are hyperbolic with non-intersecting axes that span an ultra-ideal triangle in $\mathbf{H}^2$. This condition is equivalent to $\chi([\rho]) \in (-\infty, -2)^3 \cup (2, +\infty)^3$, such that odd number of components of $\chi([\rho])$ are negative, or equivalently

$$|x| > 2 \tag{3.3}$$

$$y^2 + z^2 - xyz + 2 < -2 \tag{3.4}$$

But characters in $\mathcal{X}_{0,1}$ that lie in the fibers of ϕ^* correspond to discrete embeddings of $\pi_1(S^\sharp_{1,1})$ in $\mathcal{R}_{0,1}$ with quotient a PK-Bottle, and thus to marked hyperbolic structures on $int(S^\sharp_{1,1})$. Note that inequality 3.4 already implies that $|x| > 2$. We have thus proved the first part of the following

Proposition 3.1. *Let $x, y, z \in \mathbb{R}$ be the coordinate functions on $\mathcal{X}_{0,1}$.*

(1) *The region $\Omega_0^K \subset \mathbb{R}^3$ parametrized by the inequality*

$$y^2 + z^2 - xyz + 4 < 0$$

corresponds to discrete geometric K-embeddings inside $\mathcal{R}_{0,1}$.

(2) *Let Γ_K be the image of the mapping class group $\mathrm{Map}(S^\sharp_{1,1}))$ in Γ under the Nielsen homomorphism and let $\Gamma_0^K = \Gamma/\Gamma_K$ be the coset space of Γ_K in Γ. The space of all discrete K-embeddings identifies with*

$$\Omega^K = \coprod_{\gamma \in \Gamma_0^K} \gamma \Omega_0^K$$

Proof. To prove part (2) we must show that if for some $\gamma \in \Gamma$ the intersection $\Omega_0^K \cap \gamma \Omega_0^K$ is nonempty, then $\gamma \in \mathrm{Map}(S^\sharp_{1,1})$.Suppose that $[\rho] \in \Omega_0^K \cap \gamma \Omega_0^K$. The automorphism γ of the character space $\mathbb{C}^3$ is induced by an automorphism $\tilde{\gamma}$ of $\pi_1(S^\sharp_{1,1})$ such that

$$[\rho_0 \circ \tilde{\gamma}] = \gamma \circ [\rho]$$

for some $\rho_0 \in \Omega_0^K$. Thus $\rho_0 \circ \tilde{\gamma}$ is also a discrete geometric K-embedding with respect to the original peripheral structure. This means that $\tilde{\gamma}$ preserves the conjugacy class of each boundary component, and in particular that it represents an equivalence class in the mapping class group of $S^\sharp_{0,2}$. $\square$

Once we have computed the Fricke space of the PK-bottle inside $\mathfrak{X}_{0,1}$, it is easy to obtain the Fricke spaces sitting in the other two non-zero components of the ISL$(2,\mathbb{R})$-character variety. We show how to do this for $\mathfrak{X}_{1,1}$. The class of the automorphism

$$\psi : \alpha \longmapsto \alpha\beta^{-1}$$

of π induces the transposition

$$t_{xz} : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \longrightarrow \begin{bmatrix} z \\ y \\ x \end{bmatrix}$$

on the character variety $\mathfrak{X}$. In particular t_{xz} interchanges $\mathfrak{X}_{0,1}$ and $\mathfrak{X}_{1,1}$ and consequently the image of Ω_0^K in $\mathfrak{X}_{1,1}$ will be given by

$$x^2 + y^2 - xyz + 4 < 0 \tag{3.5}$$

This corresponds to a change of marking, with respect to which the new presesntation of $\pi_1(S^\sharp_{1,1})$ becomes

$$\langle \gamma, \beta, \delta \mid \gamma\beta^2\gamma = \delta^{-1} \rangle$$

3.2. **The Fricke Space of the Punctured Möbius Band.** We now determine the characters of the discrete geometric embeddings inside the $(1,1)$-component of $\mathrm{Hom}(\pi, \mathrm{ISL}(2,\mathbb{R}))$. To this end, consider the inclusion of the fundamental group of the quadruply punctured sphere $\pi_1(S_{0,4})$ into $\pi_1(S^\sharp_{0,2})$ induced by the (double) covering map

$$q : S_{0,4} \longrightarrow S^\sharp_{0,2}$$

Suppose $\pi_1(S^\sharp_{0,2})$ is given the presentation discussed in section (2.7), namely:

$$\pi = \langle \beta, \gamma, \delta_1, \delta_2 \mid \delta_1 = \beta\gamma, \delta_2 = \beta\gamma^{-1} \rangle \tag{3.6}$$

The orientable double cover of $S^\sharp_{0,2}$, which we denote for the moment by $\widetilde{S^\sharp_{0,2}}$, is a connected surface of genus 0 with 4 boundary components and thus homeomorphic to $S_{0,4}$. The two sheets of the covering map, denoted L and L', are represented schematically in Fig. 3.2. To obtain the covering surface, L and L' are glued via identifying the two pairs of edges, respectively marked with $\tilde{a}'_2$ and $\tilde{a}''_2$ (these are respectively the lifts of a'_2 and a''_2 from Fig. 1(b)). After this identification, the geodesic segments d and $\hat{d}$ concatenate to form a boundary loop, denoted by $\tilde{\delta}_2$. Similarly, the geodesic segments d' and $\hat{d}'$ concatenate to form a boundary loop, denoted by $\tilde{\delta}'_2$. Thus the boundary of $\widetilde{S^\sharp_{0,2}}$ consists of four components, namely $\tilde{\delta}_1$, $\tilde{\delta}'_1$, $\tilde{\delta}_2$, and $\tilde{\delta}'_2$. Their corresponding elements in the fundamental group of $\widetilde{S^\sharp_{0,2}}$, denoted again by $\tilde{\delta}_1$, $\tilde{\delta}'_1$, $\tilde{\delta}_2$, and $\tilde{\delta}'_2$, are in fact the lifts of δ_1, $\delta_2 \in \pi_1(S^\sharp_{0,2})$.

More precisely, fix a base point $x_0 \in S^\sharp_{0,2}$ and let $\tilde{x}_0$ be the lift of x_0 to L. Consider the fundamental group $\tilde{\pi}$ of $\widetilde{S^\sharp_{0,2}}$ with base point $\tilde{x}_0$. Let $\tilde{x}'_0$ be the lift of x_0 to L'. In

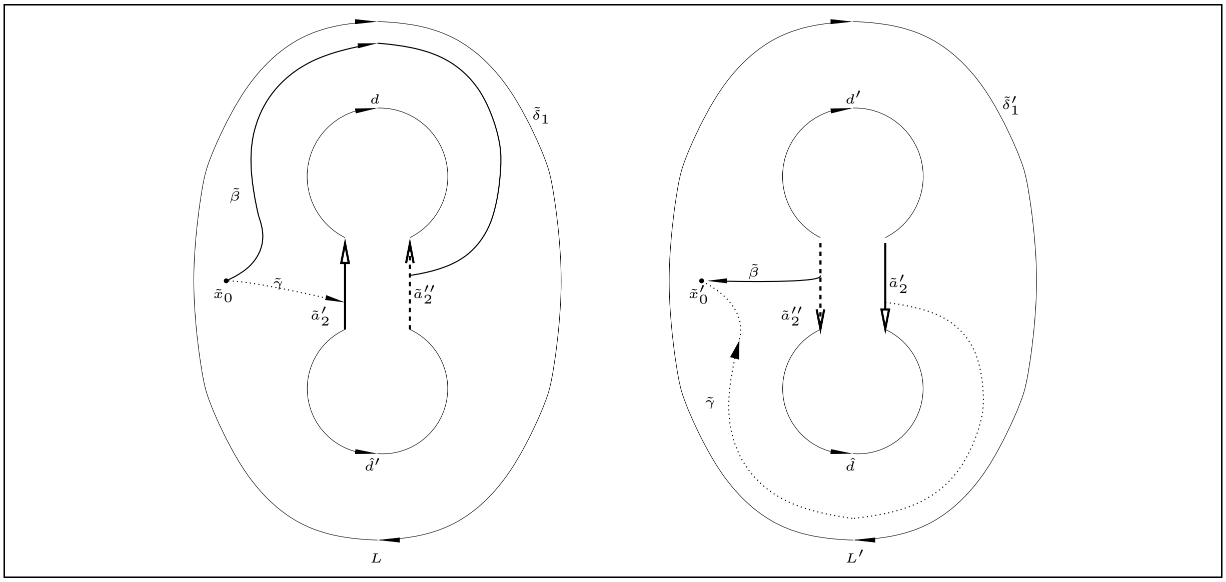


Figure 3: The Two Sheets of $S_{0,4}$ as a Double Cover of $S^\sharp_{0,2}$

$\widetilde{S^\sharp_{0,2}}$ there are two homotopy classes of paths with initial point $\tilde{x}_0$ and end point $\tilde{x}_0'$; one class projects to the generator $\beta \in \pi$, the other one projects to β^{-1}. Let $\tilde{\beta}$ and $\tilde{\beta}'$ be representatives of each class respectively. Similarly, let $\tilde{\gamma}$ and $\tilde{\gamma}'$ be representatives of the two homotopy classes of paths that project to γ and γ^{-1} respectively (see Fig. (3.2)). With this notation, the homotopy classes of the boundary components of $\widetilde{S^\sharp_{0,2}}$ can be written as follows:

$$\tilde{\delta}_1 = [\tilde{\beta} * (\tilde{\gamma}')^{-1}] \qquad \tilde{\delta}_1' = [\tilde{\gamma} * (\tilde{\beta}')^{-1}]$$
$$\tilde{\delta}_2 = [\tilde{\beta} * \tilde{\gamma}^{-1}] \qquad \tilde{\delta}_2' = [\tilde{\gamma}' * (\tilde{\beta}')^{-1}]$$

where '$*$' denotes path concatenation. From these equations, the images of $\tilde{\delta}_1, \tilde{\delta}_1', \tilde{\delta}_2$, and $\tilde{\delta}_2'$ under the induced covering projection map q_* can be easily computed:

$$q_* : \pi_1(\widetilde{S^\sharp_{0,2}}) \longrightarrow \pi_1(S^\sharp_{0,2})$$
$$\begin{bmatrix} \tilde{\delta}_1 \\ \tilde{\delta}_2 \\ \tilde{\delta}_1' \\ \tilde{\delta}_2' \end{bmatrix} \longmapsto \begin{bmatrix} \beta\gamma \\ \beta\gamma^{-1} \\ \gamma\beta \\ \gamma^{-1}\beta \end{bmatrix}$$

But $\tilde{\delta}_1, \tilde{\delta}_1', \tilde{\delta}_2$, and $\tilde{\delta}_2'$ satisfy the relation

$$\tilde{\delta}_1\tilde{\delta}_2'(\tilde{\delta}_1')^{-1}(\tilde{\delta}_2)^{-1} = I$$

and thus if we set

$$A = \tilde{\delta}_1, \quad B = \tilde{\delta}_2', \quad C = (\tilde{\delta}_1')^{-1}, \quad D = (\tilde{\delta}_2)^{-1}$$

we obtain a presentation of $\pi_1(\widetilde{S^\sharp_{0,2}})$ such that

$$\pi_1(\widetilde{S^\sharp_{0,2}}) \equiv \pi_1(S_{0,4}) = \langle A, B, C, D \mid ABCD = I \rangle$$

The induced map q_* is then uniquely determined by the images of the three generators A, B, and C

$$q_* : \begin{bmatrix} A \\ B \\ C \end{bmatrix} \longmapsto \begin{bmatrix} \beta\gamma \\ \gamma^{-1}\beta \\ \beta^{-1}\gamma^{-1} \end{bmatrix} \tag{3.7}$$

The image of q_* in $\pi_1(S^\sharp_{0,2})$ is the index-2 subgroup consisting of words of even length, and generated by $\beta\gamma$, $\gamma^{-1}\beta$, and $\beta^{-1}\gamma^{-1}$. The elements of $\pi_1(\widetilde{S^\sharp_{0,2}})$ whose traces generate the coordinate ring of the character variety of $\widetilde{S^\sharp_{0,2}}$, are (see for instance Magnus [12])

$$A, B, C, D, AB, BC, CA$$

From (3.7) we compute the images under q_* of the last four elements:

$$q_* : \begin{bmatrix} D \\ AB \\ BC \\ CA \end{bmatrix} \longmapsto \begin{bmatrix} \beta\gamma^{-1} \\ \beta^2 \\ \gamma^{-2} \\ \beta^{-1}\gamma^{-1}\beta\gamma \end{bmatrix}$$

