Projective geometry on manifolds
Lecture Notes (v.0.3) — Mathematics 748B Spring 1988

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11 March 1988

Affine, Projective and non-Euclidean Geometry

According to Felix Klein's Erlanger program (1872), a (classical) geometry is the study of properties of a space X invariant under a group G of transformations of X. In practice G will be a Lie group which acts transitively on X, so that X is represented as a homogeneous space G/H, where $H \subset G$ is a closed subgroup. For example Euclidean geometry is the geometry of n-dimensional Euclidean space $\mathbf{R}^n$ invariant under its group $\mathrm{Euc}(\mathbf{R}^n)$ of isometries (i.e. rigid motions, congruence transformations). In Euclidean geometry (or more generally any Riemannian geometry) we can speak of points, lines, parallelism of lines, angles between lines, distance between points, area, volume, etc. It is not difficult to show that all of these concepts can be derived from the notion of distance, i.e. from the metric structure of Euclidean geometry. Thus any isometry preserves all of these geometric entities. Other "weaker" geometries are obtained by removing some of these concepts. For example, *similarity geometry* — Euclidean geometry where the equivalence relation of congruence is replaced by the broader equivalence relation of similarity is the geometry invariant under similarity transformations — arises if one doesn't speak of distance, but does speak of angles (and lines, parallelism) etc. *Affine geometry* arises when one speaks only of points, lines and the relation of parallelism. And when one removes the notion of parallelism and only studies lines, points and the relation of incidence between them (e.g. three points being *collinear* or three lines being *concurrent*) one is led to *projective geometry*.

Here is a basic example illustrating the differences between the various geometries. Consider a particle moving along a smooth path; it has a well-defined velocity vector field (this uses only the differentiable structure of $\mathbf{R}^n$). In Euclidean geometry, it makes sense to discuss its "speed," so "motion at unit speed" (i.e. "arc-length-parametrized geodesic") is a meaningful concept there. But in affine geometry, the concept of "speed" or "arc-length" must be abandoned: yet "motion at constant speed" remains meaningful since the property of moving at constant speed can be characterized as parallelism of the velocity vector field (zero acceleration). In projective geometry this notion of "constant speed" (or "parallel velocity") must be further broadened to the concept of "projective parameter" introduced by J.H.C. Whitehead.

The development of synthetic projective geometry was begun by the French architect Desargues in 1636–1639 out of attempts to understand the geometry of perspective. Two hundred years later non-Euclidean (hyperbolic) geometry was developed independently and practically simultaneously by Bolyai in 1833 and Lobachevsky in 1826–1829. These geometries were unified in 1871 by Klein who noticed that Euclidean, affine, hyperbolic and elliptic geometry were all "present" in projective geometry.

In the first lecture, affine geometry is introduced as the geometry of parallelism. The second lecture introduces projective space as a natural compactification of affine space; coordinates are introduced as well as the "dictionary" between geometric objects in projective space and algebraic objectes in a vector space. The collineation group is compactified as a projective space of "projective endomorphisms;" this will be useful for studying limits of sequences of projective transformations. The third lecture discusses, first from the point of view of polarities, the Cayley-Beltrami-Klein model for hyperbolic geometry. The Hilbert metric on a properly convex domain in projective space is introduced and is shown to be equivalent to the categorically defined Kobayashi metric.

§1: Affine geometry

1.1 We wish to capture the geometry of Euclidean n-space $\mathbf{R}^n$ in which "parallelism" plays the central role. If $X, X' \subset \mathbf{R}^n$ one might say that they are "parallel" if one can be obtained from the other by (parallel) translation, i.e. if there is a vector v such that $X' = X + v$. This motivates the following definition.

An *affine space* is a set E provided with a simply transitive action of a vector group τ_E. We call τ_E the *vector space of translations* of E. Of course every vector space has the underlying structure of an affine space and an affine space with a distinguished point ("an origin") has the natural structure of a vector space. If $x, y \in E$, we denote by $\tau_{x,y} \in \tau_E$ the unique translation taking x to y. If E is a vector space then $\tau_{x,y}$ is more familiarly denoted by $y - x$ and the effect of translating $x \in E$ by $t \in \tau_E$ is denoted by $x + t$.

