Algebraic varieties of surface group representations

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HIRZ80
A Conference in Algebraic Geometry
Honoring F. Hirzebruch's 80th Birthday
Emmy Noether Institute, Bar-Ilan University, Israel
22 May 2008

Outline

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- The *mapping class group* $\mathrm{Mod}(\Sigma) \cong \mathrm{Aut}(\pi)/\mathrm{Inn}(\pi)$ acts on $\mathrm{Hom}(\pi, G)/G$.
- Representations $\pi \xrightarrow{\rho} G$ arise from *locally homogeneous geometric structures* on Σ, modelled on homogeneous spaces of G.

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- Topological invariants of this bundle define invariants of the representation.

Characteristic classes

- The first characteristic invariant corresponds to the *connected components* of G:

$$\mathrm{Hom}(\pi, G) \longrightarrow \mathrm{Hom}\big(\pi, \pi_0(G)\big) \cong H^1\big(\Sigma, \pi_0(G)\big)$$

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- $G = \mathsf{GL}(n,\mathbb{R}), \mathsf{O}(n)$: the *first Stiefel-Whitney class* detects orientability of the associated vector bundle.

Compact and complex semisimple groups

- Now suppose G is connected. The next invariant *obstructs* lifting ρ to the universal covering group $\tilde{G} \longrightarrow G$:

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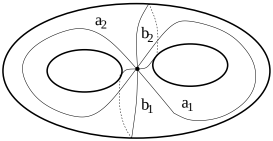
- When G is a connected *complex* or *compact semisimple* Lie group, then $\mathfrak{o}_2$ defines an isomorphism

$$\pi_0\big(\mathrm{Hom}(\pi, G)\big) \xrightarrow{\cong} \pi_1(G).$$

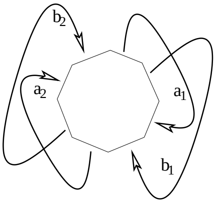
(Narasimhan–Seshadri, Atiyah–Bott, Ramanathan, Goldman, Jun Li, Rapinchuk–Chernousov–Benyash-Krivets, . . .)

Closed orientable surfaces

Decompose a surface of genus g



as a $4g$-gon with its edges identified in $2g$ pairs and all vertices identified to a single point.



Presentation of $\pi_1(\Sigma)$

$$\langle A_1, \ldots, B_g \mid A_1 B_1 A_1^{-1} B_1^{-1} \ldots A_g B_g A_g^{-1} B_g^{-1} = 1 \rangle$$

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$$(\alpha_1, \dots, \beta_g) \in G^{2g}$$

satisfying

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  Take $\alpha_i = \rho(A_i)$ and $\beta_i = \rho(B_i)$.

- To compute $\mathfrak{o}_2(\rho)$, lift the images of the generators

$$\widetilde{\alpha_1}, \dots, \widetilde{\beta_g} \in \tilde{G}.$$

The second obstruction

- Evaluate the relation:

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- Equals $\mathfrak{o}_2(\rho) \in \pi_1(G)$.

Euler class

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- Uniformization: *maximal component* of $\mathrm{Hom}(\pi, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$ identifies with *Teichmüller space* $\mathfrak{T}_\Sigma$ of *marked hyperbolic structures* on Σ.

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- Component of $\mathrm{Hom}(\pi, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PGL}(2,\mathbb{R})$ consisting *exactly* of discrete embeddings.