Thus the induced map q^* on representation spaces:

$$\begin{aligned} q^* : \operatorname{Hom}(\pi_1(S^\sharp_{0,2}), G) &\longrightarrow \operatorname{Hom}(\pi_1(S_{0,4}), G) \\ \rho &\longmapsto \rho \circ q_* \end{aligned}$$

can be expressed in terms of the images $X = \rho(\beta)$ and $Y = \rho(\gamma)$ of the generators β and γ as follows

$$q^* : \operatorname{Hom}(\pi_1(S^\sharp_{0,2}), G) \longrightarrow \operatorname{Hom}(\pi_1(S_{0,4}), G)$$

$$\begin{bmatrix} X \\ Y \end{bmatrix} \longmapsto \begin{bmatrix} XY \\ Y^{-1}X \\ X^{-1}Y^{-1} \\ XY^{-1} \\ X^2 \\ Y^{-2} \\ X^{-1}Y^{-1}XY \end{bmatrix}$$

Consequently, the induced map on the $\mathrm{SL}(2,\mathbb{C})$-character varieties, denoted again by q^*, is defined in terms of trace coordinates as follows:

$$q^* : \mathrm{Hom}(\pi_1(S^\sharp_{0,2}), G)/\!\!/G \longrightarrow \mathrm{Hom}(\pi_1(S_{0,4}), G)/\!\!/G$$

$$\mathbb{C}^3 \ni \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \longmapsto \begin{bmatrix} \zeta \\ \xi\eta - \zeta \\ \zeta \\ \xi\eta - \zeta \\ \xi^2 - 2 \\ \eta^2 - 2 \\ \kappa(\xi,\eta,\zeta) \end{bmatrix} \in \mathbb{C}^7 \tag{3.8}$$

where

$$\xi = \mathrm{tr}(X),\ \eta = \mathrm{tr}(Y),\ \zeta = \mathrm{tr}(XY)$$

Recall that the $(1,1)$-component $\mathfrak{X}_{1,1}$ of the $\mathrm{ISL}(2,\mathbb{R})$-character variety of $\widetilde{S^\sharp_{0,2}}$ is isomorphic to $i\mathbb{R} \times i\mathbb{R} \times \mathbb{R}$, and therefore identifies with $\mathbb{R}^3$ by setting

$$\xi = ix, \quad \eta = iy, \quad \zeta = z$$

with $x, y, z \in \mathbb{R}$. The restriction of q^* to $\mathfrak{X}_{1,1}$ can be expressed in terms of the x, y, z coordinates in the following way:

$$\begin{aligned} q^* : \quad \mathfrak{X}_{1,1} \cong \mathbb{R}^3 \quad &\longrightarrow \qquad \mathbb{R}^7 \\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad &\longmapsto \quad \begin{bmatrix} z \\ -xy - z \\ z \\ -xy - z \\ -x^2 - 2 \\ -y^2 - 2 \\ \kappa(ix, iy, z) \end{bmatrix} \end{aligned} \tag{3.9}$$

The coordinate ring of the $\mathrm{SL}(2,\mathbb{C})$-character variety of $S_{0,4}$ is generated by the traces

$$a = \mathrm{tr}(A),\ b = \mathrm{tr}(B),\ c = \mathrm{tr}(C),\ d = \mathrm{tr}(D)$$
$$t_{AB} = \mathrm{tr}(AB),\ t_{BC} = \mathrm{tr}(BC),\ t_{CA} = \mathrm{tr}(CA)$$

subject to the relation (see for instance Magnus ([12])):

$$\begin{aligned} t_{AB}^2 + t_{BC}^2 + t_{CA}^2 + t_{AB}t_{BC}t_{CA} \\ &= (ab + cd)t_{AB} + (ad + bc)t_{BC} + (ac + bd)t_{CA} \\ &\quad - (a^2 + b^2 + c^2 + d^2 + abcd - 4). \end{aligned}$$

Thus the map

$$\mathfrak{X}^{\mathbb{C}}_{S_{0,4}} = \mathrm{Hom}(\pi_1(S_{0,4}), \mathrm{SL}(2,\mathbb{C}))/\!\!/\,\mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathbb{C}^7$$

defined by $(a, b, c, d, t_{AB}, t_{BC}, t_{CA})$ embeds $\mathfrak{X}$ onto a hypersurface in $\mathbb{C}^7$. The set of real points of $\mathfrak{X}^{\mathbb{C}}_{S_{0,4}}$ parametrizes equivalence classes of representations in $\mathrm{SL}(2,\mathbb{R})$. The following Proposition (compare Keen [11], Theorem 2) provides necessary and sufficient conditions for

$$(a, b, c, d, t_{AB}, t_{BC}) \in \mathbb{R}^6$$

to represent an equivalence class of discrete geometric Q-embeddings of $\pi_1(S_{0,4})$ into $\mathrm{SL}(2, \mathbb{R})$.

Proposition 3.2. *Given $a < -2$, $b < -2$, $c < -2$, $d < -2$, $t_{AB} < -2$, and $t_{BC} < -2$, there exist elements $A^*, B^*, C^*, D^* \in \mathrm{SL}(2, \mathbb{R})$ such that*

$$a = \mathrm{tr}(A^*),\ b = \mathrm{tr}(B^*),\ c = \mathrm{tr}(C^*),\ d = \mathrm{tr}(D^*)$$
$$t_{A^*B^*} = \mathrm{tr}(A^*B^*),\ t_{B^*C^*} = \mathrm{tr}(B^*C^*)$$

and such that $F = \langle A^*, B^* \rangle$ and $F' = \langle C^*, D^* \rangle$ are Fuchsian groups of signature $(0; 3)$. Moreover, the group $H = F *_{\mathbb{Z}} F'$, which is the amalgamated product of F and F' over the cyclic subgroups

$$\mathbb{Z} \cong \langle A^*B^* \rangle \subset F, \qquad \mathbb{Z} \cong \langle C^*D^* \rangle \subset F',$$

is Fuchsian and represents a marked *surface of signature $(0; 4)$. Every such marked surface is so representable.*

But a Fuchsian group H, whose quotient $\mathbf{H}^2/H$ is a marked hyperbolic surface S, gives rise to a discrete geometric embedding $\rho \in \mathrm{Hom}(\pi_1(S), \mathrm{SL}(2, \mathbb{R}))$ determined (up to conjugation) by the correspondence between the generators of $\pi_1(S_{0,4})$ and those of H. Thus we have the following

Corollary 3.3. *Given $a, b, c, d, t_{AB}, t_{BC} \in \mathbb{R}$, such that*

$$a < -2,\ b < -2,\ c < -2,\ d < -2,\ t_{AB} < -2,\ t_{BC} < -2, \tag{3.10}$$

there exists a hypebolic surface $S_{0,4}$ of signature $(0; 4)$ with marking $\Lambda = \{A, B, C, D\}$ and a discrete Q-geometric embedding $\rho \in \mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{R}))$, such that

$$a = \mathrm{tr}(\rho(A)),\ b = \mathrm{tr}(\rho(B)),\ c = \mathrm{tr}(\rho(C)),\ d = \mathrm{tr}(\rho(D))$$
$$t_{AB} = \mathrm{tr}(\rho(AB)),\ t_{BC} = \mathrm{tr}(\rho(BC))$$

All such discrete Q-geometric embeddings are obtained this way.

It follows from Corollary 3.3 that a real point on the character variety $\mathfrak{X}^{\mathbb{C}}_{S_{0,4}}$, that is,. a point $p = (a, b, c, d, t_{AB}, t_{BC}, t_{CA}) \in \mathbb{R}^7$ whose coordinates satisfy the Fricke relation, corresponds to a discrete geometric Q-embedding if and only if its coordinates satisfy inequalities (3.10).

This conclusion together with the observation that some of the expressions that define the image of q^* in (3.9) satisfy the given inequalities trivially, prove the first part of the following

Proposition 3.4. *Let $x, y, z \in \mathbb{R}$ be the coordinate functions on $\mathfrak{X}_{1,1}$.*

(1) *The region* Ω_0^M *of* $\mathbb{R}^3$ *parametrized by the inequalities:*

$$\begin{aligned} xy + z &> 2 \\ z &< -2 \end{aligned}$$

corresponds to discrete geometric M*-embeddings inside* $\mathcal{R}_{1,1}$.

(2) *Let* Γ_M *be the image of the mapping class group* $\mathrm{Map}(S_{0,2}^\sharp))$ *in* Γ *under the Nielsen homomorphism and let* $\Gamma_0^M = \Gamma/\Gamma_M$ *be the coset space of* Γ_M *in* Γ. *The space of all discrete* M*-embeddings identifies with*

$$\Omega^M = \coprod_{\gamma \in \Gamma_0^M} \gamma \Omega_0^M$$

The proof of the second part of the Proposition is analogous to the proof of Proposition 3.1

4. Classification of Characters and the τ-reduction Algorithm

The next two sections discuss the Γ-action on the components

$$\mathfrak{X}_{0,1} \cup \mathfrak{X}_{1,0} \cup \mathfrak{X}_{1,1}$$

of the $\mathrm{ISL}(2,\mathbb{R})$-character variety of π. We shall mainly work in $\mathfrak{X}_{1,1}$, but the results carry over verbatim to $\mathfrak{X}_{0,1}$ and $\mathfrak{X}_{1,0}$.

Denote by "$\sim$" the equivalence relation induced by the Γ-action on characters: For $u, v \in \mathfrak{X}_{1,1}$

$$u \sim v \Longleftrightarrow \exists \gamma \in \Gamma \text{ such that } \gamma u = v$$

Since κ is Γ-invariant, $u \sim v$ implies $\kappa(u) = \kappa(v)$. In this section we show that when $c < -14$ there exist essentially two types of equivalence classes of characters upon each of which Γ acts in a substantially different manner.

Theorem 4.1. *Suppose that $u \in \mathfrak{X}_{1,1}$ satisfies $\kappa(u) < -14$. Then $\exists u' = (x', y', z') \sim u$ such that either:*

- **M:** *$u' \in \Omega_0^M$, in which case u' (and therefore u) is a character of a Fuchsian representation whose quotient is homeomorphic to a a once-punctured Möbius band; or*
- **E:** *$\bar{z}' \in (-2, 2)$ in which case u' is the character of a representation mapping the peripheral element $\beta\gamma^{-1} \in \pi$ to an elliptic element of* $\mathrm{SL}(2,\mathbb{R})$.

4.1. **Notation.** For any $(x, y, z) \in \mathbb{R}^3$, let

$$\bar{z} := -xy - z$$

Since the quadratic reflection

$$Q_z : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ y \\ -xy - z \end{bmatrix}$$

preserves every $\kappa^{-1}(c)$ for $c \in \mathbb{R}$, it interchanges z and $\bar{z}$. Fix $x, y \in \mathbb{R}$. Then z and $\bar{z}$ are the two (necessarily real) roots of the quadratic polynomial

$$z^2 + (xy)z - x^2 - y^2 - 2 - c \tag{4.1}$$

since

$$\begin{aligned} z + \bar{z} &= -xy \\ z\bar{z} &= -x^2 - y^2 - 2 - c. \end{aligned} \tag{4.2}$$

Thus Q_z is the deck transformation of the double covering

$$\Pi : \kappa^{-1}(c) \longrightarrow \Pi(\kappa^{-1}(c))$$

obtained by restriction of the projection $\Pi(x, y, z) = (x, y)$ to the xy-plane.

Denote by $\mathcal{E}_c$ the region

$$\{(x, y, z) \in \kappa^{-1}(c) \mid \bar{z} \in (-2, 2)\}$$

4.2. **The level sets of κ and the Fricke space of $S^{\sharp}_{0,2}$.** Next we find a necessary condition for Ω_0^M and $\kappa^{-1}(c)$ to have a non-empty intersection.

Lemma 4.2. *Suppose $u = (x, y, z) \in \Omega_0^M \cap \kappa^{-1}(c)$. Then $c < -14$*

Proof. Since $u \in \Omega_0^M$, by defintion $z < -2$, and $\bar{z} < -2$. Therefore $z + \bar{z} < -4$ and $z\bar{z} > 4$. Then (4.2) imply that

$$xy > 4 \tag{4.3}$$
$$x^2 + y^2 < -c - 6$$

Since any $x, y \in \mathbb{R}$ satisfy $x^2 + y^2 \geq 2xy$,

$$c = -6 - (-c - 6) \leq -6 - (x^2 + y^2) \leq -6 - 2xy < -14.$$

□

Corollary 4.3. $\Omega_0^M \cap \kappa^{-1}(c) \neq \varnothing$ *if and only if* $c < -14$.

