ENERGY OF TWISTED HARMONIC MAPS OF RIEMANN SURFACES

WILLIAM M. GOLDMAN AND RICHARD A. WENTWORTH

Abstract. The energy of harmonic sections of flat bundles of nonpositively curved (NPC) length spaces over Riemann surfaces is a function on Teichmüller space which is a qualitative invariant of the holonomy representation. Adapting ideas of Sacks-Uhlenbeck, Schoen-Yau and Tromba, we show that the energy function is proper for any convex cocompact representation of the fundamental group. As a corollary, the mapping class group acts properly on the subset of convex cocompact representations.

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Introduction

Let S a closed orientable surface and G a Lie group. This paper discusses an analytic invariant of a representation $\pi_(S) \xrightarrow{\rho} G$, and applies to the action of the mapping class group $\pi_0\mathsf{Diff}^+(S)$ of S on the space of representations $\mathrm{Hom}(\pi_1(S), G)/G$.

We assume G is a reductive real algebraic group with maximal compact subgroup K and symmetric space $X = G/K$. Suppose that ρ is *reductive,* that is, a representation whose image is Zariski dense in a reductive subgroup of G. Then according to Corlette [10], for every conformal structure σ on S, there is a ρ-equivariant twisted harmonic map

$$\widetilde{S} \xrightarrow{f_{\rho,\sigma}} X,$$

which is unique up to isometries of X. In particular its *energy*

$$E_\rho(\sigma) \in \mathbb{R}$$

is well-defined. Letting σ vary over *Teichmüller space* $\mathcal{T}_S$ defines a function

$$\mathcal{T}_S \xrightarrow{E_\rho} \mathbb{R}.$$

The starting point of our paper is the following result:

Theorem A. *Suppose that ρ is convex cocompact. Then E_ρ is a proper function on $\mathcal{T}_S$.*

Recall that a discrete subgroup $\Gamma \subset G$ is *convex cocompact* if there exists a Γ-invariant closed geodesically convex subset $N \subset X$ such that N/Γ is compact. A representation ρ is *convex cocompact* if ρ is an isomorphism of $\pi_1(S)$ onto a convex cocompact discrete subgroup of G.

From this theorem follows the example which motivated this study. Let $\mathfrak{C}$ be the subset of $\mathrm{Hom}(\pi_1(S), G)/G$ consisting of equivalence classes of convex cocompact representations.

Corollary B. *$\pi_0\mathsf{Diff}^+(S)$ acts properly on $\mathfrak{C}$.*

When $G = \mathsf{PSL}(2,\mathbb{C})$, a convex cocompact representation is *quasi-Fuchsian,* that is a discrete embedding whose action on $S^2 = \partial\mathsf{H}^3$ is topologically conjugate to the action of a discrete subgroup of $\mathsf{PSL}(2,\mathbb{R})$. The corollary is just the known fact that $\pi_0\mathsf{Diff}^+(S)$ acts properly on the space $\mathcal{QF}_S$ of quasi-Fuchsian embeddings. Bers's simultaneous uniformization theorem [3] provides a $\pi_0\mathsf{Diff}^+(S)$-equivariant homeomorphism

$$\mathcal{QF}_S \longrightarrow \mathcal{T}_S \times \bar{\mathcal{T}}_S.$$

Properness of the action of $\pi_0\mathsf{Diff}^+(S)$ on $\mathcal{T}_S$ implies properness on on $\mathcal{QF}_S$.

However, as first shown by Kleiner and Leeb [25] (see also Quint [37]), convex cocompactness is highly restrictive, only interesting when G has $\mathbb{R}$-rank one. A more general condition guaranteeing properness of the action of $\pi_0\mathsf{Diff}^+(S)$ is given by the notion of *Anosov representations* introduced by Labourie [28].

We generalize these results in two directions. First, we extend the isometric actions of surface groups on surface groups to *isometric actions on non-positively curved metric spaces* as developed by Korevaar-Schoen [26, 27]. Second, following a suggestion of Bruce Kleiner, we consider embeddings of surface groups onto *normal subgroups* of a convex cocompact group Γ of isometries of an NPC space. The quotient group $Q = \Gamma/\rho(\pi_1(S))$ acts on $\mathcal{T}_S$. Since E_ρ is Q-invariant, it induces a function E'_ρ on $\mathcal{T}_S/Q$ and we show:

Theorem C. *The mapping*

$$\mathcal{T}_S/Q \xrightarrow{E'_\rho} \mathbb{R}$$

is proper.

This generalization was motivated by hyperbolic 3-manifolds fibering over the circle. The hyperbolic 3-manifold determines a representation ρ of the fundamental group of the fiber surface S. Furthermore the monodromy of the fibration determines an automorphism ϕ of $\pi_1(S)$ such that ρ is conjugate to $\rho \circ \phi$.

According to Thurston [44] (see also Otal [35]), Φ is a *pseudo-Anosov* or *hyperbolic* mapping class, and generates a proper $\mathbb{Z}$-action on $\mathcal{T}_S$. In particular every orbit is an infinite discrete subset of $\mathcal{T}_S$. Since E_ρ is ϕ-invariant and constant on each infinite discrete orbit, E_ρ is not proper. Kleiner observed that E_ρ induces a proper map on the cyclic quotient $\mathcal{T}_S/\langle\phi\rangle$.