Of course this is a rather fancy way of saying fairly well known facts. An affine space is just a vector space "with the origin forgotten." There is no distinguished point — like 0 in a vector space — in affine space since the translations act transitively. "Choosing an origin" in an affine space E (which of course can be an arbitrary point in E) turns E into a vector space.

If E, E' are affine spaces a map $f : E \longrightarrow E'$ is *affine* if for each $v \in \tau_E$, there exists a translation $v' \in \tau_{E'}$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ {\scriptstyle v}\downarrow & & {\scriptstyle v'}\downarrow \\ E & \xrightarrow{f} & E' \end{array}$$

commutes. Necessarily v' is unique and it is easy to see that the correspondence $\mathbf{L}(f) : v \mapsto v'$ is a homomorphism $\tau_E \longrightarrow \tau_{E'}$, i.e. a linear map. We denote the space of all affine maps $E \longrightarrow E'$ by $\mathfrak{aff}(E, E')$. Thus $\mathbf{L}$ defines a map (called the *linear part*) $\mathfrak{aff}(E, E') \longrightarrow \mathrm{Hom}(\tau_E, \tau_{E'})$ where $\mathrm{Hom}(\tau_E, \tau_{E'})$ denotes the vector space of linear maps $\tau_E \longrightarrow \tau_{E'}$. The set of affine *endomorphisms* of an affine space E will be denoted by $\mathfrak{aff}(E)$ and the group of affine *automorphisms* of E will be denoted $\mathrm{Aff}(E)$.

Exercise 1.2. *Show that* $\mathfrak{aff}(E, E')$ *has the natural structure of an affine space and that its group of translations may be identified with the vector space* $\mathrm{Hom}(\tau_E, \tau_{E'}) \oplus \tau_{E'}$. *Show that* $\mathrm{Aff}(E)$ *is a Lie group and its Lie algebra may be identified with* $\mathfrak{aff}(E)$. *Show that* $\mathrm{Aff}(E)$ *is isomorphic to the semidirect product* $\mathrm{Aut}(\tau_E) \cdot \tau_E$ *where* τ_E *is the normal subgroup consisting of translations and* $\mathrm{Aut}(\tau_E) = \mathrm{GL}(E)$ *is the group of linear automorphisms of the vector space* τ_E.

The kernel of $\mathbf{L} : \mathfrak{aff}(E, E') \longrightarrow \mathrm{Hom}(\tau_E, \tau_{E'})$ (i.e. the inverse image of 0) is the vector space $\tau_{E'}$ of translations of E'. Choosing an origin $x \in E$, we write, for $f \in \mathfrak{aff}(E, E')$,

$$f(y) = f(x + (y - x)) = \mathbf{L}f(y - x) + t$$

Since every affine map $f \in \mathfrak{aff}(E, E')$ may be written as

$$f(x) = \mathbf{L}(f)(x) + f(0)$$

we call $\mathbf{L}(f)$ the *linear part* of f and $f(0) \in E'$ the *translational part* of f.

Affine geometry is the study of affine spaces and affine maps between them. If $U \subset E$ is an open subset, then a map $f : U \longrightarrow E'$ is *locally affine* if for each connected component $U_i \subset U$ there exists an affine map $f_i \in \mathfrak{aff}(E, E')$ such that the restrictions of f and f_i to U_i are identical. (Note that two affine maps which agree on a nonempty open set are identical.)

Exercise 1.3. *If* E *is an affine space show that there is a flat torsionfree connection* ∇ *on* E *such that if* $U, V \subset E$ *are open, and* $f : U \longrightarrow V$ *is a diffeomorphism, then* f *preserves* $\nabla \iff f$ *is locally affine. Show that a map* $\gamma : (-\epsilon, \epsilon) \longrightarrow E$ *is a geodesic* $\iff$ *it is locally affine.*

1.4 There are various extra structures on affine spaces which are preserved by notable subgroups of the affine group. Let $\mathbf{B}$ be an inner product on E and $O(E; \mathbf{B}) \subset \mathrm{GL}(E)$ the corresponding orthogonal group. The $\mathbf{B}$ defines a flat Riemannian metric on E and the inverse image $\mathbf{L}^{-1}(O(E; \mathbf{B})) \cong O(E; \mathbf{B}) \cdot \tau_E$ is the full group of isometries, i.e. the *Euclidean group*. If $\mathbf{B}$ is a nondegenerate indefinite form, then there is a corresponding flat pseudo-Riemannian metric on E and the inverse image $\mathbf{L}^{-1}(O(E; \mathbf{B}))$ is the full group of isometries of this pseudo-Riemannian metric.