Remark. The sign change transformations

$$\sigma_{xz} : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} -x \\ y \\ -z \end{bmatrix}, \quad \sigma_{yz} : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} x \\ -y \\ -z \end{bmatrix} \tag{4.4}$$

preserve κ and commute with Q_z. Since

$$\bar{z}(\sigma_{xz}(u)) = \bar{z}(\sigma_{yz}(u)) = -\bar{z}(u)$$

it suffices to consider $u \in \mathcal{X}_{1,1}$ such that $\kappa(u) < -14$, $z < -2$ and $\bar{z} > 2$.

The proof of Theorem 4.1 proceeds in two steps: first we construct a function τ that is non-decreasing along certain subsets of the orbit of each character in Ω_0^M. Then, (always assuming $c < -14$ and $u \notin \Omega_0^M \cup \mathcal{E}_c$), we find a finite sequence of characters $u_0 = u, u_1, \ldots u_N$, and a constant $T > 0$ such that $\tau(u_i) > \tau(u_{i+2}) + T$, and either $u_N \in \Omega_0^M$, or $u_N \in \mathcal{E}_c$.

4.3. **The Quadratic Reflections.** Recall that, apart from Q_z, two other quadratic reflections act on characters as automorphisms of κ:

$$Q_x : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \longrightarrow \begin{bmatrix} yz - x \\ y \\ z \end{bmatrix} \quad \text{and} \quad Q_y : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ xz - y \\ z \end{bmatrix}$$

They are induced, respectively by the following automorphisms of π (cf. Goldman [8])

$$q_x : \begin{array}{rcl} X & \longmapsto & (XY)Y^2X(Y^{-1}X^{-1}) \\ Y & \longmapsto & (XYX^{-1})Y^{-1}(XY^{-1}X^{-1}) \\ XY & \longmapsto & (XY^2)(XY)(Y^{-2}X^{-1}) \end{array}$$

and

$$q_y : \begin{array}{rcl} X & \longmapsto & (XY)X(Y^{-1}X^{-1}) \\ Y & \longmapsto & (XY)X^{-1}Y^{-1}X^{-1}(Y^{-1}X^{-1}) \\ XY & \longmapsto & (XY)^{-1} \end{array}$$

On the other hand Q_z is induced by the automorphism

$$q_z : \begin{array}{rcl} X & \longmapsto & X \\ Y & \longmapsto & Y^{-1} \\ XY & \longmapsto & XY^{-1} \end{array}$$

which interchanges the two boundary components of $S^{\sharp}_{0,2}$. Therefore its coset in $\mathrm{Out}(\pi)$ lies in the image of the mapping class group under the Nielsen homomorphism (2.5). In contrast q_x and q_y do not represent elements of the mapping class group of $S^{\sharp}_{0,2}$.

Clearly Q_z preserves Q_x and Q_y preserves Ω_0^M. Indeed, for each $u \in \Omega_0^M$

$$\bar{z}(Q_x(u)) = -y(yz - x) - z = -z(y^2 + 1) + xy > 6$$

since (4.3) implies $xy > 4$ and $z < -2$. Similarly,

$$\bar{z}(Q_y(u)) > 6$$

and therefore $Q_x(u), Q_y(u) \notin \Omega_0^M \cup \mathcal{E}_c$.

4.4. **The orbit as a binary tree.** Consider the subgroup $\Lambda \subset \Gamma$ generated by Q_x, Q_y, Q_z. Since $\Lambda \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$, it can be naturally associated with a trivalent tree, $T_\Lambda(V, E)$, defined as follows. Each node $v \in V$ represents a group element, that is, a reduced word on the generators. Nodes corresponding to words w_1 and w_2 respectively, are linked by an edge $e \in E$ if there is a generator λ such that $w_1 = \lambda w_2$.

For any $u \in \Omega_0^M$, the tree $T_\Lambda(V, E)$ imparts a binary forest structure on the orbit $\Lambda \cdot u$

Definition. *Let $u \in \Omega_0^M$. Define a binary tree $B_\Lambda(u)$ inductively as follows*

- ▷ *u is the root of $B_\Lambda(u)$*
- ▷ *$Q_x(u)$ and $Q_y(u)$ are, respectively, the left and the right descendent of u.*

$\triangleright$ *Suppose v is an arbitrary node and let $\hat{v}$ be its parent. Let $\lambda \in \Lambda$ be the generator such that $v = \lambda\hat{v}$. Then the descendents of v are $\lambda_1 v$, and $\lambda_2 v$, where $\lambda_1, \lambda_2 \neq \lambda$.*

For every $u \in \Omega_0^M$ there is a "dual" tree $B_\Lambda(Q_z(u))$ rooted at $Q_z(u)$. Since q_x, Q_y, Q_z freely generate Λ, the orbit $\Lambda \cdot u$ is the disjoint union of two binary trees

$$B_\Lambda(u) \amalg B_\Lambda(Q_z(u))$$

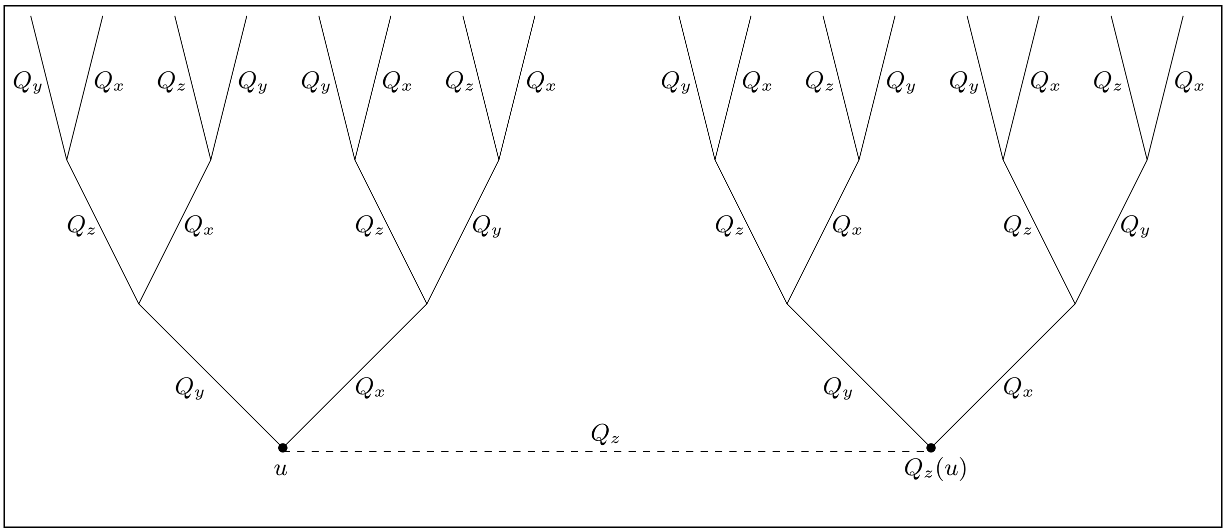


Figure 4: The binary trees rooted at u and $Q_z(u)$

4.5. **The τ-function.**

Proposition 4.4. *For every $u \in \Omega_0^M$, the function*

$$\tau : \mathcal{X}_{1,1} \longrightarrow \mathbb{R}$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto -z\bar{z}$$

does not decrease along the depth levels of $B_\Lambda(u)$. More precisely, if v is a node of $B_\Lambda(u)$, and v_l, v_r are its left and right descendents respectively, then

$$\tau(v) \leq \tau(v_l), \quad \tau(v) \leq \tau(v_r)$$

with at least one of the inequalities strict.

Proof. Fix $u_0 = (x_0, y_0, z_0) \in \Omega_0^M$ and let $u_1 = (x_1, y_1, z_1) = Q_x(u_0)$. By definition $z_0 < -2$ and $\bar{z} < -2$ and therefore

$$\tau(u_0) = -z\bar{z} < 0$$

Then, $x_0 y_0 = -(z_0 + \bar{z}_0)$ implies

$$\bar{z}_1 = -y_0^2 z_0 + x_0 y_0 - z_0 = -z_0(y^2 + 2) - \bar{z}_0 > -2z_0 - \bar{z}_0 > 0 \tag{4.5}$$

and consequently:

$$\tau(u_1) > 0 > \tau(u_0)$$

Similar estimates apply verbatim in the case when $u_1 = Q_y(u_0)$ and therefore τ is strictly increasing at the root of $B_\Lambda(u_0)$. However, the same type of algebraic argument does not extend to other nodes of $B_\Lambda(u_0)$ directly. In order to proceed with the induction step we analyse the orbits of Λ geometrically.

Let $L_{x-x_0}, L_{y-y_0}, L_{z-z_0}$ denote the level sets of the x-, y-, and z-coordinate functions respectively. For example

$$L_{x-x_0} = \{(x, y, z) \in \mathbb{R}^3 \mid x - x_0 = 0\}$$

Consider the conic $h_{c,z_0} = L_{z-z_0} \cap \kappa^{-1}(c)$ and its projection $\Pi(h_{c,z_0})$ to the xy-plane:

$$\Pi(h_{c,z_0}) = \{(x, y) \in \mathbb{R}^2 \mid k(x, y, z_0) = c\}$$

Lemma 4.5. *Let $c < 2$ and $|z| > 2$. Then $h_{c,z}$ is a hyperbola with principal axes*

$$a_1 := \{(x, y) \mid x + y = 0\}, \quad a_2 := \{(x, y) \mid x - y = 0\} \tag{4.6}$$

and asymptotes

$$\begin{aligned} l_1 &:= \{(x, y) \mid y = \frac{z - \sqrt{z^2 - 4}}{2} x\} \\ l_2 &:= \{(x, y) \mid y = \frac{z + \sqrt{z^2 - 4}}{2} x\}. \end{aligned} \tag{4.7}$$

When $z < -2$, the asymptotes l_1 and l_2 lie entirely in quadrant II and IV, and

$$h_{c,z} \cap a_1 = \varnothing, \quad h_{c,z} \cap a_2 \neq \varnothing$$

When $z > 2$, the asymptotes l_1 and l_2 lie entirely in quadrant I and III, and

$$h_{c,z} \cap a_1 \neq \varnothing, \quad h_{c,z} \cap a_2 = \varnothing$$

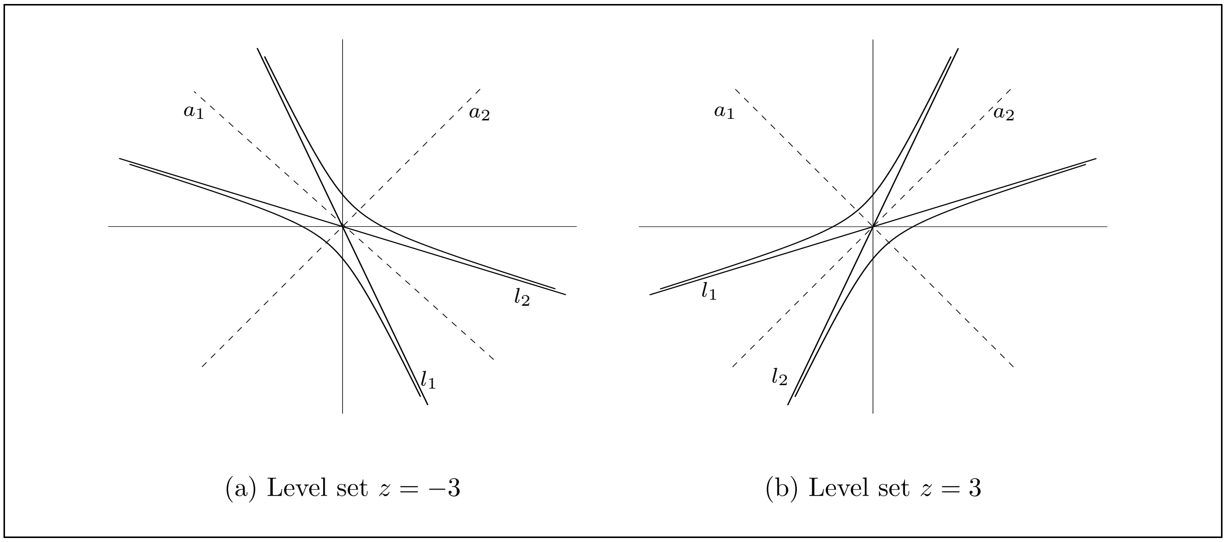


Figure 5: Intersections of $\kappa^{-1}(c)$ with z-coordinate level sets

Proof. Assume $c < 2$ and fix z_0 such that $|z_0| > 2$. The quadratic form

$$S_{z_0}(x, y) = -x^2 - y^2 + z_0 xy$$

has discriminant $D_{z_0} = 1 - z_0^2/4$, which is negative for $|z_0| > 2$. Therefore S_{z_0} is indefinite and its level sets must be hyperbolae. The eigenvalues of S_{z_0} are

$$s_1 = -\frac{z_0}{2} - 1 \quad \text{and} \quad s_2 = \frac{z_0}{2} - 1$$

with eigenvectors

$$\mathbf{e}_1 = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \mathbf{e}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Therefore with respect to the orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2)$, the form S_{z_0} has the following presentation

$$S_{z_0}(\tilde{x}, \tilde{y}) = -(\frac{z_0}{2} + 1)\tilde{x}^2 + (\frac{z_0}{2} - 1)\tilde{y}^2$$