Properness of the energy function fails for surface group representations containing “accidental parabolics”. Such representations are discrete embeddings mapping some nontrivial simple loop c to a parabolic isometry. One can find a sequence of bounded energy mappings for which the conformal structures σ degenerate as to shorten c, contradicting properness. Using the Marden Tameness Conjecture in Kleinian groups (recently established by Agol [2], Calegari-Gabai [8] and Choi [9]), we obtain a sharp converse to Theorem for discrete embeddings into $\mathsf{PSL}(2,\mathbb{C})$:

Theorem D. *Let $\rho : \pi_1(S) \to \mathsf{PSL}(2, \mathbb{C})$ be a discrete embedding. Then E_ρ is proper if and only if ρ is convex cocompact (that is, quasi-Fuchsian).*

We make no claim to the originality of the results, which can probably be found in the literature, or known to experts. However, we hope that this work will clarify and isolate these ideas and will be useful for researchers in this field.

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Notation and Terminology

If X is a metric space, we denote the distance function by d_X. If c is a curve in a length space X, we denote its length by $L_X(c)$. We denote by $[a]$ the equivalence class of a, in various contexts. We denote the identity transformation by $\mathbf{I}$. We shall sometimes implicitly assume a fixed basepoint in discussing the fundamental group $\pi_1(S)$ and the corresponding universal covering space $\widetilde{S} \longrightarrow S$.

1. Flat bundles and harmonic maps

Let S be a closed oriented surface with $\chi(S) < 0$, and let $\pi_1(S)$ be its fundamental group. Let (X, d) be a complete nonpositively curved length space (an *NPC space*). In this note we discuss an invariant of isometric actions of $\pi_1(S)$ on X.

Choose a universal covering space $\widetilde{S} \longrightarrow S$ with group of deck transformations $\pi_1(S)$. An isometric action of $\pi_1(S)$ on X is a homomorphism $\pi_1(S) \xrightarrow{\rho} G$, where G is the isometry group of X. Such a homomorphism defines a *flat* (G, X)*-bundle* X_ρ over S, whose total space is the quotient $\widetilde{S} \times X$ by the (diagonal) π-action by deck transformations on $\widetilde{S}$ and by ρ on X. Since every flat bundle over a simply connected space is trivial, a section over the universal covering space $\widetilde{S}$ is the graph of a mapping $\widetilde{S} \longrightarrow X$. Sections of the bundle over S correspond to ρ-equivariant mappings

$$\widetilde{S} \xrightarrow{u} X.$$

X is contractible (see, for example, Bridson-Haefliger [5]), and therefore sections always exist.

For us, a *conformal structure* on S will be an *almost complex structures*σ on S, that is, an automorphism of the tangent bundle TS satisfying $\sigma^2 = -\mathbf{I}$. A Riemannian metric g is *in the conformal class of* σ if and only if

$$g(\sigma v_1, \sigma v_2) = g(v_1, v_2)$$

for tangent vectors v_1, v_2.

Choose a conformal structure σ on S. Let $\widetilde{S} \xrightarrow{f} X$ be a continuously differentiable ρ-equivariant mapping. Its differential defines a continuous section df of the vector bundle $T^*\widetilde{S} \otimes f^*TX$, Choose a Riemannian metric g on S in the conformal class of σ. Denote by $\tilde{g}$ its pullback to $\widetilde{S}$ and $\widetilde{dA}$ the corresponding area form on $\widetilde{S}$. Let $\| \cdot \|_{g,X}$ denotes the Hilbert-Schmidt norm with respect to the metric on $\widetilde{S}$ induced by g and the metric on X. Define *energy density of* f *with respeect to* g on $\widetilde{S}$ as

$$\tilde{e}(f) = \|df\|_{g,X}^2 \, dA,$$

The energy density $\tilde{e}(f)$ a $\pi_1(S)$-invariant exterior 2-form on $\widetilde{S}$ and hence defines an exterior 2-form, the *energy density* $e(f)$ on S. Define the *energy* $E_{\rho,g}(f)$ as its integral over S. Alternatively, $E_{\rho,g}(f)$ is the integral of the energy density on $\widetilde{S}$ over a fundamental domain for the $\pi_1(S)$-action on $\widetilde{S}$. Since S is two-dimensional, $E_{\rho,g}(f)$ (if finite) depends only on the conformal structure σ, and we denote it $E_{\rho,\sigma}(f)$.

When the target X is only a metric space, define the energy density via ball averages of the approximate derivative

$$\frac{d_X(f(x), f(y))}{d_{\widetilde{S}}(x, y)} \, .$$

(See Korevaar-Schoen[26] or Jost [22].)

For finite energy maps the energy density $e(f)$ is a measure which is absolutely continuous with respect to Lebesgue measure. The Radon-Nikodym derivative plays the role of $\|df\|^2$. For more details, see [26]. Finite energy maps always exist, and [26, Theorem 2.6.4] guarantees an energy minimizing sequence of uniformly Lipschitz equivariant mappings. In addition to providing a definition of energy minimizing maps to metric spaces, this construction also defines a Sobolev completion of the continuously differentiable maps to Riemannian targets that does not appeal to an isometric embedding of X into euclidean space.