Exercise 1.5. *Show that an affine automorphism* g *of Euclidean* n*-space* $\mathbf{R}^n$ *is conformal (i.e. preserves angles)* $\iff$ *its linear part is the composition of an orthogonal transformation and multiplication by a scalar (i.e. a homothety).*

Such a transformation will be called a *similarity transformation* and the group of similarity transformations will be denoted $\mathrm{Sim}(\mathbf{R}^n)$.

1.6 We shall be interested in putting affine geometry on a manifold, i.e. finding a coordinate atlas on a manifold M such that the coordinate changes are locally affine. Such a structure will be called an *affine structure* on M. We say that the manifold is *modelled* on an affine space E if its coordinate charts map into E. Clearly an affine structure determines a differential structure on M. A manifold with an affine structure will be called an *affinely flat manifold*, or just an *affine manifold*. If M, M' are affine manifolds (of possibly different dimensions) and $f : M \longrightarrow M'$ is a map, then f is *affine* if in local affine coordinates, f is locally affine. If $G \subset \mathrm{Aff}(E)$ then we recover more refined structures by requiring that the coordinate changes are locally restrictions of affine transformations from G. For example if G is the group of Euclidean isometries, we obtain the notion of a *Euclidean structure* on M.

EXERCISE. *Let M be a smooth manifold. Show that there is a natural correspondence between affine structures on M and flat torsionfree affine connections on M. In a similar vein, show that there is a natural correspondence between Euclidean structures on M and flat Riemannian metrics on M.*

1.7 If M is a manifold, we denote the Lie algebra of vector fields on M by $\mathfrak{X}(M)$. A vector field X on E is said to be *affine* if it generates a one-parameter group of affine transformations. A vector field X on E (or more generally on an affine manifold M) is said to be *parallel* if it generates a flow of translations. A vector field X is said to be *radiant* if for each $Y \in \mathfrak{X}(M)$ we have $\nabla_Y X = Y$. We obtain equivalent criteria for these conditions in terms of the covariant differential operation $\nabla : \mathcal{T}^p(M;TM) \longrightarrow \mathcal{T}^{p+1}(M;TM)$ (here $\mathcal{T}^p(M;TM)$ denotes the space of TM-valued covariant p-tensor fields on M,i.e. the tensor fields of type $(1,p)$. Thus $\mathcal{T}^0(M;TM) = \mathfrak{X}(M)$, the space of vector fields on M).

EXERCISE. *Let E be an affine space and let X be a vector field on E.*

(1) X is parallel $\Longleftrightarrow \nabla_Y X = 0$ for all $Y \in \mathfrak{X}(E) \Longleftrightarrow \nabla X = 0 \Longleftrightarrow X$ has constant coefficients (i.e. is a "constant vector field"). One may identify τ_E with the parallel vector fields on E. The parallel vector fields form an abelian Lie algebra of vector fields on E.

(2) X is affine $\Longleftrightarrow$ for all $Y, Z \in \mathfrak{X}(E)$, $\nabla_Y \nabla_Z X = \nabla_{\nabla_Y Z} X \Longleftrightarrow \nabla\nabla X = 0 \Longleftrightarrow$ the coefficients of X are affine functions, i.e.

$$X = \sum_{i,j=1}^{n} (a^i_j x^j + b^i) \frac{\partial}{\partial x^i}$$

for constants $a^i_j, b^i \in \mathbf{R}$. We may write

$$\mathbf{L}(X) = \sum_{i,j=1}^{n} a^i_j x^j \frac{\partial}{\partial x^i}$$

for the linear part (which corresponds to the matrix $(a^i_j) \in \mathfrak{gl}(\mathbf{R}^n)$) and

$$X(0) = \sum_{i=1}^{n} b^i \frac{\partial}{\partial x^i}$$

for the translational part (the translational part of an affine vector field is a parallel vector field). The Lie bracket of two affine vector fields is given by

$$\mathbf{L}([X,Y]) = [\mathbf{L}(X), \mathbf{L}(Y)] = \mathbf{L}(X)\mathbf{L}(Y) - \mathbf{L}(X)\mathbf{L}(Y) \qquad \textit{(matrix multiplication)}$$
$$[X,Y](0) = \mathbf{L}(X)Y(0) - \mathbf{L}(Y)X(0).$$

In this way the space $\mathfrak{aff}(E) = \mathfrak{aff}(E, E)$ of affine endomorphisms $E \longrightarrow E$ is a Lie algebra.