The principal axes a_1 and a_2 of $h_{c,z}$ are, by definition, collinear with $\mathbf{e}_1$ and $\mathbf{e}_2$ respectively, which implies the first part of Lemma 4.5

Since by assumption $z_0^2 - c - 2 > 0$, the equation of $h_{c,z}$

$$S_{z_0}(\tilde{x}, \tilde{y}) + (z_0^2 - c - 2) = 0$$

in terms of $(\mathbf{e}_1, \mathbf{e}_2)$, has the following canonical form

$$\begin{cases} -\dfrac{\tilde{x}^2}{a_c^2(z_0)} + \dfrac{\tilde{y}^2}{b_c^2(z_0)} = 1, & \text{if} \quad z_0 < -2 \\ \dfrac{\tilde{x}^2}{a_c^2(z_0)} - \dfrac{\tilde{y}^2}{b_c^2(z_0)} = 1, & \text{if} \quad z_0 > 2 \end{cases} \tag{4.8}$$

where

$$a_c(z_0) = \sqrt{2\left|\frac{(-z_0^2 + c + 2)}{z_0 + 2}\right|}$$

$$b_c(z_0) = \sqrt{2\left|\frac{(-z_0^2 + c + 2)}{z_0 - 2}\right|}$$

Therefore in $\tilde{x}, \tilde{y}$-coordinates, the asymptotes of $h_c(z_0)$ are

$$\tilde{l}_{1,2} : a_c(z_0)\tilde{y} \mp b_c(z_0)\tilde{x} = 0$$

from which equations (4.7) follow after applying the appropriate coordinate transformation. Similarly, the principal axes of $h_c(z_0)$ are:

$$a_1 : \tilde{y} = 0, \quad a_2 : \tilde{x} = 0$$

implying (4.6). The assertions about the intersections $h_c(z_0) \cap a_1$ and $h_c(z_0) \cap a_2$ follow from (4.8). $\square$

Next consider the restriction of τ to $h_{c,z}$.

Definition. *Let l be a line in $\mathbb{R}^2$ with equation $l(x, y) = 0$. Then l partitions $\mathbb{R}^2$ into half-planes*

$$H_l^+ := \{(x, y) \in \mathbb{R}^2 \mid l(x, y) > 0\}$$

$$H_l^- := \{(x, y) \in \mathbb{R}^2 \mid l(x, y) < 0\}$$

the positive, *and the* negative half-plane associated with l *respectively.*

Lemma 4.6. *Suppose $c < 2$, $|z| > 2$ and let $h_{c,z}$ be as in Lemma 4.5. Let*

$$\mathbf{a} := \begin{cases} a_1, & \text{if } a_1 \cap h_{c,z} \neq \varnothing \\ a_2, & \text{if } a_2 \cap h_{c,z} \neq \varnothing \end{cases}$$

and $H_{\mathbf{a}}^+$ and $H_{\mathbf{a}}^-$ as above. Then

(1) *The restriction $\tau|_{h_{c,z}}$ is invariant with respect to reflections in a_1 and a_2.*
(2) *Let $\{P_1, P_2\} = h_{c,z} \cap \mathbf{a}$. Then $\tau|_{h_{c,z}}$ has global minima at P_1, P_2*
(3) *$\tau|_{h_{c,z}}$ decreases along $h_{c,z} \cap H_{\mathbf{a}}^-$ and increases along $h_{c,z} \cap H_{\mathbf{a}}^+$.*

Proof. With respect to the standard basis in $\mathbb{R}^2$ reflection in a_1 and a_2 respectively are given by matrices

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Each preserves $\bar{z} = -xy - z$ as well as $\tau = -z\bar{z}$.

Next assume $z < -2$. Parametrize $h_{c,z}$ as:

$$h_{c,z}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} -a_c(z)\sinh t \pm b_c(z)\cosh t \\ a_c(z)\sinh t \pm b_c(z)\cosh t \end{bmatrix}, \quad t \in \mathbb{R}. \tag{4.9}$$

For fixed z, the restriction $\tau|_{h_{c,z}}$ is a function of t alone:

$$\tau|_{h_{c,z}}(t) = -z\bar{z} = x^2 + y^2 + c + 2 = b_c^2(z)\cosh^2 t + a_c^2(z)\sinh^2 t + c + 2.$$

The sign of the derivative of τ

$$\frac{d\tau}{dt} = \sinh 2t(a_c^2(z) + b_c^2(z))$$

depends only on the sign of $\sinh 2t$. Thus

$$\frac{d\tau}{dt} \begin{cases} < 0 & \text{if and only if } t < 0 \\ = 0 & \text{if and only if } t = 0 \\ > 0 & \text{if and only if } t > 0 \end{cases}$$

Since

$$h_{c,z} \cap H_{\mathbf{a}}^- = \{h_{c,z}(t) \mid t < 0\}, \quad h_{c,z} \cap H_{\mathbf{a}}^+ = \{h_{c,z}(t) \mid t > 0\}$$

and $\{P_1, P_2\} = h_{c,z}(0)$, the last two parts of Lemma 4.6 follow. A similar argument applies when $z > 2$. $\square$

We complete the proof of Proposition 4.4. Assume $c < 2$ and $|z_0| > 2$. Let $u_0 = (x_0, y_0, z_0)$ lie on h_{c,z_0}. Since Q_x interchanges the two (necessarily real) roots of the quadratic polynomial

$$\kappa(x, y_0, z_0) - c = -x^2 + (y_0 z_0)x - y_0^2 + z_0^2 - 2 - c$$

Q_x acts as a deck transformation of the double covering $h_{c,z_0} \longrightarrow \mathbb{R}$.. Thus $l_x = L_{y-y_0} \cap L_{z-z_0}$ intersects h_{c,z_0} precisely at two points: u_0 and $Q_x(u_0)$. Similarly $l_y = L_{x-x_0} \cap L_{z-z_0}$ intersects h_{c,z_0} at u_0 and $Q_y(u_0)$.

Claim. *The points u_0 and $Q_x(u_0)$ lie on opposite branches of h_{c,z_0}. The points u_0 and $Q_y(u_0)$ lie on opposite branches of h_{c,z_0}.*

The line at infinity l_∞ in the projective completion $\mathbb{RP}^2$ of the affine plane L_{x-x_0} intersects h_{c,z_0} precisely at two points, the points of tangency of $h_{c,z_0}$with its asymptotes l_1 and l_2. Denote these points by P_1^∞ and P_2^∞. The point $O = l_1 \cap l_2$ corresponds to the origin $(0, 0, z_0)$ in L_{z-z_0}. The coordinate axes correspond to lines a_x and a_y respectively, both of which must intersect at O. Let $P_x^\infty = l_\infty \cap a_x$ and $P_y^\infty = l_\infty \cap a_y$. Since in the affine plane h_{c,z_0} intersects both the x- and y-axis for any $|z_0| > 2$, it intersects a_x and a_y in $\mathbb{RP}^2$. Therefore P_x^∞ and P_y^∞ lie in the interior of the segment $P_1^\infty P_2^\infty$ (see Figure 6). l_∞ partitions h_{c,z_0} into two disjoint arcs corresponding to the two branches of h_{c,z_0} in the affine plane. A line in $\mathbb{R}^2$ intersects both branches of h_{c,z_0} if and only if the corresponding line in $\mathbb{RP}^2$ intersects l_∞ in the interior of the segment $P_1^\infty P_2^\infty$. But u_0 and $Q_x(u_0)$ span a line parallel to the x-axis, so its corresponding line in $\mathbb{RP}^2$ intersects a_x at P_x^∞. Similarly, u_0 and $Q_y(u_0)$ span a line in $\mathbb{RP}^2$ intersecting a_y at P_y^∞. The proof of the claim is complete..

Therefore the action of

$$\Lambda_{x,y} = \langle Q_x, Q_y \rangle \subset \Lambda$$

on the plane $L_{z-z_0} \subset \mathcal{X}_{1,1}$ reduces to a linear action of the infinite dihedral group on hyperbolae h_{c,z_0} (see Figure 7).

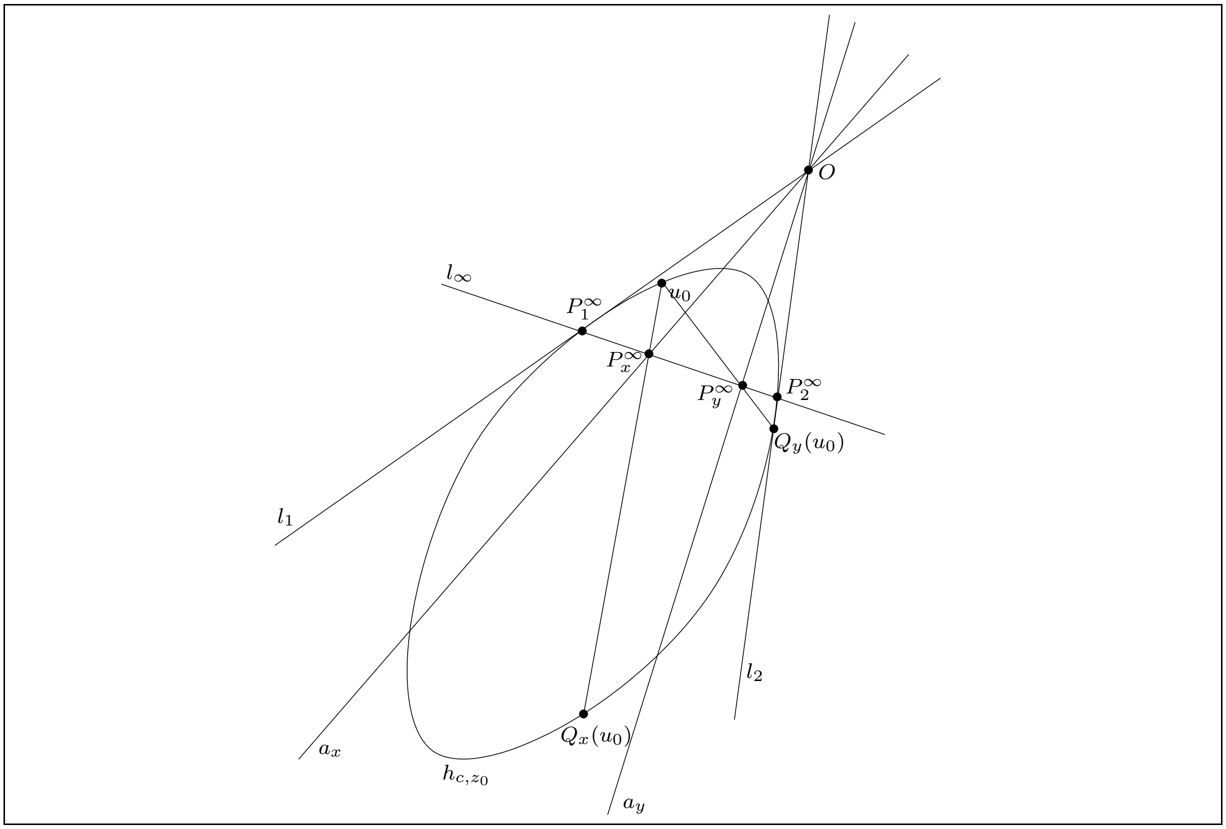


Figure 6: Projective model of h_{c,z_0} and associated objects

Definition. *We call the sets*

$$\mathcal{O}_x(u) = \{u, Q_x(u), Q_yQ_x(u) \ldots\}$$
$$\mathcal{O}_y(u) = \{u, Q_y(u), Q_xQ_y(u) \ldots\}$$

respectively the x-forward *and the* y-forward $\Lambda_{x,y}$-orbit *of* u.