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  - Equality $\iff$ f homotopic to a covering-space.
- Components of $\mathsf{Hom}\big(\pi, \mathsf{PSL}(2,\mathbb{R})\big)$ are the $4g-3$ nonempty preimages of

$$\mathsf{Hom}(\pi, \mathsf{PSL}(2,\mathbb{R})) \xrightarrow{e} \mathbb{Z}.$$

(G, Hitchin)

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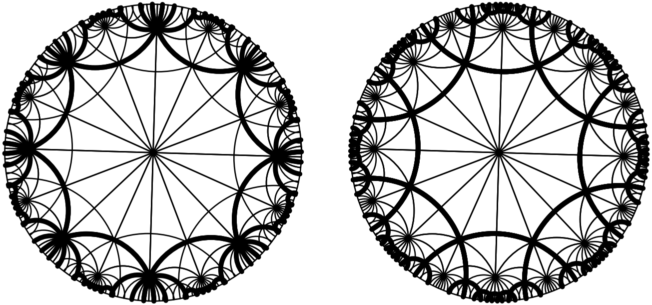
- For example, such structures arise from identifying polygons in H^2 If the sum of the interior angles is $2\pi k$, where $k \in \mathbb{Z}$, then quotient space is a hyperbolic surface with one singularity (the image of the vertex) with cone angle $2\pi k$.

A hyperbolic surface of genus two



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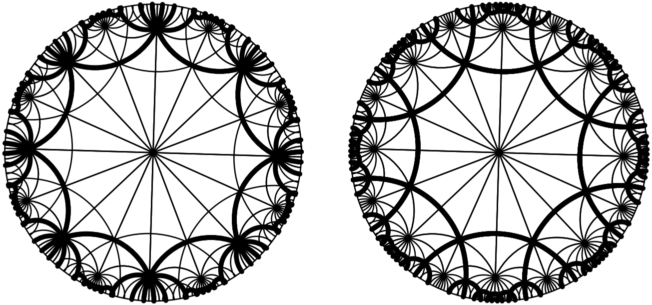

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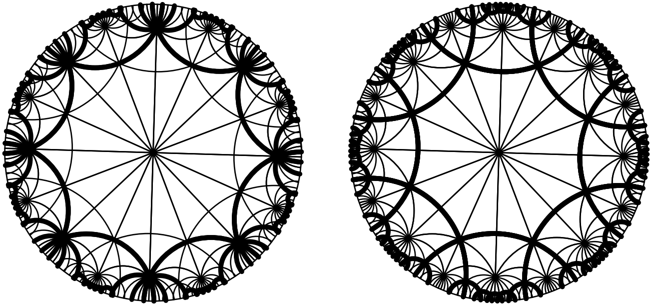

- Identifying a regular octagon with angles $\pi/4$ yields a nonsingular hyperbolic surface with $e(\rho) = \chi(\Sigma) = -2$.
- But when the angles are $\pi/2$, the surface has one singularity with cone angle 4π and

$$e(\rho) = 1 + \chi(\Sigma) = -1.$$

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- If $\Sigma \xrightarrow{f} \Sigma_1$ is a degree one map not homotopic to a homeomorphism, and Σ_1 is a hyperbolic structure with holonomy ϕ_1, then the composition

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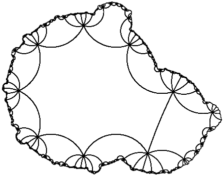
- *Conjecture:* every representation with dense image occurs as the holonomy of a branched hyperbolic structure.

Quasi-Fuchsian groups

The group of orientation-preserving isometries of $\mathsf{H}^3_{\mathbb{R}}$ equals $\mathsf{PSL}(2,\mathbb{C})$. Close to Fuchsian representations in $\mathsf{PSL}(2,\mathbb{R})$ are *quasi-Fuchsian representations*.

- Quasi-fuchsian representations are discrete embeddings.
- $\mathcal{QF} \approx \mathfrak{T}_\Sigma \times \overline{\mathfrak{T}_\Sigma}$ (Bers 1960)
- The closure of $\mathcal{QF}$ consists of all discrete embeddings $\pi \hookrightarrow \mathsf{PSL}(2,\mathbb{C})$ (Thurston-Bonahon 1984)
- The discrete embeddings are *not open* and do not comprise a component of $\mathsf{Hom}(\pi, G)/G$.