(3) X is radiant $\iff \nabla X = I_E$ (where $I_E \in \mathcal{T}^1(E; TE)$ is the identity map $TE \longrightarrow TE$, regarded as a tangent bundle-valued 1-form on E) $\iff$ there exists $b^i \in \mathbf{R}$ for $i = 1, \ldots, n$ such that

$$X = \sum_{i=1}^{n} (x^i - b^i) \frac{\partial}{\partial x^i}$$

Note that the point $b = (b^1, \ldots, b^n)$ is the unique zero of X and that X generates the one-parameter group of homotheties fixing b. (Thus a radiant vector field is a special kind of affine vector field.) Furthermore X generates the center of the isotropy group of $\mathrm{Aff}(E)$ at b, which is conjugate (by translation by b) to $\mathrm{GL}(E)$. Show that the radiant vector fields on E form an affine space isomorphic to E.

1.8 Suppose that $\iota : E_1 \hookrightarrow E$ is an injective affine map; then we say that $\iota(E_1)$ (or with slight abuse, ι itself) is an affine subspace. If E_1 is an affine subspace then for each $x \in E_1$ there exists a linear subspace $V_1 \subset \tau_E$ such that E_1 is the orbit of x under V_1 (i.e. "an affine subspace in a vector space is just a coset (or translate) of a linear subspace $E_1 = x + V_1$.") An affine subspace of dimension 0 is thus a point and an affine subspace of dimension 1 is a line.

Exercise. *Show that if l, l' are (affine) lines and $x, y \in l$ and $x', y' \in l'$ are pairs of distinct points, then there is a unique affine map $f : l \longrightarrow l'$ such that $f(x) = x'$ and $f(y) = y'$. If $x, y, z \in l$ (with $x \neq y$), then define $[x, y, z]$ to be the image of z under the unique affine map $f : l \longrightarrow \mathbf{R}$ with $f(x) = 0$ and $f(y) = 1$. Show that if $l = \mathbf{R}$, then $[x, y, z]$ is given by the formula*

$$[x, y, z] = \frac{z - x}{y - x}.$$

1.9 Although an affine automorphism of an affine space E need not preserve a natural measure on E, Euclidean volume nonetheless does behave rather well with respect to affine maps. The Euclidean volume form ω can almost be characterized affinely by its parallelism: it is invariant under all translations. Moreover two τ_E-invariant volume forms differ by a scalar multiple but there is no natural way to normalize. Such a volume form will be called a *parallel volume form*. If $g \in \mathrm{Aff}(E)$, then the distortion of volume is given by

$$g^* \omega = \det \mathbf{L}(g) \cdot \omega.$$

Thus although there is no canonically normalized volume or measure there is a natural affinely invariant line of measures on an affine space. The subgroup $\mathrm{SAff}(E)$ of $\mathrm{Aff}(E)$ consisting of volume-preserving affine transformations is the inverse image $\mathbf{L}^{-1}(\mathrm{SL}(E))$, sometimes called the *special affine group* of E. Here $\mathrm{SL}(E)$ denotes, as usual, the *special linear group*

$$\mathrm{Ker}\det : \mathrm{GL}(E) \longrightarrow \mathbf{R}^* = \{g \in \mathrm{GL}(E) \mid \det(g) = 1\}.$$

1.10 Given a finite subset $F \subset E$ of an affine space, its center of gravity $\bar{F} \in E$ is an affinely invariant notion. That is, given an affine map $\phi : E \longrightarrow E'$ we have

$$(\phi(\bar{F})) = \phi(\bar{F}).$$

This operation can be generalized as follows.