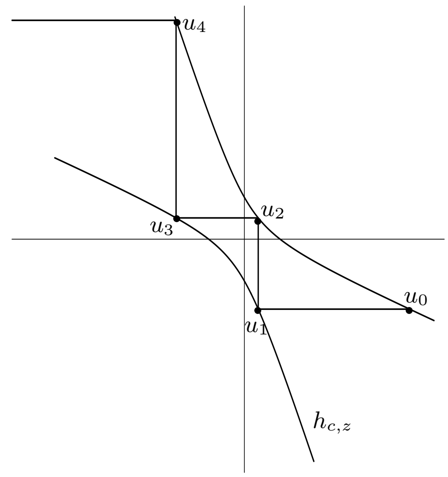


Figure 7: Points on the $\Lambda_{x,y}$-orbit: u_0, $u_1 = Q_x(u_0)$, $u_2 = Q_y(u_1)$, etc

Lemma 4.6 implies that points on $h_{c,z}$ may be partitioned into four types

(1) type `(+,+)`: points $u \in h_{c,z}$ such that

$$\tau(Q_x(u)) - \tau(u) > 0, \quad \tau(Q_y(u)) - \tau(u) > 0$$

(2) type `(+,-)`: points $u \in h_{c,z}$ such that

$$\tau(Q_x(u)) - \tau(u) > 0, \quad \tau(Q_y(u)) - \tau(u) < 0$$

(3) type `(-,+)`: points $u \in h_{c,z}$ such that

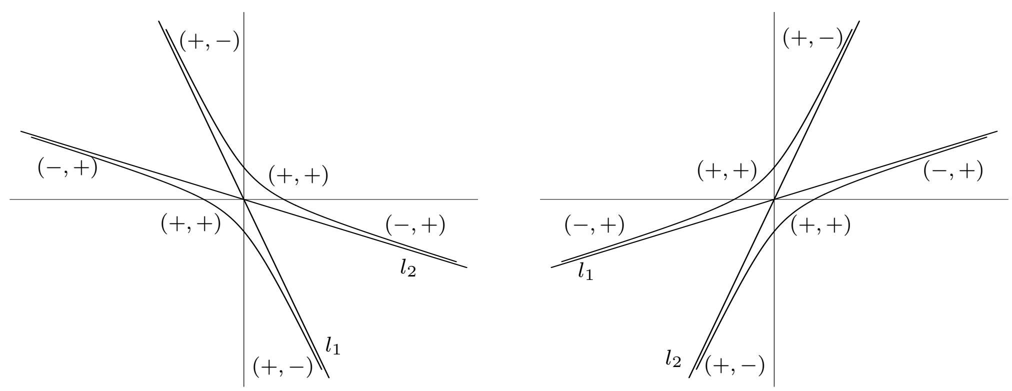
$$\tau(Q_x(u)) - \tau(u) < 0, \quad \tau(Q_y(u)) - \tau(u) > 0$$

(4) type `(0|0)`: points $u \in h_{c,z}$ such that

$$\text{either} \quad \tau(Q_x(u)) - \tau(u) = 0, \quad \text{or} \quad \tau(Q_y(u)) - \tau(u) = 0$$

Let $\mathcal{T}$ denote the set $\{$`(+,+)`, `(+,-)`, `(-,+)`, `(0|0)`$\}$ of point types. Define a function $\theta : h_{c,z} \longrightarrow \mathcal{T}$ mapping each $u \in h_{c,z}$ to the element of $\mathcal{T}$ representing the type of u. We call θ the *type assignment function* on $h_{c,z}$.

Lemmas 4.5,4.6 provide a classification for the subsets of points of each type along $h_{c,z}$. Exactly four points have type `(0|0)`: the x- and the y-intercepts of $h_{c,z}$. Let $V_{l_i,x}$ (respectively $V_{l_i,y}$) denote the double cone spanned by the asymptote l_i and the x-axis (respectively y-axis)[3]. Assume $z < -2$. Then all points of type (+,+) lie in quadrant I and III. Points of type (+,-) and (-,+) lie inside $V_{l_1,y}$ and $V_{l_2,x}$ respectively. Assume $z > 2$. Then all points of type (+,+) lie in quadrant II and IV. Points of type (+,-) and (-,+) lie inside $V_{l_2,y}$ and $V_{l_1,x}$ respectively. Clearly, reflection in the principal axes a_1 and a_2 interchages the (+,-) and (-,+) types and leaves the (+,+) type invariant. A diagram representing points of different types is shown on Figure 8(a) for the case $z < -2$ and on Figure 8(b) for the case $z > 2$.



(a) A z-level set when $z < -2$ (b) A z-level set when $z > 2$

Figure 8: Classification of Points With Respect to the Monotonicity of τ

Lemma 4.2 implies that Ω_0^M lies in the cylinder

$$x^2 + y^2 < -c - 6, \ z \in \mathbb{R}$$

Therefore, if for some $z < -2$ and $c < -14$,

$$L_z \cap \Omega_0^M \cap \kappa^{-1}(c) \neq \emptyset,$$

then $L_z \cap \Omega_0^M \cap \kappa^{-1}(c)$ consists of two arcs of $h_{c,z}$ inside the disk

$$D_c^\Omega : \quad x^2 + y^2 < -c - 6.$$

[3]There are, naturally, two double cones associated with a pair of intersecting lines; we choose the one with the smaller cone angle

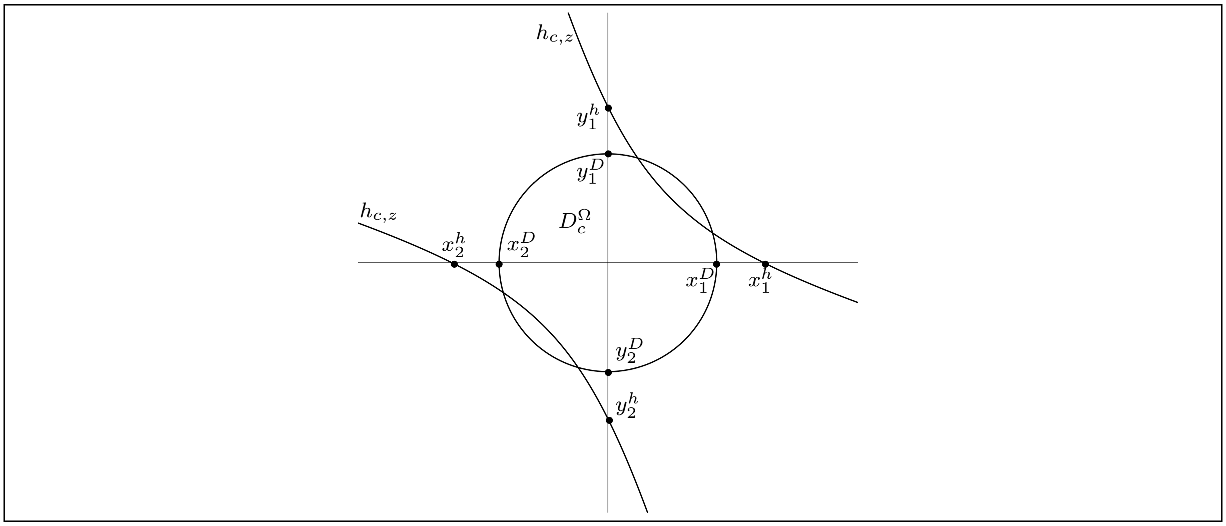


Figure 9: The set $L_z \cap \Omega_0^M \cap \kappa^{-1}(c)$ inside D_c^Ω

The x- and the y-intercepts of $h_{c,z}$ are:

$$x_{1,2}^h = y_{1,2}^h = \pm\sqrt{z^2 - c - 2}$$

while the values of the x and the y-intercepts of ∂D_c^Ω are:

$$x_{1,2}^D = y_{1,2}^D = \pm\sqrt{-c-6}$$

Since $\sqrt{z^2 - c - 2} > \sqrt{-c-6}$, the arcs $h_{c,z} \cap D_c^\Omega$ do not intersect the coordinate axes. Therefore all points in $h_{c,z} \cap D_c^\Omega$ have type $\texttt{(+,+)}$, and consequently

$$\tau(u) < \tau(Q_x(u)), \quad \tau(u) < \tau(Q_y(u)), \quad \forall u \in h_{c,z} \cap D_c^\Omega$$

Thus, when $u \in \Omega_0^M \cap \kappa^{-1}(c)$, τ strictly increases at the initial point of each orbit $\mathcal{O}_x(u)$ and $\mathcal{O}_y(u)$. (Compare Proposition 4.4.) Furthermore suppose that $u \in h_{c,z}$ satisfies

$$\theta(Q_x(u)) \neq \texttt{(+,+)} \quad \text{and} \quad \theta(Q_y(u)) \neq \texttt{(+,+)}.$$

Then

$$\theta(u) = \texttt{(-,+)} \Rightarrow \theta(Q_x(u)) = \theta(Q_y(u)) = \texttt{(+,-)}$$

$$\theta(u) = \texttt{(+,-)} \Rightarrow \theta(Q_x(u)) = \theta(Q_y(u)) = \texttt{(-,+)}$$

Therefore, when $u \in \Omega_0^M \cap \kappa^{-1}(c)$ the image of $\mathcal{O}_x(u)$ under θ is the sequence

$$\texttt{(+,+)}, \texttt{(-,+)}, \texttt{(+,-)}, \texttt{(-,+)}, \ldots.$$

The image of $\mathcal{O}_y(u)$ under θ is the sequence

$$\texttt{(+,+)},\texttt{(+,-)},\texttt{(-,+)},\texttt{(+,-)}\ldots$$

Consequently, τ strictly increases along both $\mathcal{O}_x(u)$ and $\mathcal{O}_y(u)$.

We return to the the proof of Proposition 4.4. Let v be an arbitrary node of $B_\Lambda(u)$. Let

$$P_{u,v} = \{v_0 = u, v_1, v_2, \ldots, v_N = v\}$$

be the (unique) path from u to v in $B_\Lambda(u)$. Partition $P_{u,v}$ into subsets

$$U_0 = \{v_{i_0} = v_0, \ldots, v_{i_1}\}, U_1 = \{v_{i_1+1}, \ldots, v_{i_2}\}, \ldots, U_m = \{v_{i_m+1}, \ldots, v_{i_{m+1}} = v_N\}$$

such that $v_{i_j+1} = Q_z(v_{i_j})$ and U_j is a subset of either $\mathcal{O}_x(v_{i_j+1})$, or $\mathcal{O}_y(v_{i_j+1})$, for each $j = 0, \ldots, m$. Since either $U_0 \subset \mathcal{O}_x(u)$, or $U_0 \subset \mathcal{O}_y(u)$, τ is strictly increasing along U_0. Furthermore, since τ is Q_z-invariant,

$$\tau(v_{i_j}) = \tau(v_{i_j+1})$$

for each $j = 0, \ldots, m$. Also, by assumption $z(v_{i_1}) = z(v_0) < -2$, and $\bar{z}(v_{i_1}) = \bar{z}(v_0) > 2$. Consequently

$$z(v_{i_1+1}) = z(Q_z(v_{i_1})) = \bar{z}(v_{i_1}) > 2$$

Recall that the hyperbola $h_c(z_{i+1})$ has the same principal axes as $h_c(z_i)$, as prescribed by Lemma 4.5

$$a_1 : \ x - y = 0$$

$$a_2 : \ x + y = 0$$

However the intersection properties of a_j with $h_c(z_i)$ and $h_c(z_{i+1})$ are different, namely:

$$a_1 \cap h_c(z_i) \neq \varnothing, \quad a_2 \cap h_c(z_i) = \varnothing$$

while

$$a_1 \cap h_c(z_{i+1}) = \varnothing, \quad a_2 \cap h_c(z_{i+1}) \neq \varnothing$$

Thus all points of type `(+,+)` on $h_c(v_{i_1+1})$ lie in quadrant II and IV. Recall that v_{i_1} is a point of type either `(+,-)` or `(-,+)` and hence lies in quadrant II or IV. But then v_{i_1+1}, which is the image of v_{i_1} under Q_z, must also lie in quadrant II or IV and therefore it must be of type (+,+). Hence τ must be strictly increasing along $\mathcal{O}_x(v_{i_1+1})$ and $\mathcal{O}_y(v_{i_1+1})$, and therefore along U_1. Similar argument applies to U_j, for $1 < j \leq m$. The proof of Proposition 4.4 is complete. $\square$

4.6. **Growth of the τ function.** So far we have shown that for each $u \in \Omega_0^M$, the τ function is non-decreasing along the path $P_{u,v}$ from u to an arbitrary node v in $B_\Lambda(u)$. Moreover, τ is strictly increasing along $P_{u,v}$ except for a (possibly empty) subset of nodes at which τ is constant. Next, we estimate the variation of τ among any pair of characters

$$(w_0, w_1) \in \Delta_c := \{(w_0, w_1) \in \kappa^{-1}(c) \times \kappa^{-1}(c) \mid w_1 = \lambda w_0,\ \lambda \in \{Q_x, Q_y\}\}$$

such that $|z(w_0)| > 2$.