Perhaps the most interesting situation occurs when the infimum of the energy is realized. In the context of NPC targets, recall that a map

f is called *harmonic* if it minimizes $E_{\rho,\sigma}$ among all ρ-equivariant maps of finite energy.

The fundamental existence theorem for harmonic maps to nonpositively curved Riemannian manifolds is due to Eells-Sampson [14]. In the equivariant setting there are various conditions on ρ which guarantee existence. When X is a symmetric space of noncompact type, ρ is said to be *reductive* if its Zariski closure has trivial unipotent radical. Existence of a harmonic map for reductive ρ was proven by Corlette [10], Donaldson [11], and Jost-Yau [24]. A geometric notion of reductivity involving stabilizers of flat totally geodesic subspaces was used in [28] (see also Jost [22]). Korevaar and Schoen [27] introduced the notion of a *proper action* (not to be confused with the more standard use of the term *proper* below) which is the condition that the sublevel sets of the displacement function associated to a generating set of $\pi_1(S)$ are bounded. This condition guarantees the existence of an energy minimizer when X is a general NPC space (see also Jost [23]).

The basic idea goes back to work of Sacks-Uhlenbeck [38] and Schoen-Yau [40]. When ρ is a Fuchsian representation (corresponding to a hyperbolic structure on S), Tromba [45] proved that E_ρ is proper and has a unique critical point (necessarily a minimum). When ρ is a quasi-Fuchsian $\mathsf{PSL}(2,\mathbb{C})$-representation, E_ρ is proper. Uhlenbeck [46] gave an explicit criterion for when E_ρ has a unique minimum. Generally E_ρ admits more than one critical point, for quasi-Fuchsian ρ.

2. Bounded geometry

Let $\gamma \in G$. Its translation length $|\gamma|$ is defined by:

$$|\gamma| := \inf_{x\in X} d(x,\gamma x)\ . \tag{2.1}$$

Lemma 2.1. *Let $\Gamma \subset G$ be a convex cocompact discrete subgroup. Then $\exists\varepsilon_0 > 0$ such that $|\gamma| \geq \varepsilon_0$ for all $\gamma \in \Gamma \setminus \{\mathbf{I}\}$.*

Proof. Suppose not. Then $\exists\gamma_i \in \Gamma$ such that $|\gamma_i| \neq 0$ for all i, and $|\gamma_i| \to 0$. Let N be a closed convex Γ-invariant subset such that N/Γ is compact. Since N is convex and Γ-invariant, $\exists x_i \in N$ such that

$$d(x_i, \gamma_i x_i) \longrightarrow 0\ .$$

Since N/Γ is compact, $\exists\lambda_i \in \Gamma$ and $x \in N$ such that, after passing to a subsequence, $\lambda_i x_i \to x$. Set $\tilde\gamma_i = \lambda_i\gamma_i\lambda_i^{-1}$. Then

$$d(\lambda_i x_i, \tilde\gamma_i\lambda x_i) \longrightarrow 0\ .$$

Properness of the action of Γ near $x \in N$ implies that for only finitely many i does $|\tilde{\gamma}_i| = |\gamma_i|$ This contradicts the assumption that

$$0 \neq |\gamma_i| \to 0.$$

$\square$

Lemma 2.2. *Let ε_0 satisfy Lemma 2.1. Let $\gamma_1, \gamma_2 \in \Gamma$ and $x, y \in X$. If*

- $d(x, y) < \varepsilon_0/2$;
- $d(\gamma_1 x, \gamma_2 y) < \varepsilon_0/2$,

then $\gamma_1 = \gamma_2$.

Proof.

$$\begin{aligned} |\gamma_2^{-1}\gamma_1| &\leq d(\gamma_2^{-1}\gamma_1 x, x) \\ &= d(\gamma_1 x, \gamma_2 x) \\ &\leq d(\gamma_1 x, \gamma_2 y) + d(\gamma_2 y, \gamma_2 x) \\ &= d(\gamma_1 x, \gamma_2 y) + d(x, y) \ < \ \varepsilon_0 \ . \end{aligned}$$

Now apply Lemma 2.1. $\square$

3. Existence of harmonic maps

Proposition 3.1. *Suppose that $\pi_1(S) \xrightarrow{\rho} G$ is convex cocompact. Then there exists a ρ-equivariant harmonic map $\widetilde{S} \xrightarrow{u} X$.*

We deduce this proposition as an immediate corollary of the following more general proposition, which we state here for later applications.

Proposition 3.2. *Suppose that $\pi_1(S) \xrightarrow{\rho} G$ is an embedding onto a normal subgroup of a convex cocompact subgroup $\Gamma \subset \mathrm{Iso}(X)$ such that $\rho(\pi_1(S))$ has trivial centralizer in Γ. Then there exists a ρ-equivariant harmonic map $\widetilde{S} \xrightarrow{u} X$.*

Proof. For any NPC space X and compact surface S, there exists an *energy minimizing* sequence u_i of uniformly Lipschitz ρ-equivariant mappings $\widetilde{S} \longrightarrow X$ (Korevaar-Schoen [26, Theorem 2.6.4]). Let $N \subset X$ be a ρ-invariant convex set such that N/Γ is compact. Projection $X \longrightarrow N$ decreases distances, and therefore decreases energy. Thus we may assume that the image of u_i lies in N.