PROPOSITION. *Let μ be a probability measure on an affine space E. Then there exists a unique point $\bar{x} \in E$ (the centroid of μ) such that for all affine maps $f : E \longrightarrow \mathbf{R}$,*

$$f(x) = \int_E f \, d\mu$$

PROOF: Let $(x^1, \dots, x^n)$ be an affine coordinate system on E. Let $\bar{x} \in E$ be the points with coordinates $(\bar{x}^1, \dots, \bar{x}^n)$ given by

$$\bar{x}^i = \int_E x^i d\mu.$$

This uniquely determines $\bar{x} \in E$; we must show that () is satisfied for all affine functions. If $f : E \longrightarrow \mathbf{R}$ is any affine function, then there exist $a_1, \dots, a_n, b$ such that $f = a_1 x^1 + \cdots + a_n x^n + b$ and thus

$$f(\bar{x}) = a_1 \int_E \bar{x}^1 \, d\mu + \cdots + a_n \int_E \bar{x}^n \, d\mu + b \int_E d\mu = \int_E f \, d\mu$$

as claimed. ∎

We call $\bar{x}$ the *center of mass* of μ and denote it by $\bar{x} = \mathrm{com}(\mu)$.

Now let $C \subset E$ be a *convex body*, i.e. a convex open subset having compact closure. Then Ω determines a probability measure μ_C on E by

$$\mu_C(X) = \frac{\int_{X \cap C} \omega}{\int_C \omega}$$

where ω is any parallel volume form on E. The center of mass of μ_C is by definition the centroid $\bar{C}$ of C.

PROPOSITION. *Let $C \subset E$ be a convex body. Then the centroid of C lies in C.*

PROOF: By [] every convex body C is the intersection of half-spaces, i.e.

$$C = \{x \in E \mid f(x) < 0 \text{ for all affine maps } f : E \longrightarrow \mathbf{R} \text{ such that} f|_C > 0\}$$

Thus if f is such an affine map, then clearly $f(\bar{C}) > 0$ and thus $\bar{C} \in C$. ∎

1.11 We have been working entirely over $\mathbf{R}$, but it is clear one may study affine geometry over any field. If $\mathbf{k} \supset \mathbf{R}$ is a field extension, then every $\mathbf{k}$-vector space is a vector space over $\mathbf{R}$ and thus every $\mathbf{k}$-affine space is an $\mathbf{R}$-affine space. In this way we obtain more refined geometric structures on affine spaces by considering affine maps whose linear parts are linear over $\mathbf{k}$.

EXERCISE. *Show that 1-dimensional complex affine geometry is the same as (orientation-preserving) 2-dimensional similarity geometry.*

§2: PROJECTIVE GEOMETRY

2.1 Projective geometry may be construed as a way of "closing off" (i.e. compactifying) affine geometry. To develop an intuitive feel for projective geometry, consider how points in $\mathbf{R}^n$ may "degenerate," i.e. "go to infinity." Naturally it takes the least work to move to infinity along straight lines moving at constant speed (zero acceleration) and two such geodesic paths go to the "same point at infinity" if they are parallel. Imagine two railroad tracks running parallel to each other; they meet at "infinity." We will thus force parallel lines to intersect by attaching to affine space a space of "points at infinity," where parallel lines intersect.

Let E be an affine space; then the relation of two lines in E being parallel is an equivalence relation. We define an *ideal point* of E to be a parallelism class of lines in E. The *ideal set* of an affine space E is the space $\mathbf{P}_\infty(E)$ of ideal points, with the quotient topology. If $l, l' \subset E$ are parallel lines, then the point in $\mathbf{P}_\infty$ corresponding to their parallelism class is defined to be their intersection. So two lines are parallel $\Longleftrightarrow$ they intersect at infinity.