Proposition 4.7. *Suppose $c < 2$. Let $w_0 = (x_0, y_0, z_0) \in \kappa^{-1}(c)$ be such that $|z_0| > 2$. Then*

(1) *if $w_1 = (x_1, y_0, z_0) = Q_x(w_0)$*

$$|\tau(w_1) - \tau(w_0)| \geq |y_0 z_0| \max\left(|y_0|\sqrt{z_0^2 - 4}, 2\sqrt{z_0^2 - c - 2}\right)$$

(2) *if $w_1 = (x_0, y_1, z_0) = Q_y(w_0)$*

$$|\tau(w_1) - \tau(w_0)| \geq |x_0 z_0| \max\left(|x_0|\sqrt{z_0^2 - 4}, 2\sqrt{z_0^2 - c - 2}\right)$$

Proof. Since $|z_0| > 2$, the z_0-level set $L_{z-z_0} \cap \kappa^{-1}(c)$ is the hyperbola h_{c,z_0}. We prove case (2) first. Let A_{l_1} and A_{l_2} be the intersection points of the line $l_x : x = x_0$ with l_1 and l_2 respectively. Clearly

$$|y_1 - y_0| > \mathrm{d}(A_{l_1}, A_{l_2}) = \left|\frac{z_0 + \sqrt{z_0^2 - 4}}{2} x_0 - \frac{z_0 - \sqrt{z_0^2 - 4}}{2} x_0\right| = |x_0|\sqrt{z_0^2 - 4}$$

where $\mathrm{d}(\cdot\ ,\cdot)$ denotes the (Euclidean) distance function in $\mathbb{R}^2$. On the other hand, since the minimum vertical distance between pairs of points on h_{c,z_0} is achieved at $x_0 = 0$

$$|y_1 - y_0| > \min_{w \in h_{c,z_0}} |y(w) - y(Q_y(w))| = 2\sqrt{z_0^2 - c - 2}$$

Therefore

$$|y_1 - y_0| \geq \max\left(|x_0|\sqrt{z_0^2 - 4}, 2\sqrt{z_0^2 - c - 2}\right) \tag{4.10}$$

By the defintion of τ

$$\begin{aligned}|\tau(w_1) - \tau(w_0)| =& |z_0||(\bar{z}_1 - \bar{z}_0)| \\ =& |z_0||y_0||x_1 - x_0|\end{aligned}$$

which together with inequality (4.10) implies part (2) of Proposition 4.7. Similar argument applies when $w_1 = Q_x(u)$, in which case

$$|x_1 - x_0| \geq \max\left(|y_0|\sqrt{z_0^2 - 4}, 2\sqrt{z_0^2 - c - 2}\right) \tag{4.11}$$

and

$$\begin{aligned}|\tau(w_1) - \tau(w_0)| =& |z_0||(\bar{z}_1 - \bar{z}_0)| \\ =& |z_0||x_0||y_1 - y_0|\end{aligned}$$

□

The assumption that $|z| > 2$ and $c < 2$ eliminates z from the estimates for the increment of τ.

Corollary 4.8. *Let $c < 2$ be fixed, and let*

$$(w_0, w_1) \in \Delta_c$$

such that $|z(w_0)| > 2$, and $|z(w_1)| > 2$. Then the variation of τ is bounded from below by a quantity depending on x_0 or y_0 alone. In particular

$$|\tau(Q_x(w_0)) - \tau(w_0)| \geq 16|y_0|\sqrt{2 - c} \tag{4.12}$$

and

$$|\tau(Q_y(w_1)) - \tau(w_0)| \geq 16|x_0|\sqrt{2 - c} \tag{4.13}$$

4.7. **τ-reduction for Fricke-space characters.** We have seen so far that the τ-function is non-decreasing along the levels of $B_\Lambda(u)$ for any

$$u = (x, y, z) \in \Omega_0^M \cap \kappa^{-1}(c).$$

Also for every $w \in B_\Lambda(u) \cap \Delta_c$, and fixed c, the increment is bounded from below by a quantity depending on $|x|$ or $|y|$ alone. Since $z < -2$ and $\bar{z} < -2$,

$$\begin{aligned}z\bar{z} &= -x^2 - y^2 - 2 - c > 4 \\ z + \bar{z} &= -xy < -4.\end{aligned}$$

The first inequality implies that $|x| < \sqrt{-c-6}$ and $|y| < \sqrt{-c-6}$, which together with the second inequality yield

$$\begin{aligned} |x| &> \frac{4}{\sqrt{-c-6}} \\ |y| &> \frac{4}{\sqrt{-c-6}}. \end{aligned}$$

Thus if $\kappa^{-1}(c) \cap \Omega_0^M \neq \varnothing$ (or equivalently, if $c < -14$), there is a constant T_c such that

$$|\tau(w_1) - \tau(w_0)| \geq T_c \tag{4.14}$$

for every $u \in \Omega_0^M$ and a pair w_0, w_1 of successive nodes in $B_\Lambda(u) \cap \Delta_c$. The uniform lower bound for the increment of τ along the depth levels of $B_\Lambda(u)$ guarantees the following. The path $P_{v,u}$ from an arbitrary character v in $\Lambda \cdot \Omega_0^M$ to the unique character $u \in \Omega_0^M$ such that $v = B_\Lambda(u)$, can be recovered in finite number of steps via reduction of τ by a definite amount at each step. More precisely,

Lemma 4.9. *Let $c < -14$ and let $v \in \kappa^{-1}(c)$. Assume $v \in B_\Lambda(u)$ for some $u \in \Omega_0^M$. Then*

$$P_{v,u} = \{v_0 = v, v_1, \ldots, v_n = u\}$$

where $\{v_i\}_{i=0}^n$ is the (unique) sequence such that for each $0 \leq i \leq n-1$

$$v_{i+1} = \begin{cases} Q_x(v_i) & \textit{if} \quad \tau(Q_x(v_i)) < \tau(v_i) \\ Q_y(v_i) & \textit{if} \quad \tau(Q_y(v_i)) < \tau(v_i) \\ Q_z(v_i) & \textit{otherwise} \end{cases}$$

Moreover n depends only on c and v.

4.8. **τ-reduction for arbitrary characters.** This type of reduction process extends to arbitrary characters

$$v \in \mathcal{H}_c := \kappa^{-1}(c) \cap \big(\mathcal{X}_{1,1} - \Omega_0^M \cup \mathcal{E}_c \cup Q_z(\mathcal{E}_c)\big)$$

By definition, $v \in \mathcal{H}_c$ implies that $|z(v)| > 2$ and $|\bar{z}(v)| > 2$. For such v, the proof of Proposition 4.4 produces a sequence

$$\mathfrak{m}_c(v): \quad v_0 = v, v_1, \ldots, v_n, \ldots$$

such that for each $i = 1, 2, \ldots$:

- ▷ $v_{i+1} = \lambda_i v_i$, where $\lambda_i \in \{Q_x, Q_y, Q_z\}$
- ▷ $\lambda_i \neq \lambda_{i+1}$
- ▷ $\tau(v_{i+1}) \leq \tau(v_i)$

Definition. *We call such a sequence* τ-minimizing. *A τ-minimizing sequence is said to* terminate, *if there exists n, such that one of the following occurs*

- ▷ $v_n \in \mathcal{E}_c$, *or*
- ▷ $v_n \in \Omega_0^M$, *or*
- ▷ $v_n \in \Sigma_0^c := \{(x, y, z) \in \kappa^{-1}(c) \mid x = 0, \ \mathit{or}\, y = 0\}$

The element v_{n_t} indexed by the smallest such n, will be called a terminator.

The sequence $\mathfrak{m}_c(v)$ can be constructed as follows. Let $v_i = (x_i, y_i, z_i)$, and assume $|z_i| > 2$ and $|\bar{z}_i| > 2$. Then v_i lies on the hyperbola $h_c(z_i)$. There are four possibilities based on the type of v_i

(1) v_i is of type `(-,+)`; then $v_{i+1} := Q_x(v_i)$
(2) v_i is of type `(+,-)`; then $v_{i+1} := Q_y(v_i)$
(3) v_i is of type `(+,+)`; then $v_{i+1} := Q_z(v_i)$
(4) v_i is of type `(0|0)`; then $v_i \in \Sigma_0^c$ and is therfore a terminator.

Proposition 4.10. *After a finite number of steps, every τ-minimizing sequence terminates.*

Proof. Clearly, this is true if $v \in \Omega^M$ (Lemma 4.9). Thus assume that $v \notin \Omega^M$.

Recall that at each v_i where τ is strictly decreasing, Proposition 4.7 provides a lower bound on $|\tau(v_i) - \tau(v_{i+1})|$ that depends on x_i or y_i alone. As long as the x_i's and the y_i's do not accumulate at 0, the decrement of τ by a definite amount at each step guarantees that after finitely many steps there will be an element v_n such that $\tau(v_n) < 4$ and hence

v_n will lie in one of Σ_0^c, $\mathcal{E}_c$, or $Q_z(\mathcal{E}_c)$. Thus it suffices to show that if $v \notin \Lambda \cdot \Sigma_0^c$, then the x and the y coordinates of characters on the Λ-orbit of v but outside of $\mathcal{E}_c$ are bounded away from 0 by some positive constant depending on v.

Lemma 4.11. *For every $v \notin \Omega^M \cup \mathcal{E}_c \cup \Lambda \cdot \Sigma_0^c$ there exists positive constants $\varepsilon_x(v)$ and $\varepsilon_y(v)$ such that*

$$\inf_{\lambda\in\Lambda}\{x(\lambda v) \mid \lambda v \notin \mathcal{E}_c\} = \varepsilon_x(v) > 0$$

$$\inf_{\lambda\in\Lambda}\{y(\lambda v) \mid \lambda v \notin \mathcal{E}_c\} = \varepsilon_y(v) > 0$$

Proof. We prove this by contradiction. Suppose no such constants exist. If $|z| > 2$ and $|\bar{z}| > 2$, but $h_{c,z} \cap \Omega_0^M = \varnothing$, the set of points on $h_{c,z}$ of type `(+,+)` partitions into three disjoint subsets

$$\bar{Z}_x = \{u \in h_{c,z} \mid Q_z(u) \in (-,+)\}$$

$$\bar{Z}_y = \{u \in h_{c,z} \mid Q_z(u) \in (+,-)\}$$

$$\bar{Z}_e = \{u \in h_{c,z} \mid -2 < z(Q_z(u)) < 2\}$$

Definition. *Points in $\bar{Z}_x$ (respectively $\bar{Z}_y$, $\bar{Z}_e$) will be called* points of type $\bar{z}_{(-+)}$ *(respectively $\bar{z}_{(+-)}$, $\bar{z}_{(e)}$). Notice that by definition $\bar{Z}_e \subset \mathcal{E}_c$.*

Fix $c < 2$ and consider the projection of $\kappa^{-1}(c)$ onto the xy-plane

$$\Pi : \kappa^{-1}(c) \ni (x,y,z) \longmapsto (x,y) \in \mathbb{R}^2.$$

The level set $L_{z+2} \cap \kappa^{-1}(c)$ projects onto the pair of lines

$$h^{\pm}_{-2} : x + y \pm \sqrt{-c+2} = 0$$

and the set $L_{z-2} \cap \kappa^{-1}(c)$ projects onto the pair of lines

$$h^{\pm}_{+2} : x - y \pm \sqrt{-c+2} = 0.$$

The pair $h^{\pm}_{-2}$ partitions the xy-plane into two regions:

$$H^{-}_{-2} = \{(x,y) \in \mathbb{R}^2 \mid (x + y + \sqrt{-c+2})(x + y - \sqrt{-c+2}) < 0\}$$

and

$$H^+_{-2} = \{(x,y) \in \mathbb{R}^2 \mid (x+y+\sqrt{-c+2})(x+y-\sqrt{-c+2}) > 0\}$$

The first one, H^-_{-2} is the "strip" bounded by $h^\pm_{-2}$, and the second one, H^+_{-2} is the

