Locally Homogeneous Geometric Manifolds

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Geometry through symmetry

In his 1872 *Erlangen Program*, Felix Klein proposed that a *geometry* is the study of properties of an abstract space X which are invariant under a transitive group G of transformations of X.

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- (Ehresmann 1936): Geometric manifold M modeled on X.

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- (Thurston 1976): 3-manifolds canonically decompose into *locally homogeneous Riemannian* pieces (8 types). (proved by Perelman)

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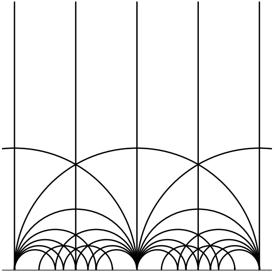

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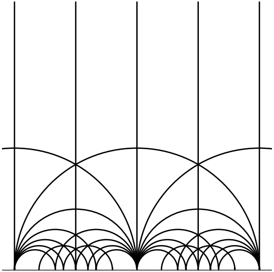

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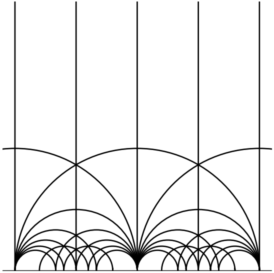
  - *Example:* The 2-torus admits a *moduli space* of Euclidean structures.



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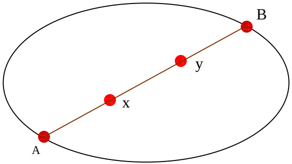

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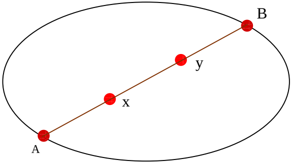

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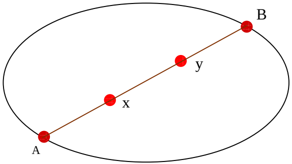
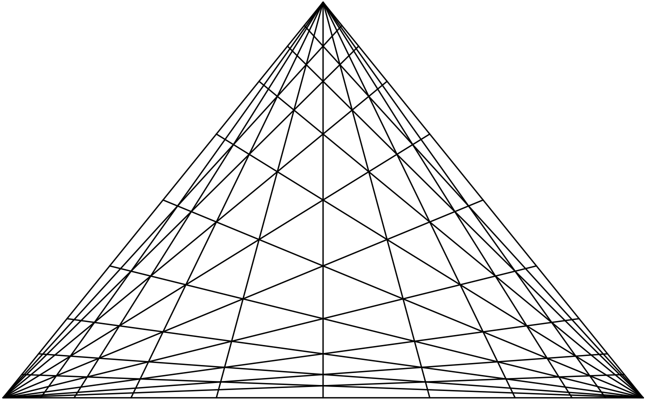


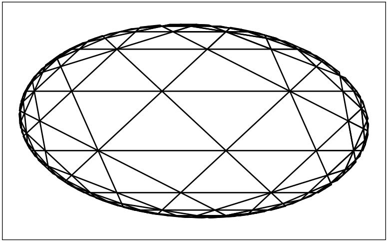
- Projective geometry *contains* hyperbolic geometry.
  - Hyperbolic structures *are* convex $\mathbb{RP}^n$-structures.

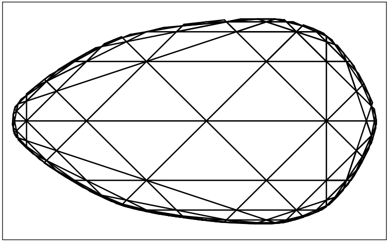
Another example: Projective tiling of $\mathbb{RP}^2$ by equilateral 60^o-triangles



This tesselation of the open triangular region is equivalent to the tiling of the Euclidean plane by equilateral triangles.

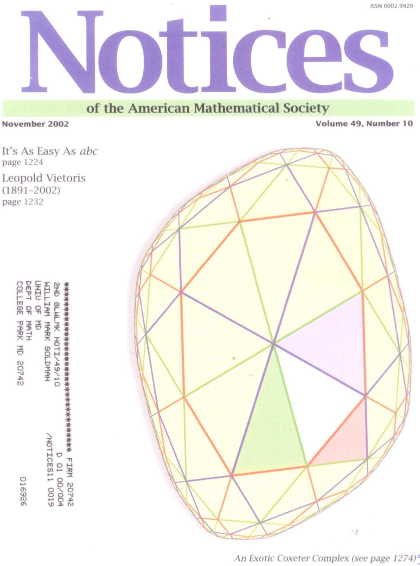
Example: A projective deformation of a tiling of the hyperbolic plane by (60^o,60^o,45^o)-triangles.





Both domains are tiled by triangles, invariant under a Coxeter group $\Gamma(3,3,4)$. First domain bounded by a conic (hyperbolic geometry), second domain bounded by $C^{1+\alpha}$-convex curve where $0 < \alpha < 1$.

Into the mainstream media



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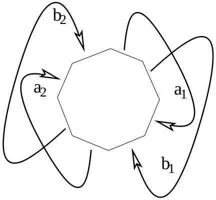

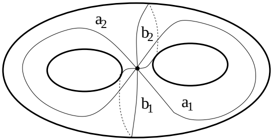

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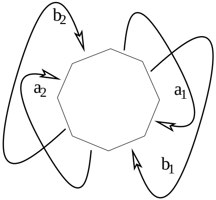
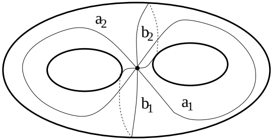
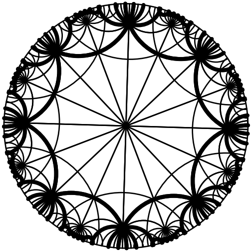




- Realize these identifications isometrically for a regular 45^o-octagon.