Projective space is defined to be the union $\mathbf{P}(E) = E \cup \mathbf{P}_\infty(E)$. The natural structure on $\mathbf{P}(E)$ is perhaps most easily seen in terms of an alternate, maybe more familiar description. We may embed E as an affine hyperplane in a vector space $V = V(E)$ as follows. Let $V = \tau_E \oplus \mathbf{R}$ and choose an origin $x_0 \in E$; then the map $E \longrightarrow V$ which assigns to $x \in E$ the pair $(x - x_0, 1)$ embeds E as an affine hyperplane in V which misses 0. Let $\mathbf{P}(V)$ denote the space of all lines through $0 \in V$ with the quotient topology; the composition

$$\iota : E \longrightarrow V - \{0\} \longrightarrow \mathbf{P}(V)$$

is an embedding of E as an open dense subset of $\mathbf{P}(V)$. Now the complement $\mathbf{P}(V) - \iota(E)$ consists of all lines through the origin in $\tau_E \oplus \{0\}$ and is in natural bijective correspondence with $\mathbf{P}_\infty(E)$: given a line l in E, the 2-plane span(l) it spans meets $\tau_E \oplus \{0\}$ in a line corresponding to a point in $\mathbf{P}_\infty(E)$; conversely lines l_1, l_2 in E are parallel if

$$\operatorname{span}(l_1) \cap (\tau_E \oplus \{0\}) = \operatorname{span}(l_2) \cap (\tau_E \oplus \{0\}).$$

In this way we topologize projective space $\mathbf{P}(E) = E \cup \mathbf{P}_\infty(E)$ in a natural way.

Projective geometry arose historically out of the efforts of artisans during the Renaissance to understand perspective. Imagine a one-eyed painter looking at a 2-dimensional canvas (the affine plane E), his eye being the origin in the 3-dimensional vector space V. As he moves around or tilts the canvas, the metric geometry of the canvas as he sees it changes. As the canvas is tilted, parallel lines no longer appear parallel (like railroad tracks viewed from above ground) and distance and angle are distorted. But lines stay lines and the basic relations of collinearity and concurrence are unchanged. The change in perspective given by "tilting" the canvas or the painter changing position is determined by a linear transformation of V, since a point on E is determined completely by the 1-dimensional linear subspace of V containing it. (One must solve systems of linear equations to write down the effect of such transformation.) Projective geometry is the study of points, lines and the incidence relations between them.

2.2 A point of $\mathbf{P}^n$ then corresponds to a nonzero vector in $\mathbf{R}^{n+1}$, uniquely defined up to a nonzero scalar multiple. If $a^1, \dots, a^{n+1} \in \mathbf{R}$ and not all of the a^i are zero, then we denote the point in $\mathbf{P}^n$ corresponding to the nonzero vector $(a^1, \dots, a^{n+1}) \in \mathbf{R}^{n+1}$ by $[a^1, \dots, a^{n+1}]$; the a^i are called the *homogeneous coordinates* of the corresponding point in $\mathbf{P}^n$. The original affine space $\mathbf{R}^n$ is the subset comprising points with homogeneous coordinates $[a^1, \dots, a^n, 1]$ where $(a^1, \dots, a^n)$ are the corresponding (affine) coordinates.

EXERCISE. *Let $E = \mathbf{R}^n$ and let $\mathbf{P} = \mathbf{P}^n$ be the projective space obtained from E as above. Exhibit $\mathbf{P}^n$ as a quotient of the unit sphere $S^n \subset \mathbf{R}^{n+1}$ by the antipodal map. Thus $\mathbf{P}^n$ is compact and for $n > 1$ has fundamental group of order two. Show that $\mathbf{P}^n$ is orientable $\iff n$ is odd.*

Thus to every projective space $\mathbf{P}$ there exists a vector space $V = V(\mathbf{P})$ such that the points of $\mathbf{P}$ correspond to the lines through 0 in V. We denote the quotient map by $\Pi : V - \{0\} \longrightarrow \mathbf{P}$. If $\mathbf{P}, \mathbf{P}'$ are projective spaces and $U \subset \mathbf{P}$ is an open set then a map $f : U \longrightarrow \mathbf{P}'$ is *locally projective* if for each component $U_i \subset U$ there exists a linear map $\tilde{f}_i : V(\mathbf{P}) \longrightarrow V(\mathbf{P}')$ such that the restrictions of $f \circ \Pi$ and $\Pi \circ \tilde{f}_i$ to $\Pi^{-1}U_i$ are identical. A *projective automorphism* or *collineation* of $\mathbf{P}$ is an invertible locally projective map $\mathbf{P} \longrightarrow \mathbf{P}$. We denote the space of locally projective maps $U \to \mathbf{P}'$ by $\mathrm{Proj}(U, \mathbf{P}')$.