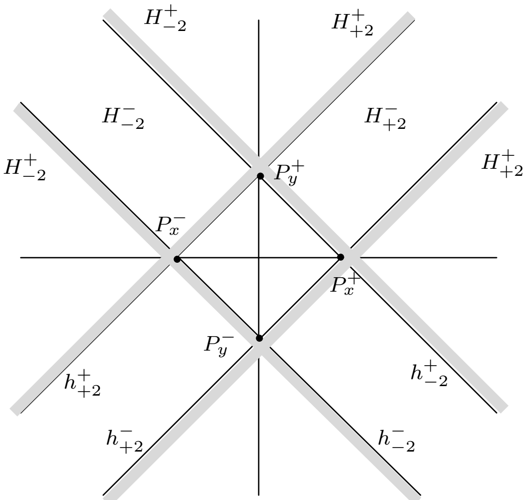


Figure 10: Lines, cones, and half-planes

complement of H^-_{-2} in $\mathbb{R}^2$ (see Figure 10). Similarly, the pair $h^\pm_{+2}$ partitions the plane into regions

$$H^-_{+2} = \{(x,y) \in \mathbb{R}^2 \mid (x-y+\sqrt{-c+2})(x-y-\sqrt{-c+2}) < 0\}$$

and

$$H^+_{+2} = \{(x,y) \in \mathbb{R}^2 \mid (x-y+\sqrt{-c+2})(x-y-\sqrt{-c+2}) > 0\}$$

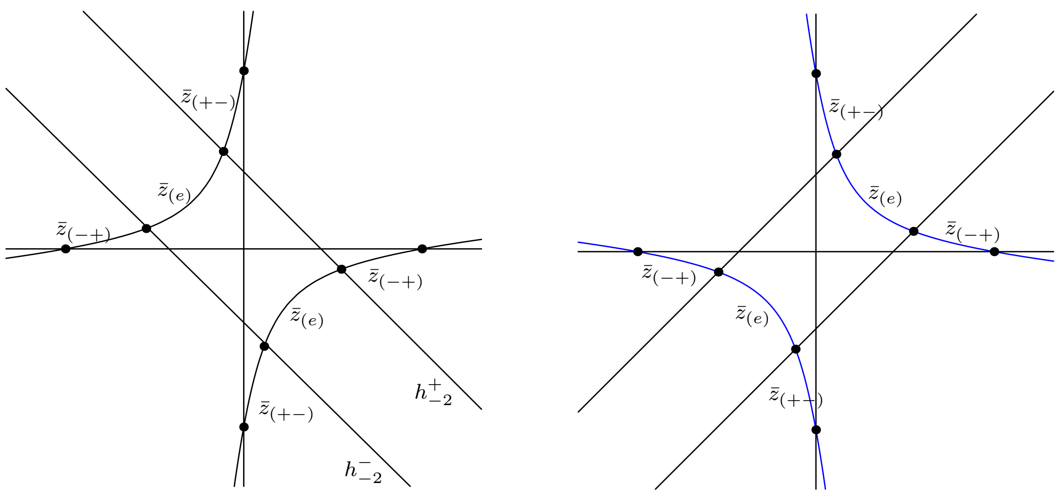
When $z_0 < -2$ (respectively $z_0 > 2$), the level sets $L_{z-z_0} \cap \kappa^{-1}(c) = h_{c,z_0}$ project to a family of hyperbolae contained in H^+_{-2} (respectively H^+_{+2}). The regions H^-_{-2} and H^-_{+2} contain a family of ellipses, which are the projections of the level sets $L_{z-z_0} \cap \kappa^{-1}(c)$ for $-2 < z_0 < 2$. Hence the set of $\bar{z}_{(e)}$ points projects onto the union

$$H^-_{-2} \cup H^-_{+2}$$

If $|z_0| > 2$, the set of $\bar{z}_{(-+)}$ and $\bar{z}_{(+-)}$points on h_{c,z_0} projects onto

$$H^+_{-2} \cap H^+_{+2}$$

which is a union of four cones with vertices $P_x^{\pm} = (\pm\sqrt{-c+2}, 0)$ and $P_y^{\pm} = (0, \pm\sqrt{-c+2})$ (see Figure 10 and 11). In particular, the set of $\bar{z}_{(-+)}$ points projects onto the cones with



(a) for $z < -2$ (b) for $z > 2$

Figure 11: Points of type $\bar{z}_{(-+)}$, $\bar{z}_{(+-)}$, and $\bar{z}_{(e)}$

vertices $P_x^{\pm}$. The set of $\bar{z}_{(+-)}$ points projects onto the cones with vertices $P_y^{\pm}$ (see Figure 12). Observe that the strip

$$F_x = \mathbb{R} \times (-\sqrt{-c+2}, \sqrt{-c+2})$$

does not contain any projections of $\bar{z}_{(+-)}$ points. Similarly, the strip

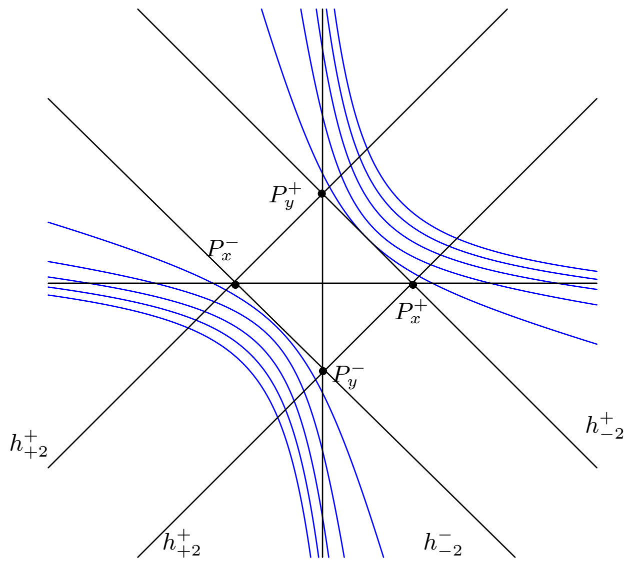
$$F_y = (-\sqrt{-c+2}, \sqrt{-c+2}) \times \mathbb{R}$$

does not contain any projections of $\bar{z}_{(-+)}$ points. Consequently, if $v_{n_1} \notin \mathcal{E}_c$ of the τ-minimizing sequence $\{v_i\}$ enters the slab

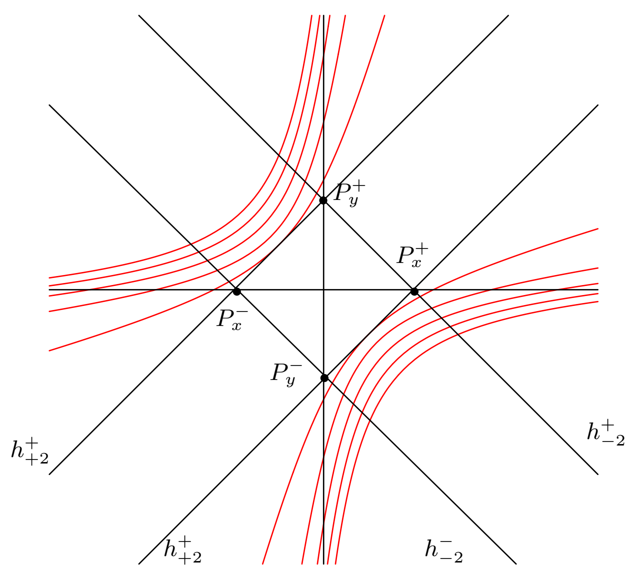
$$F_x \times \mathbb{R}$$

then it must remain on the same y-level set for all $i \geq n_1$. Similarly, if an element $v_{n_2} \notin \mathcal{E}_c$ enters the slab

$$F_y \times \mathbb{R}$$



(a) for $z < -2$



(b) for $z > 2$

Figure 12: Families of hyperbolae as projections of $\kappa^{-1}(c) \cap L_z$

then it must remain on the same x-level set for all $i \geq n_2$. Since we had assumed that either the x or the y-coordinates of characters in $\{v_i\}$ accumulate at 0, there exists $v_n \notin \mathcal{E}_c \cup \Sigma_0^c$, $n > 0$, such that either

$$v_n \in F_x \times \mathbb{R}$$

or

$$v_n \in F_y \times \mathbb{R}$$

Suppose $v_n \in F_y \times \mathbb{R}$. This means that $x(v_i)$ must be constant and $|y(v_i)| > \sqrt{-c+2}$ for all $i > n$, a contradiction. Similarly, $v_n \in F_x \times \mathbb{R}$ implies a contradiction. This concludes the proof of Lemma 4.11 and hence of Proposition 4.10. $\square$

The proof of Lemma 4.11 suggests that if a τ-minimizing sequence enters one of the regions $F_x \times \mathbb{R}$, or $F_y \times \mathbb{R}$, its behavior can be determined explicitly. More precisely, let $\{v_i\}$, be a τ-minimizing sequence.

Definition. *A* terminal x-plane *(respectively y-plane) is a level set L_{x-x_0} (respectively L_{y-y_0}) such that if $v_n \in L_{x-x_0}$ (respectively $v_n \in L_{y-y_0}$) for some n, then $v_i \in L_{x-x_0}$ (respecitvely $v_i \in L_{y-y_0}$), for all $n \leq i \leq n_t$, where n_t is the index of the terminating element of $\{v_i\}$.*

Thus all elements of a τ-minimizing sequence that belong to

$$F_x \times \mathbb{R} \cup F_y \times \mathbb{R}$$

must lie on a terminal plane. Figure 13 shows several elements of the minimizing sequence of the character $u = (-0.2, 12, -10)$ on its terminal x-plane. Successive points are joined by line segments to facilitate visualization.

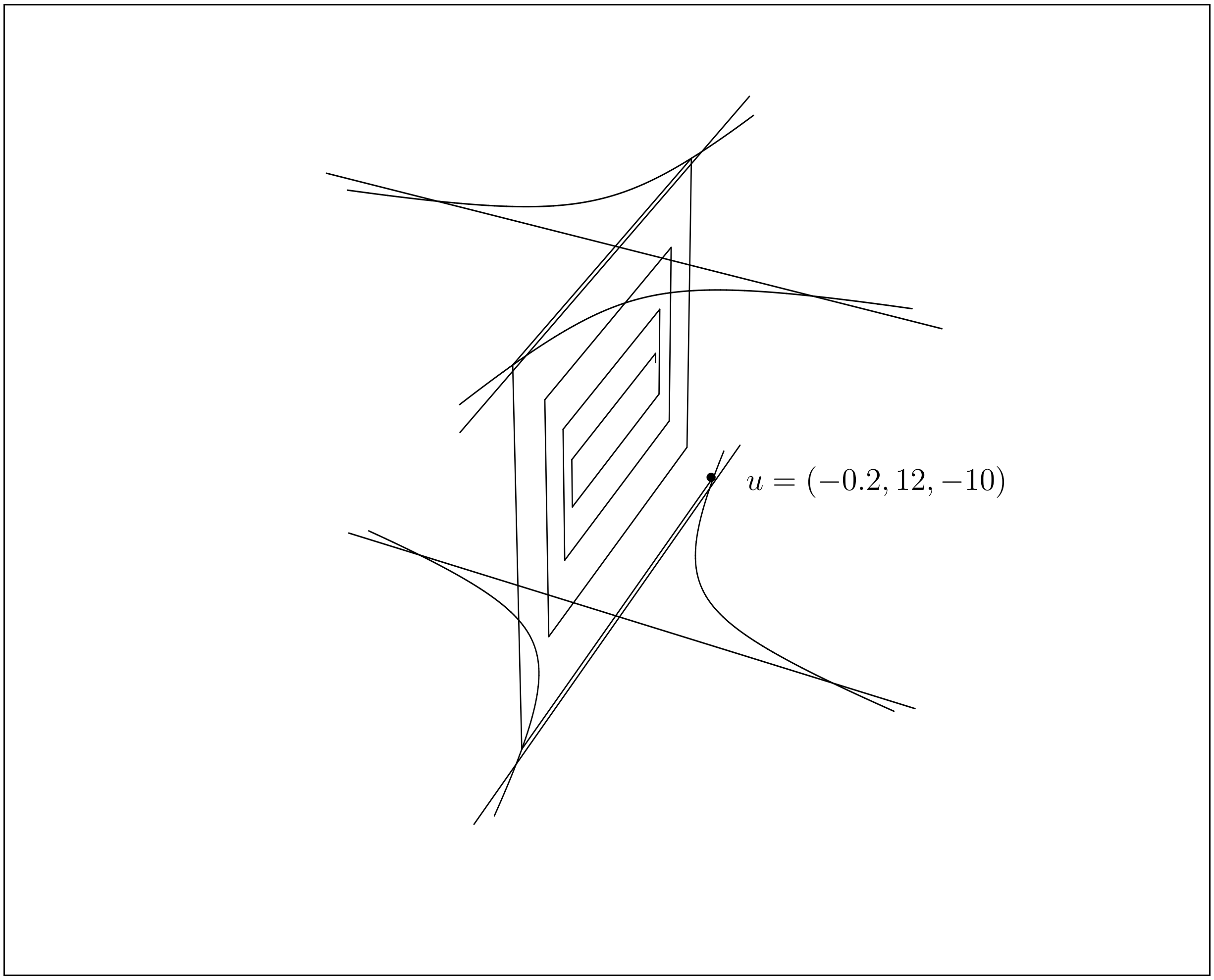


Figure 13: The terminal plane of the character $u = (-0.2, 12, -10)$

4.9. **The τ-Reduction Algorithm.** We use the results obtained so far to construct an algorithm, which implements the method of τ-reduction for distinguishing among characters on $\kappa^{-1}(c)$ when $c < -14$.