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- *Mapping class group*

$$\text{Mod}(\Sigma) := \pi_0\big(\text{Diff}(\Sigma)\big)$$

acts on $\mathfrak{D}_{(G,X)}(\Sigma)$.

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  - For quotient structures, hol is an embedding.

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  - All subsumed in *Anosov representations* (Labourie).

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- $\mathfrak{F}(\Sigma) \approx \mathbb{R}^{6g-6}$ and $\mathsf{Mod}(\Sigma)$ acts properly discretely.

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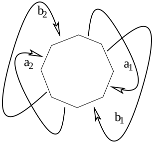

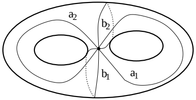

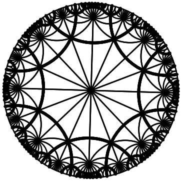

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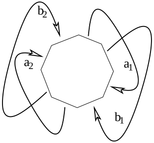

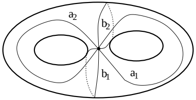

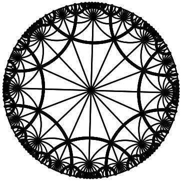

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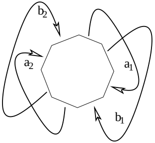

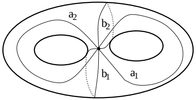

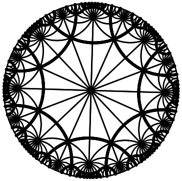

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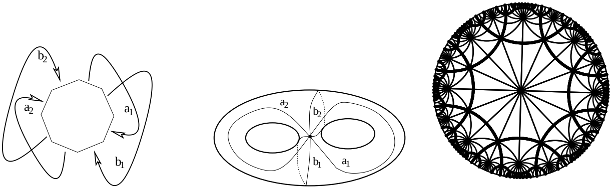

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  - Corroborates Morse theory description of topology of maximal components extending Hitchin's Higgs bundle methods (Bradlow, Garcia-Prada, Gothen 2005).

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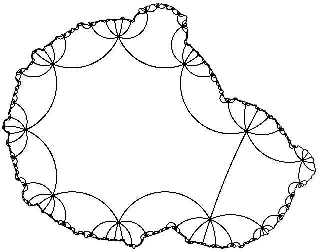

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- (Choi-G 1990) Deformation space of *all* $\mathbb{RP}^2$-structures on Σ homeomorphic to $\mathbb{R}^{-8\chi(\Sigma)} \times \mathbb{Z}$.

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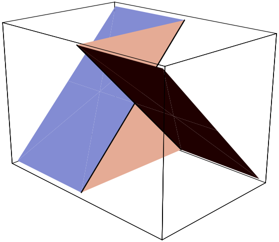
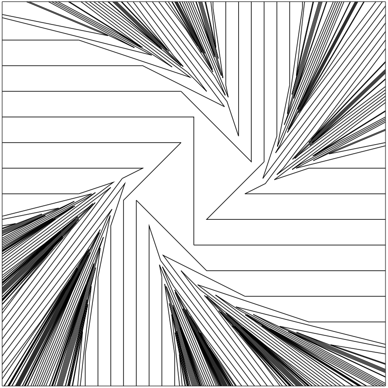
- Drumm (1990) Every noncompact complete hyperbolic surface of finite type admits a proper affine deformation.

Drumm's Schottky groups

The classical construction of Schottky groups fails using affine half-spaces and slabs. Drumm's geometric construction uses *crooked planes*, PL hypersurfaces adapted to the Lorentz geometry which bound fundamental polyhedra for Schottky groups.



Affine action of level 2 congruence subgroup of $GL(2, \mathbb{Z})$



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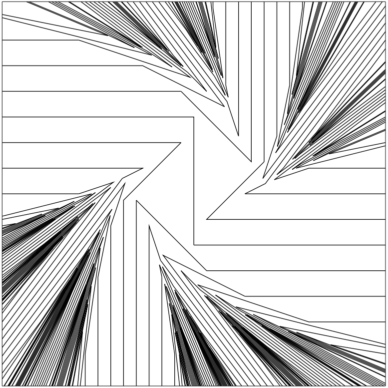

Proper affine deformations exist even for *lattices* (Drumm).

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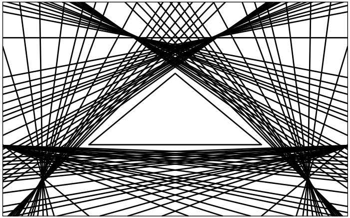
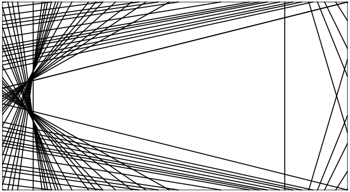
  - (Charette-Drumm-G 2010): Proved for $\chi(\Sigma) = -1$.

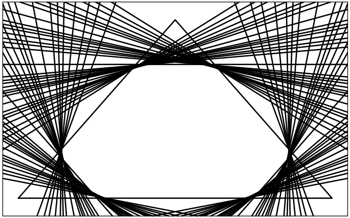
Deformation spaces for surfaces with $\chi(\Sigma)$



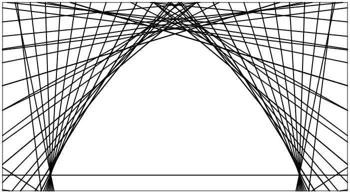
(u) Three-holed sphere



(v) Two-holed $\mathbb{RP}^2$



(w) One-holed torus



(x) One-holed Klein bottle