Locally projective maps (and hence also locally affine maps) satisfy the *Unique Extension Property:* if $U \subset U' \subset \mathbf{P}$ are open subsets of a projective space with U nonempty and U' connected, then any two locally projective maps $f_1, f_2 : U' \longrightarrow \mathbf{P}'$ which agree on U must be identical.

EXERCISE 2.3. *Show that the projective automorphisms of $\mathbf{P}$ form a group and that this group (which we denote $\mathrm{Aut}(\mathbf{P})$) has the following description. If $f : \mathbf{P} \longrightarrow \mathbf{P}$ is a projective automorphism, then there exists a linear isomorphism $\tilde{f} : V \longrightarrow V$ inducing f. Indeed there is a short exact sequence*

$$1 \longrightarrow \mathbf{R}^* \longrightarrow \mathrm{GL}(V) \longrightarrow \mathrm{Aut}(\mathbf{P}) \longrightarrow 1$$

where $\mathbf{R}^* \longrightarrow \mathrm{GL}(V)$ is the inclusion of the group of multiplications by nonzero scalars. (Sometimes this quotient $\mathrm{GL}(V)/\mathbf{R}^* \cong \mathrm{Aut}(\mathbf{P}^n)$ (the projective general linear group) is denoted by $\mathrm{PGL}(V)$ or $\mathrm{PGL}(n+1, \mathbf{R})$.) Show that if n is even, then $\mathrm{Aut}(\mathbf{P}^n) \cong \mathrm{SL}(n+1; \mathbf{R})$ and if n is odd, then $\mathrm{Aut}(\mathbf{P}^n)$ has two connected components, and its identity component is doubly covered by $\mathrm{SL}(n+1; \mathbf{R})$.

If V, V' are vector spaces with associated projective spaces $\mathbf{P}, \mathbf{P}'$ then a linear map $\tilde{f} : V \longrightarrow V'$ always maps lines through 0 to lines through 0. But $\tilde{f}$ only induces a map $f : \mathbf{P} \longrightarrow \mathbf{P}'$ if it is injective, since $f(x)$ can only be defined if $\tilde{f}(\tilde{x}) \neq 0$ (where $\tilde{x}$ is a point of $\Pi^{-1}(x) \subset V - \{0\}$). Suppose that $\tilde{f}$ is a (not necessarily injective) linear map and let $N(f) = \Pi(\mathrm{Ker}(\tilde{f}))$. The resulting *projective endomorphism* of $\mathbf{P}$ is defined on the complement $\mathbf{P} - N(f)$; if $N(f) \neq \emptyset$, then the corresponding projective endomorphism is by definition a *singular projective transformation* of $\mathbf{P}$.

A projective map $\iota : \mathbf{P}_1 \longrightarrow \mathbf{P}$ corresponds to a linear map $\tilde{\iota} : V_1 \longrightarrow V$ between the corresponding vector spaces (well-defined up to scalar multiplication). Since ι is defined on all of $\mathbf{P}_1$, $\tilde{\iota}$ is an injective linear map and hence corresponds to an embedding. Such an embedding (or its image) will be called a *projective subspace*. Projective subspaces of dimension k correspond to linear subspaces of dimension $k+1$. (By convention the empty set is a projective space of dimension -1.) Note that the "bad set" $N(f)$ of a singular projective transformation is a projective subspace. Two projective subspaces of dimensions k, l where $k + l \geq n$ intersect in a projective subspace of dimension at least $k + l - n$. The *rank* of a projective endomorphism is defined to be the dimension of its image.

EXERCISE 2.4. *Let* $\mathbf{P}$ *be a projective space of dimension* n. *Show that the (possibly singular) projective transformations of* $\mathbf{P}$ *form themselves a projective space of dimension* $(n+1)^2 - 1$. *We denote this projective space by* $\mathrm{End}(\mathbf{P})$. *Show that if* $f \in \mathrm{End}(\mathbf{P})$, *then*

$$\dim N(f) + \mathrm{rank}(f) = n - 1.$$

Show that $f \in \mathrm{End}(\mathbf{P})$ *is nonsingular (i.e. a collineation)* $\Longleftrightarrow \mathrm{rank}(f) = n \Longleftrightarrow N(f) = \emptyset$.