Fix any point $\widetilde{s_0} \in \widetilde{S}$ with image $\tilde{s} \in S$. Since N/Γ is compact, after passing to a subsequence, $\exists \gamma_i \in \Gamma$ such that $v_i(\widetilde{s_0})$ converges to a point in N, where

$$v_i := \rho(\gamma_i) \circ u_i.$$

The v_i are uniformly Lipschitz and $v_i(\widetilde{s_0})$ converges. The Arzéla-Ascoli theorem implies that a subsequence of v_i converges uniformly on compact subsets of $\widetilde{S}$. Choose $\varepsilon_0 > 0$ satisfying Lemma 2.1. For each compact $K \subset \widetilde{S}$, there exists $I > 0$ so that

$$d(v_i(w), v_j(w)) < \varepsilon_0/2 \tag{3.1}$$

whenever $i, j \geq I$ and $w \in K$.

Each v_i is equivariant with respect to $\rho_i = \rho \circ \mathbf{Inn}_{\gamma_i}$, where $\mathbf{Inn}_{\gamma_i}$ denotes the inner automorphism of $\pi_1(S)$ defined by γ_i. Fix $i, j \geq I$, and set $x = v_i(\widetilde{s_0})$ and $y = v_j(\widetilde{s_0})$.

Choose a finite generating set $\Pi \subset \pi_1(S)$. Applying (3.1) to the finite set $K = \Pi\widetilde{s_0}$,

$$d(\rho_i(c)x, \rho_j(c)y) = d(v_i(c\widetilde{s_0}), v_j(c\widetilde{s_0})) < \varepsilon_0/2$$

whenever $c \in \Pi$. Since

$$d(x, y) = d(v_i(\widetilde{s_0}), v_j(\widetilde{s_0})) < \varepsilon_0/2 \ ,$$

Lemma 2.2 implies $\rho_i(c) = \rho_j(c)$ for all $c \in \Pi$. As Π generates $\pi_1(S)$ it follows $\rho_i = \rho_j$ if $i, j \geq I$. Since ρ is injective and the centralizer of $\pi_1(S)$ in Γ is trivial, $c_i = c_j$ for all $i, j \geq I$. Therefore u_i itself converges locally uniformly to the desired minimizer. $\square$

4. Properness of the action on $\mathcal{T}_S$

Our proof is based on the following classical fact. Using the uniformization theorem, it is a special case of Corollary B.

Lemma 4.1. $\pi_0\mathsf{Diff}^+(S)$ *acts properly on* $\mathcal{T}_S$.

This theorem is commonly attributed to Fricke. Fricke's original proof uses the fact that $\pi_0\mathsf{Diff}^+(S)$ acts isometrically on $\mathcal{T}_S$ with respect to a metric (say, the Teichmüller metric), and that the marked length spectrum is discrete. Many good expositions of these ideas exist. See Abikoff [1], §2.2, Farb-Margalit [15], Harvey [19], §2.4.1, or Buser [7], §6.5.6 (p.156), Imayoshi-Tanigawa [21],§6.3, Nag [33],§2.7, Theorem II of Bers-Gardiner [4] for this proof.

Another proof follows from the properness of the diffeomorphism group $\mathsf{Diff}(S)$ of an arbitrary smooth manifold S on the space $\mathsf{Met}(S)$ of Riemannian metrics on S. This properness is due to Ebin [13] Palais (unpublished, and Earle-Eels [12]. (See also Tromba [45].) In dimension two the uniformization theorem identifies $\mathcal{T}_S$ with the quotient of the $\mathsf{Diff}(S)$-invariant subset $\mathsf{Met}_{-1}(S)$ of $\mathsf{Met}(S)$ comprising metrics of curvature -1. Since $\mathsf{Diff}(S)$ acts properly on $\mathsf{Met}(S)$, it acts properly on $\mathsf{Met}_{-1}(S)$. Thus

$$\pi_0\mathsf{Diff}^+(S) = \mathsf{Diff}(S)/\mathsf{Diff}^0(S)$$

acts properly on $\mathsf{Met}_{-1}(S)/\mathsf{Diff}^0(S)$, which identifies with $\mathcal{T}_S$.

5. Properness of the energy function

We now prove that for ρ convex cocompact, the function E_ρ on $\mathcal{T}_S$ is proper. With little extra effort, we prove a more general theorem (first noticed by Bruce Kleiner), concerning homomorphisms

$$\pi_1(S) \xrightarrow{\rho} \Gamma \subset G$$

where Γ is convex cocompact and $\rho(\pi_1(S))$ is a normal subgroup $\Gamma_1 \lhd \Gamma$. Furthermore we assume that the centralizer of Γ_1 in Γ is trivial. Let

$$\Gamma \xrightarrow{\psi} \mathrm{Aut}(\pi_1(S))$$

be the homomorphism induced by the inclusion $\Gamma_1 \hookrightarrow \Gamma$ and the isomorphism $\pi_1(S) \xrightarrow{\rho} \Gamma_1$. By Γ_1 has trivial centralizer, ψ is injective. Thus ψ induces a monomorphism

$$Q \hookrightarrow \mathrm{Out}(\pi_1(S))$$

where $Q := \Gamma/\Gamma_1$. Hence Q acts on $\mathcal{T}_S$ via (5.5). Furthermore E_ρ is Q-invariant and hence induces a map $\mathcal{T}_S/Q \xrightarrow{E'_\rho} \mathbb{R}$