Algorithm τ-REDUCTION(u)

Input. A character $u \in \kappa^{-1}(c)$.

Output. A character in Ω_0^M, $\mathcal{E}_c$, or Σ_0^c, which is Γ-equivalent to u.

$u \leftarrow u_0$;

while $|\bar{z}| > 2$ **do**

if x(u) = 0 **or** y(u) = 0 **then** $u \in \Sigma_0^c$ **return** u

if $z(u) < -2$ **and** $\bar{z}(u) < -2$ **then** $u \in \Omega_0^M$ **return** u

if $z(u) > 2$ **and** $\bar{z}(u) > 2$ **then** $\sigma_{xz}(u) \in \Omega_0^M$ **return** $\sigma_{xz}(u)$

if $\tau(Q_x(u)) < \tau(u)$ **then** $u \leftarrow Q_x(u)$

elseif $\tau(Q_y(u)) < \tau(u)$ **then** $u \leftarrow Q_y(u)$

else $u \leftarrow Q_z(u)$

end do

$u \in \mathcal{E}_c$; **return** u.

5. The Action of the Modular Group on Characters

In Chapter 3 we proved that the action of Γ on

$$\Omega^M = \coprod_{\gamma\in\Gamma} \gamma\Omega_0^M$$

is wandering. In this chapter we show that the action of Γ on the complement of Ω^M in $\mathfrak{X}_{1,1}$ is ergodic in the following sense. Recall that the action of Γ induces a measurable equivalence relation "$\sim$", which is ergodic if and only if every function that is constant on equivalence classes is constant almost everywhere. In that context, if every point in a subspace X is Γ-equivalent to a point in a subspace Y, then ergodicity on X is equivalent to ergodicity on Y (regardless of whether Y is invariant or not; compare Goldman [8]). In the previous section we have proved that when $c < 2$ every point on $(\mathfrak{X}_{1,1} - \Omega^M \cup \mathcal{E}_c) \cap \kappa^{-1}(c)$ is Γ-equivalent to a point in $\mathcal{E}_c$. Thus ergodicity on $\mathfrak{X}_{1,1}$ reduces to ergodicity of "$\sim$" on $\mathcal{E}_c$, or equivalently on its "Q_z-dual"

$$\overline{\mathcal{E}}_c = Q_z(\mathcal{E}_c) = \{(x, y, z) \in \kappa^{-1}(c) \mid -2 < z < 2\}$$

Let $u = (x, y, z) \in \overline{\mathcal{E}}_c$. Then $e_c(z) = L_z \cap \kappa^{-1}(c)$ is an ellipse upon which $Q_xQ_y \in \Lambda_{x,y}$ acts by a rotation of angle

$$\alpha = 2\cos^{-1}\frac{z}{2}.$$

(Compare Goldman [6].) For almost every z (namely when $\alpha/(2\pi)$ is irrational), this transformation generates an ergodic action on $e_c(z)$. Thus a function

$$f : \overline{\mathcal{E}}_c \longrightarrow \mathbb{R}$$

that is Λ-invariant, would be constant almost everywhere on each $L_z \cap \kappa^{-1}(c)$ and would therefore depend almost everywhere on z alone. Thus $f(x, y, z) = g(z)$ almost everywhere for some function

$$g: \ [-2,2] \longrightarrow \mathbb{R}$$

It now suffices to eliminate the dependence of g on z. To this end, we parametrize $e_c(z)$ as follows (see Lemma 4.5)

$$e_c(z): \begin{cases} x = \dfrac{\sqrt{2}}{2}(-A\cos\theta + B\sin\theta) \\ y = \dfrac{\sqrt{2}}{2}(\ \ \ A\cos\theta + B\sin\theta) \end{cases}$$

where

$$A = \sqrt{\frac{z^2-c-2}{2+z}}, \quad B = \sqrt{\frac{z^2-c-2}{2-z}}$$

Consequently, for a fixed $z \in (-2,2)$ the restriction of $\bar{z}$ to $e_c(z)$ depends on θ alone

$$\bar{z}_{c,z}(\theta) = \bar{z}|_{e_c(z)} = \frac{z^2-c-2}{2+z}\cos^2\theta - \frac{z^2-c-2}{2-z}\sin^2\theta - z$$

whose extrema are attained at $\theta = k\pi/2$, for $k \in \mathbb{Z}$. The critical values of $\bar{z}_{c,z}(\theta)$ are

$$\begin{aligned} \bar{z}_{\text{odd}} &= -\frac{z^2-c-2}{2-z} - z &= \frac{-2z+c+2}{2-z} &= \quad 2 + \frac{c-2}{2-z} \\ \bar{z}_{\text{even}} &= \quad \frac{z^2-c-2}{2+z} - z &= \frac{-2z-c-2}{2+z} &= -2 - \frac{c-2}{2+z} \end{aligned} \tag{5.1}$$

respectively, depending on the parity of k. We classify the extrema of $\bar{z}_{c,z}$ by analysing the second derivative

$$\bar{z}''_{c,z}(\theta) = -\frac{8(z^2-c-2)}{4-z^2}\cos 2\theta$$

Clearly

$$\bar{z}''_{c,v}\left(\frac{k\pi}{2}\right) \quad \begin{cases} > 0, & k \text{ odd} \\ < 0, & k \text{ even} \end{cases}$$

provided that

$$z^2 - c - 2 > 0 \tag{5.2}$$

Observe that when $c < -2$, the latter is satisfied trivially for any $z \in \mathbb{R}$. On the other hand, when $-2 < c < 2$, inequality (5.2) is satisfied *a fortiori* for all points in

$$\overline{\mathcal{E}}_c = \kappa^{-1}(c) \cap \mathbb{R}^2 \times (-2,2)$$

since for any such point $u = (x,y,z)$, the quadratic form

$$S_z(x,y) = -x^2 - y^2 + xyz$$

is negative definite and therefore the set

$$\kappa^{-1}(c) \cap \mathbb{R}^2 \times (-\sqrt{c+2}, \sqrt{c+2})$$

is empty. Consequently, the critical value $\bar{z}_{\text{odd}}$ is a minimum, and the critical value $\bar{z}_{\text{even}}$ is a maximum of $\bar{z}_{c,z}$ for each $z \in (-2,2)$, such that $L_z \cap \kappa^{-1}(c)$ is non-empty. Clearly, these extrema are global. Thus for any $z \in (-2,2)$

$$\begin{aligned} \bar{z}_{min}(c,z) = \min_{\theta\in\mathbb{R}} \bar{z}_{c,z}(\theta) &= 2 + \frac{c-2}{2-z} \\ \bar{z}_{max}(c,z) = \max_{\theta\in\mathbb{R}} \bar{z}_{c,z}(\theta) &= -2 - \frac{c-2}{2+z} \end{aligned} \tag{5.3}$$

When $c < -14$

$$(-2,2) \subsetneq (\bar{z}_{min}(c,z), \bar{z}_{max}(c,z)) \tag{5.4}$$

for any $z \in (-2,2)$. The cyclic group generated by an irrational rotation of an ellipse acts ergodically and therefore, for almost $z \in (-2,2)$ and every $u \in e_c(z)$, the set

$$\{\bar{z}(\lambda u) \mid \lambda \in \Lambda_{x,y}\}$$

must be dense in $[\bar{z}_{min}(c,z), \bar{z}_{max}(c,z)]$. Therefore the Q_z-invariance of f implies that f must be constant also with respect to z, for almost every $z \in (-2,2)$.

Notice that the set

$$\Sigma^c = \Sigma^c_0 - (\Sigma^c_0 \cap \mathcal{E}_c)$$

is invariant with respect to Q_z, σ_{xz}, σ_{yz} and transpositions of x and y. Clearly, $Q_x\Sigma^c$ and $Q_y\Sigma^c$ do not intersect $\mathcal{E}_c$. Therefore $\Gamma \cdot \Sigma^c \cap \mathcal{E}_c = \varnothing$. Moreover by definition $|z(u)| > 2$ for every $u \in \Sigma^c$ and hence the set $\Lambda_{x,y}$ lies on a hyperbola $h_{c,z}$. Recall that the action of $\Lambda_{x,y}$ on $h_{c,z}$ is wandering and therefore $\Lambda_{x,y}u$ is nowhere dense in $h_{c,z}$. Consequently, the set

$$\coprod_{\gamma\in\Gamma} \gamma\Sigma^c$$

has measure 0 in $\kappa^{-1}(c)$. We now have a complete picture of the Γ-action on characters for the case when $c < -14$.

Theorem 5.1. *Suppose $c < -14$. Then the action of Γ on the level sets $\kappa^{-1}(c)$ is*

(1) *wandering on the set Ω^M of characters of discrete M-embeddings*
(2) *ergodic on the complement of $\Omega^M \cap \kappa^{-1}(c)$*

$$\kappa^{-1}(c) \cap (\mathcal{X}_{1,1} - \Omega^M)$$

Next, we extend this argument to the case when $-14 < c < 2$. In that case the inclusion (5.4) is no longer valid. However

$$\bar{z}_{min}(c,z) < 2, \quad \bar{z}_{max}(c,z) > -2$$

for each $z \in (-2,2)$. Moreover, there exists a countable open cover of the interval $(-2,2)$ by intervals

$$I_n = (\bar{z}_{min}(c,z_n), \bar{z}_{max}(c,z_n)) \cap (-2,2), \quad n = 1,2,\dots$$

(see Figure 14). In each I_n a Q_z-invariant function $g(z)$ is constant almost everywhere

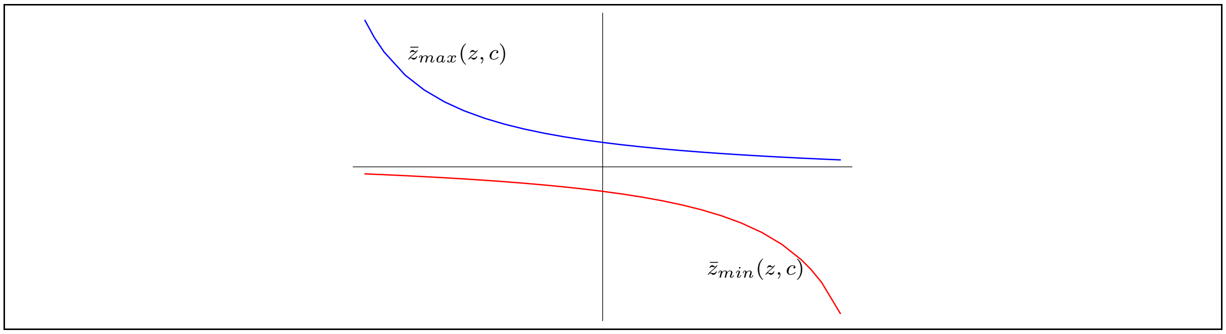


Figure 14: The functions $\bar{z}_{min}(z,c)$ and $\bar{z}_{max}(z,c)$

by the argument above, and the values of $g(z)$ agree by default on the overlaps $I_n \cap I_{n+1}$. Therefore $g(z)$ must be constant almost everywhere in $(-2,2)$.

Corollary 5.2. *For any $c < 2$ the action of Γ on $\kappa^{-1}(c) \cap (\mathfrak{X}_{1,1} - \Omega^M)$ is ergodic.*

We conclude with a remark about the special case $c = -2$. In that case the level set $\kappa^{-1}(c)$ consists of characters $u = (x,y,z)$ that satisfy the equation

$$-x^2 - y^2 + z^2 + xyz = 0$$

The coordinates of such characters are closely related to Markoff triples, which play an important role in Number Theory. In a recent paper, Bowditch ([1]) studies the action of the modular group on complex Markoff triples and proves that the action has dense orbits in a neigborhood of the origin. Thus Bowditch's result, restricted to real characters, is a special case of Corollary 5.2.

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