Complex hyperbolic geometry

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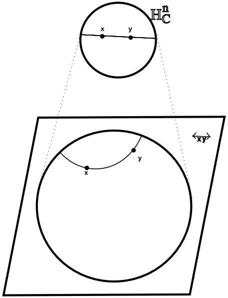

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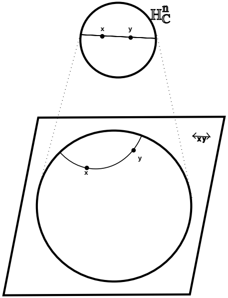

- $\mathbb{C}$-linear subspaces meet $H^n_{\mathbb{C}}$ in totally geodesic subspaces.

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- Generalized to *maximal representations* by Burger-Iozzi-Wienhard and Bradlow-Garcia-Prada-Gothen-Mundet.

Singularities in Hom(π, G)

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- In general the analytic germ of a *reductive representation* of the fundamental group of a compact Kähler manifold is defined by a system of homogeneous quadratic equations. (Goldman–Millson 1988, with help from Deligne)

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- In general the analytic germ of a *reductive representation* of the fundamental group of a compact Kähler manifold is defined by a system of homogeneous quadratic equations. (Goldman–Millson 1988, with help from Deligne)
- Deformation theory: twisted version of the *formality* of the rational homotopy type of compact Kähler manifolds (Deligne-Griffiths-Morgan-Sullivan 1975).

The deformation groupoid

- Objects in the deformation theory correspond to *flat connections*, $\mathfrak{g}_{\mathrm{Ad}\rho}$-valued 1-forms ω on Σ satisying the *Maurer-Cartan equations:*

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- Morphisms in the deformation theory correspond to *infinitesimal gauge transformations,* sections η of $\mathfrak{g}_{\mathrm{Ad}_\rho}$:

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- This groupoid is *equivalent* to the groupoid whose objects form $\mathrm{Hom}(\pi, G)$ and the morphisms $\mathrm{Inn}(G)$.

The quadratic cone

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- ω is tangent to an analytic path $\Longleftrightarrow$

$$[\omega, \omega] = 0 \in H^2(\Sigma, \mathfrak{g}_{\mathsf{Ad}\rho}).$$

- An explicit *exponential map* from the quadratic cone in $Z^1(\Sigma, \mathfrak{g}_{\mathsf{Ad}\rho})$ can be constructed from Hodge theory:

$$\omega \longmapsto \big(I + \bar{\partial}_D^* \mathsf{ad}(\omega^{(0,1)})\big)^{-1}(\omega).$$

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- $\mathrm{Ad}(\mathrm{U}(1,1))$-invariant decomposition of Lie algebras

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Complex hyperbolic surfaces

Consider a discrete embedding $\pi \xrightarrow{\rho_0} \mathsf{SU}(1,1)$ and its neighborhood in $\mathsf{Hom}(\pi, \mathsf{U}(n,1))$.

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- The quadratic form reduces to the cup-product

$$H^1(\Sigma, \mathbb{C}^{1,1}_{\rho_0}) \times H^1(\Sigma, \mathbb{C}^{1,1}_{\rho_0}) \longrightarrow H^2(\Sigma, \mathbb{R}) \cong \mathbb{R},$$

coefficients $\mathbb{C}^{1,1}_{\rho_0}$ paired by

$$(z_1, z_2) \longmapsto \mathsf{Im}\langle z_1, z_2 \rangle.$$

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  - Local rigidity.
- $\forall$ even e with $|e| \leq 2g-2$, corresponding component of $\mathrm{Hom}(\pi, \mathrm{SU}(2,1))$ contains *discrete embeddings.* (Goldman–Kapovich–Leeb 2001)

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- Eichler-Shimura isomorphisms

$$H^{0,1}(X, \mathbb{C}^{1,1}_{\rho_0}) \cong H^0(X, K^{3/2})$$
$$H^{1,0}(X, \mathbb{C}^{1,1}_{\rho_0}) \cong H^0(X, K^{3/2})$$

carries cup-product/symplectic coefficient pairing to L^2 Hermitian product on weight 3 automorphic forms.

Happy Birthday, Professor Hirzebruch!