Proposition 5.1. *The map $\mathcal{T}_S/Q \xrightarrow{E'_\rho} \mathbb{R}$ is proper.*

Suppose that $[\sigma_i] \in \mathcal{T}_S$ is a sequence whose image in $\mathcal{T}_S/Q$ diverges. Suppose further that

$$E_\rho([\sigma_i]) \leq B$$

for some constant $B > 0$, and all $i = 1, 2, \ldots$.

Our assumption that the images of $[\sigma_i]$ diverge in $\mathcal{T}_S/Q$ means the following. Choose any invariant $\pi_0\mathsf{Diff}^+(S)$-invariant metric $d_\mathcal{T}$ on $\mathcal{T}_S$, for example, the Teichmüller metric. (For a survey of invariant metrics on $\mathcal{T}_S$ see Wolpert's paper [48] in this volume.) We may assume, for each $\eta \in Q$, that

$$d_\mathcal{T}(\psi(\eta)[\sigma_i], [\sigma_j]) \geq 1 \tag{5.1}$$

for $i \neq j$.

By [26] the ρ-equivariant harmonic maps

$$(\widetilde{S}, \widetilde{g_i}) \xrightarrow{u_i} X$$

have a uniform Lipschitz constant K (depending on B), where $\widetilde{g_i}$ denotes the hyperbolic metric on $\widetilde{S}$ associated to σ_i. In particular, given a closed curve c in S, choose a lift $\tilde{c} \subset \widetilde{S}$ running from $\widetilde{s_0}$ to $[c]\widetilde{s_0}$, where $[c] \in \pi_1(S; s_0)$ denotes the deck transformation corresponding to c. Denote the length of c with respect to the metric g_i on S by $L_i(c)$. Then

$$\begin{aligned} |\rho([c])| &\leq d(u_i(\widetilde{s_0}), \rho([c])u_i(\widetilde{s_0})) \\ &= d(u_i(\widetilde{s_0}), u_i(\rho([c])\widetilde{s_0})) \\ &\leq L_X(u_i(\tilde{c})) \\ &\leq KL_i(c) \ . \end{aligned} \tag{5.2}$$

Suppose that $c \subset \Sigma$ is any closed essential curve. Since ρ is injective, the isometry $\rho(c)$ is nontrivial. Let $\varepsilon_0 > 0$ satisfy Lemma 2.1. Then (5.2) implies

$$\ell_c(\sigma_i) \geq \varepsilon_0/K$$

where $\ell_c(\sigma)$ denotes the *geodesic length function of c with respect to σ*, that is, the length of the unique closed geodesic freely homotopic to c in the hyperbolic metric corresponding to σ.

Mumford's compactness theorem [32] implies that the conformal structures $[\sigma_i]$ project to a compact subset of the *Riemann moduli space* $\mathcal{T}_S/\pi_0\mathsf{Diff}^+(S)$. Thus $[\varphi_i] \in \pi_0\mathsf{Diff}^+(S)$ and $[\sigma_\infty] \in \mathcal{T}_S$ exist such that, after passing to a subsequence,

$$[\varphi_i][\sigma_i] \longrightarrow [\sigma_\infty]. \tag{5.3}$$

As $\mathsf{Diff}(S)$ acts properly on the set of Riemannian metrics (§4), representatives $g_i \in \mathsf{Met}_{-1}(S)$ and $\varphi_i \in \mathsf{Diff}^+(S)$ exist with $\varphi_i(g_i) \longrightarrow g_\infty$, where g_∞ denotes the hyperbolic metric associated to σ_∞. Choose a base point $\widetilde{s_0} \in \widetilde{S}$ with image $s_0 \in S$. We may assume that $\varphi_i(s_0) = s_0$. Let $\widetilde{\varphi_i} \in \mathsf{Diff}(\widetilde{S})$ be the unique lift of φ_i such that $\widetilde{\varphi_i}(\widetilde{s_0}) = \widetilde{s_0}$.

The map

$$v_i := u_i \circ \widetilde{\varphi_i}^{-1} : \widetilde{S} \longrightarrow X$$

is harmonic with respect to the metric $\phi_i(g_i)$ and equivariant with respect to the homomorphism

$$\rho \circ (\varphi_i^{-1}) : \pi_1(S) \longrightarrow G.$$

The maps v_i are uniformly Lipschitz with respect to the metric $\widetilde{g_\infty}$ on $\widetilde{S}$ induced from the metric g_∞ on S. In particular the family $\{v_i\}$ is equicontinuous.

Since N/Γ is compact, $\exists \gamma_i \in \Gamma$ such that all $\gamma_i v_i(\widetilde{s_0})$ lie in a compact subset of X.

By the Arzéla-Ascoli theorem, a subsequence of

$$w_i := \gamma_i \circ v_i$$

converges uniformly on compact sets. For I sufficiently large,

$$\sup_{z \in \widetilde{S}} d_X(w_i(z), w_j(z)) \ < \ \varepsilon_0/2 \tag{5.4}$$

for $i, j \geq I$.

Each $v_i = u_i \circ \widetilde{\varphi_i}^{-1}$ is equivariant with respect to $\rho \circ (\varphi_i)_*^{-1}$ and is harmonic with respect to $\varphi_i(g_i)$. Thus each $w_i = \gamma_i \circ v_i$ is equivariant with respect to

$$\rho_i := \rho \circ (\varphi_i)_*^{-1} \circ \psi(\gamma_i)$$

and also harmonic with respect to $\varphi_i(g_i)$ (since γ_i is an isometry).

Fix $i, j \geq I$, and let $x = w_i(\widetilde{s_0})$ and $y = w_j(\widetilde{s_0})$. For every $c \in \pi_1(S)$, (5.4) implies

$$\begin{aligned} d_X(\rho_i(c)x, \rho_j(c)y) &= d_X(w_i(c\widetilde{s_0}), w_j(c\widetilde{s_0})) \\ &< \ \varepsilon_0/2 \ . \end{aligned}$$

Since $d_X(x, y) < \ \varepsilon_0/2$, Lemma 2.2 implies $\rho_i(c) = \rho_j(c)$ for all $c \in \pi_1(S)$. Since ρ is injective,

$$\psi(\gamma_i) \circ (\varphi_i)_* = \psi(\gamma_j) \circ (\varphi_j)_*$$

The Dehn-Nielsen theorem (see Nielsen [34], Stillwell [41]) implies the natural homomorphism

$$\pi_0 \mathsf{Diff}^+(S) \cong \mathrm{Out}(\pi_1(S)). \tag{5.5}$$

is an isomorphism. Its injectivity implies $\psi(\gamma_i) \circ \varphi_i$ is isotopic to $\psi(\gamma_j) \circ \varphi_j$ for all $i, j \geq I$. Call this common mapping class $[\varphi]$. Thus, for $i \geq I$,

$$\psi(\gamma)_i) \circ \varphi_i)_* = [\varphi] \tag{5.6}$$

Then for $i, j \geq I$,

$$\begin{aligned} d_{\mathcal{T}}\big(\psi(\gamma_i)^{-1}[\sigma_i], \psi(\gamma_j)^{-1}[\sigma_j]\big) &= d_{\mathcal{T}}\big([\varphi]^{-1}(\varphi_i^{-1})_*[\sigma_i], [\varphi]^{-1}(\varphi_j^{-1})_*[\sigma_j]\big) \\ &= d_{\mathcal{T}}\big((\varphi_i^{-1})_*[\sigma_i], (\varphi_j^{-1})_*[\sigma_j]\big) \\ &\longrightarrow 0 \end{aligned}$$

by (5.3), contradicting (5.1). Thus E_ρ is proper, as claimed.

6. Action of the mapping class group

Corollary B follows from the properness of the action of $\pi_0\mathsf{Diff}^+(S)$ on $\mathcal{T}_S$ and a general fact on proper actions on metric spaces. Let X be a metric space and let $\mathcal{K}(X)$ denote the space of compact subsets of X, with the Hausdorff metric.

Lemma 6.1. *A group Γ of homeomorphisms of X acts properly on X if and only if Γ acts properly on $\mathcal{K}(X)$.*

Proof. The mapping

$$\begin{aligned} X &\xrightarrow{\iota} \mathcal{K}(X) \\ x &\longmapsto \{x\} \end{aligned}$$

is a proper isometric Γ-equivariant embedding. If Γ acts properly on $\mathcal{K}(X)$, then equivariance implies that Γ acts properly on X.

Conversely, suppose that Γ acts properly on X. For any compact subset $K \subset \mathcal{K}(X)$ of $\mathcal{K}(X)$, its union

$$UK := \bigcup_{A\in K} A$$

is a compact subset of X. For $\gamma \in \Gamma$ the condition

$$\gamma(K) \cap K \;\neq\; \emptyset \tag{6.1}$$

implies the condition

$$\gamma(UK) \,\cap\, UK \;\neq\; \emptyset. \tag{6.2}$$

To show that Γ acts properly on $\mathcal{K}(X)$, let $K \subset \mathcal{K}(X)$ be a compact subset. Since Γ acts properly on X, only finitely many $\gamma \in \Gamma$ satisfy (6.2), and hence only finitely many $\gamma \in \Gamma$ satisfy (6.1). Thus Γ acts properly on $\mathcal{K}(X)$. ☐

We now prove Corollary B. Let $[\rho] \in \mathfrak{C}$. By Theorem A, E_ρ is a proper function on $\mathcal{T}_S$, and assumes a minimum $m_0(E_\rho)$ Furthermore

$$\mathsf{Min}(\rho) := \{[\sigma] \in \mathcal{T}_S \mid E_\rho(\sigma) = m_0(E_\rho)\}$$

is a compact subset of $\mathcal{T}_S$, and

$$\mathfrak{C} \xrightarrow{\mathsf{Min}} \mathcal{K}(\mathcal{T}_S)$$

is a $\pi_0\mathsf{Diff}^+(S)$-equivariant continuous mapping.

Conclusion of Proof of Corollary B . Lemmas 4.1 and 6.1 together imply $\pi_0\mathsf{Diff}^+(S)$ acts properly on $\mathcal{K}(\mathcal{T}_S)$. By equivariance, $\pi_0\mathsf{Diff}^+(S)$ acts properly on $\mathfrak{C}$. ☐

7. Accidental parabolics

Now we illustrate with a well-known construction how properness of the energy functional can fail if the action contains non-semisimple isometries. For simplicity, assume in this section that X is a simply connected nonpositively curved complete Riemannian manifold (a *Cartan-Hadamard manifold*), and let G denote the isometry group of X.

Theorem 7.1. *Fix a homomorphism $\rho : \pi_1(S) \to G$. Assume that for some simple closed curve c in S, there is a complete geodesic α in X such that*

$$d_X(\alpha(t), \rho[c]\alpha(t)) \leq Ce^{-\delta t} \ ,$$

for positive constants C, δ and all $t \geq 0$. Then the energy functional E_ρ is **not** *proper.*

Proof. It suffices to construct a family σ_t, $0 < t \leq 1$, of conformal structures on S such that the corresponding points in $\mathcal{T}_S$ diverge as $t \to 0$, and a family u_t of ρ-equivariant maps $\widetilde{S} \to X$ such that $E_{\rho,\sigma_t}(u_t)$ is uniformly bounded in t.

Fix an initial conformal structure σ_1 on S. Let A_ε denote a tubular neighborhood of the geodesic representative of c with respect to the hyperbolic metric g_1 associated to σ_1. We denote this geodesic also by c. Let $A_\varepsilon^\pm$ be the connected components of $A_\varepsilon - c$. We furthermore choose A_ε such that in the uniformization $\mathbb{H}$ of (S, g_1), $\widetilde{A}_\varepsilon^\pm$ are isometric to the strip

$$\mathbb{R} \times [\varepsilon, 1/\ell_c(\sigma_1)),$$

where ε_1 is some positive number, and $\ell_c(g_1)$ denotes the length of the geodesic. This realizes the isometry $[c] \in \pi_1(S)$ as the isometry

$$(x, y) \mapsto (x + 1, y).$$

The family σ_t is then defined by the "plumbing construction" compare Wolpert [47]).

The key feature is that the conformal structure on $S_\varepsilon = S - A_\varepsilon$ is unchanged whereas the conformal structure on $A_\varepsilon^\pm$ is equivalent to the annulus

$$A_t^\pm := \mathbb{R} \times [\varepsilon, 1/\ell_c(\sigma_t))\big/(x, y) \sim (x + 1, y),$$

where $\ell_c(\sigma_t) \to 0$ as $t \to 0$.

Next, let $\alpha_1 : [0, \infty) \to X$ be a geodesic ray with

$$d_X(\alpha(t), \rho[c]\alpha(t)) \leq Ce^{-\delta t}$$

for all $t \geq 0$ and some constants C and δ. Define

$$\alpha : \mathbb{R} \times [0, \infty) \to X$$

so that $s \mapsto \alpha(s,t)$ is the complete unit speed geodesic satisfying

$$\alpha(0,t) = \alpha_1(t)$$
$$\alpha(S_t,t) = \rho[c]\alpha_1(t)$$

for some $S_t \geq 0$. Set

$$L(t) = (1/\delta)\log(t/\varepsilon),$$

and notice that $S_{L(t)} \leq C\varepsilon/t$. Finally, define

$$\begin{aligned} [0,1] \times [\varepsilon,\infty) &\xrightarrow{\beta} X \\ (s,t) &\longmapsto \alpha(S_{L(t)}s, L(t)) \ . \end{aligned}$$

Then:

$$\|\partial_t\beta(s,t)\| \leq 1/\delta t$$

for all $0 \leq s \leq 1$, by the nonpositive curvature and the fact that

$$\|\partial_t\beta(0,t)\| = (1/\delta t)\|\alpha_1'(L)\| = 1/\delta t.$$

Also,

$$\|\partial_s\beta(s,t)\| = S_L\|(\partial_s\alpha)(S_L s, L)\| \leq C\varepsilon/t$$

We extend β to $\mathbb{R} \times [\varepsilon,\infty)$ equivariantly with respect to the $\mathbb{Z}$-action $s \mapsto s+1$ on the domain and $x \mapsto \rho(c)x$ on X. Because of the bounds on the derivatives β has finite energy as an equivariant map.

Choose a finite energy ρ-equivariant map $u : (\widetilde{S},\sigma_1) \to X$. Hence, the restriction u_1^c to $\widetilde{S}_\varepsilon$ is also finite energy. By interpolating near the boundary $\partial\widetilde{S}_\varepsilon$ we may assume that u_1 restricted to the connected components of $\partial\widetilde{S}_\varepsilon$ coincides with the geodesic $s \mapsto \alpha(s,0)$. Then for each t, u_1^c extends to a map $u_t : \widetilde{S} \to X$ by requiring

$$u_t\big|_{\widetilde{A}_t^{\pm}} = \beta\big|_{\mathbb{R}\times[\varepsilon,1/\ell_c(\sigma_t)]} \ .$$

Then u_t is equivariant and has finite energy with respect to σ_t, uniformly in t. This completes the proof. □

For discrete embeddings in $G = \mathsf{PSL}(2,\mathbb{C})$, R. Canary and Y. Minsky have explained a partial converse to Theorem A. Namely suppose that ρ is a discrete embedding of a closed surface group π into G. Let $M := \mathsf{H}^3/\rho(\pi_1(S))$ be the corresponding hyperbolic 3-manifold.

Then by Marden's Tameness Conjecture, recently established by Agol [2], Calegari-Gabai [8] and Choi [9], ρ is a strong limit of quasi-Fuchsian representations and either ρ is quasi-Fuchsian (in which case E_ρ is proper) or there exist a sequence of pleated surfaces

$$S \xrightarrow{f_n} M$$

which exhaust the ends of the hyperbolic 3-manifold M^3.

The intrinsic geometry of each f_n is that of a totally geodesic surface in H^3 and therefore its energy (computed with respect to the intrinsic hyperbolic metric) equals

$$-2\pi\chi(S) = \text{area}(S).$$

Let σ_n be the conformal structure underlying this intrinsic metric; then

$$E_\rho(\sigma_n) \leq -2\pi\chi(S)$$

is bounded.

However, the corresponding sequence $[\sigma_n] \in \mathcal{T}_S$ tends to ∞. It suffices to show that for some $c \in \pi_1(S)$, the geodesic length $l_c(\sigma_n)$ in unbounded. Since each pleated surface f_n is an isometric map, it suffices to show that for some c, the corresponding closed geodesics c_n on f_n become arbitrarily long.

Thus E_ρ is proper if and only if ρ is quasi-Fuchsian.

8. Speculation

Deformation spaces of flat bundles over a surface S are natural geometric objects upon which the mapping class group of S acts. When G is a compact group, then the action is ergodic (Goldman [17] and Pickrell-Xia [36]. At the other extreme, uniformization identifies the Teichmüller space $\mathcal{T}_S$ of S with a connected component in the deformation space of flat $\mathsf{PSL}(2,\mathbb{R})$-bundles over S, and $\pi_0\mathsf{Diff}^+(S)$ acts properly on $\mathcal{T}_S$.

In general one expects the dynamics of $\pi_0\mathsf{Diff}^+(S)$ to intermediate between these two extremes. The purpose of this paper is to survey several conditions which guarantee properness of the $\pi_0\mathsf{Diff}^+(S)$-action and discuss several open questions in this subject. In particular we discuss an analytic invariant based on harmonic maps which seems to be closely related to the dynamics of the $\pi_0\mathsf{Diff}^+(S)$-action.

As mentioned earlier, convex cocompactness excludes all higher rank examples which do not come from rank one. However it may be possible to replace geodesic convexity of the Riemannian structure by another notion. All that is needed is a compact *core* N/Γ of the locally symmetric space X/Γ in which all all harmonic mappings $S \longrightarrow X/\Gamma$ take values.

For example, when $G = \mathsf{SL}(3,\mathbb{R})$, the mapping class group $\pi_0\mathsf{Diff}^+(S)$ acts properly on the component of $\text{Hom}(\pi, G)/G$ corresponding to convex $\mathbb{R}\mathsf{P}^2$-structures (Goldman [16]). Recently using his notion of *Anosov representations,* Labourie has proved [28] that for any split real form G, the action of $\pi_0\mathsf{Diff}^+(S)$ on the *Hitchin-Teichmüller component* of $\mathrm{Hom}(\pi, G)/G$ (see Hitchin [20]) is proper.

Labourie's definition is as follows. The unit tangent buncle

$$US \xrightarrow{\Pi} S$$

induces a central extension of fundamental groups

$$\mathbb{Z} \longrightarrow \pi_1(US) \xrightarrow{\Pi_*} \pi$$

where the center $\mathbb{Z}$ of $\pi_1(US)$ corresponds to the fundamental group of the fibers of Π. A representation $\rho : \pi \longrightarrow G$ and a linear representation of G on a vector space V defines a flat vector bundle

$$V_\rho \longrightarrow US$$

with holonomy representation $\rho \circ \Pi_*$. Let $\tilde{\xi}_t$ denote the lift of the vector field on US defining the geodesic flow to the total space $\mathbb{V}_\rho$. Labourie defines an *Anosov structure* to be a continuous splitting of the vector bundle

$$V_\rho = V_+ \oplus V_0 \oplus V_-$$

so that vectors in V_+ (respectively in V_-) are exponentially expanded (respectively contracted) under $\tilde{\xi}_t$.

Labourie proves [28] that the mapping class group acts properly on all such representations. All known examples of open sets of representations upon which the mapping class group acts properly satisify Labourie's condition.

In another direction, using ideas generalizing those of Bowditch [6] Tan, Wong and Zhang [43] have shown that the action of $\pi_0\mathsf{Diff}^+(S)$ on representations satisfying the analogue of *Bowditch's Q-conditions* is proper. This also generalizes the properness of the action on the space of quasi-Fuchsian representations.

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Department of Mathematics, University of Maryland, College Park, MD 20742

E-mail address: `wmg@math.umd.edu`

Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218

E-mail address: `wentworth@jhu.edu`