An important kind of projective endomorphism is a *projection*, also called a *perspectivity*. Let $A^k, B^l \subset \mathbf{P}^n$ be disjoint projective subspaces whose dimensions satisfy $k + l = n - 1$. We define the projection onto A^k from B^l

$$\Pi_{A^k, B^l} : \mathbf{P}^n - B^l \longrightarrow A^k$$

as follows. For every $x \in \mathbf{P}^n - A^k$ there is a unique projective subspace $\mathrm{span}(\{x\} \cup B^l)$ of dimension $l + 1$ containing $\{x\} \cup B^l$ which intersects A^k in a unique point. Let $\Pi_{A^k, B^l}(x)$ be this point. (Clearly such a perspectivity is the projectivization of a linear projection $V \longrightarrow V$.) It can be shown that every projective map defined on a projective subspace can be obtained as the composition of projections.

2.5 EXERCISE. *Suppose that* n *is even. Show that a collineation of* $\mathbf{P}^n$ *which has order two fixes a unique pair of disjoint projective subspaces* $A^k, B^l \subset \mathbf{P}^n$ *where* $k + l = n - 1$. *Conversely suppose that* $A^k, B^l \subset \mathbf{P}^n$ *where* $k + l = n - 1$ *are disjoint projective subspaces; then there is a unique collineation of order two whose set of fixed points is* $A^k \cup B^l$. *If* n *is odd find a collineation of order two which has no fixed points.*

Such a collineation will be called a *projective reflection*. Consider the case $\mathbf{P} = \mathbf{P}^2$. Let R be a projective reflection with fixed line l and isolated fixed point p. Choosing homogeneous coordinates $[u^0, u^1, u^2]$ so that $l = \{[0, u^1, u^2] \mid (u^1, u^2) \neq (0, 0)\}$ and $p = [1, 0, 0]$, we see that R is represented by the diagonal matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

in $\mathrm{SL}(3;\mathbf{R})$. Note that near l the reflection looks like a Euclidean reflection in l and reverses orientation. (Indeed R is given by $R(y^1, y^2) = (-y^1, y^2)$ in affine coordinates $y^1 = u^0/u^2, y^2 = u^1/u^2$.) On the other hand, near p, the reflection looks like reflection in p (i.e. rotation of order two about p) and preserves orientation. (In affine coordinates $x^1 = u^1/u^0, x^2 = u^2/u^0$ we have $R(x^1, y^2) = (-x^1, -x^2)$.) Of course there is no global orientation on $\mathbf{P}^2$ and the fact that a single reflection can appear simultaneously as a point-symmetry and reflection in a line is an indication of the topological complexity of $\mathbf{P}^2$.

We can consider the passage between the geometry of $\mathbf{P}$ and the algebra of V as a kind of "dictionary" between linear algebra and projective geometry. Linear maps and linear subspaces correspond geometrically to projective maps and projective subspaces; inclusions, intersections and linear spans correspond to incidence relations in projective geometry. In this way we can either use projective geometry to geometrically picture linear algebra or linear algebra to prove theorems in geometry.

2.6 We shall be interested in the singular projective transformations since they occur as limits of nonsingular projective transformations. The collineation group $\mathrm{Aut}(\mathbf{P})$ of $\mathbf{P} = \mathbf{P}^n$ is a large noncompact group which is naturally embedded in the projective space $\mathrm{End}(\mathbf{P})$ as an open dense subset. Thus it will be crucial to understand precisely what it means for a sequence of collineations to converge to a (possibly singular) projective transformation.

PROPOSITION. *Let $g_m \in \mathrm{Aut}(\mathbf{P})$ be a sequence of collineations of $\mathbf{P}$ and let $g_\infty \in \mathrm{End}(\mathbf{P})$. Then the sequence g_m converges to g_∞ in $\mathrm{End}(\mathbf{P})$ $\iff$ the restrictions $g_m|_K$ converge uniformly to $g_\infty|_K$ for all compact sets $K \subset \mathbf{P} - N(g_\infty)$.*

PROOF: Convergence in $\mathrm{End}(\mathbf{P})$ may be described as follows. Let $\mathbf{P} = \mathbf{P}(V)$ where $V \cong \mathbf{R}^{n+1}$ is a vector space. Then $\mathrm{End}(\mathbf{P})$ is the projective space associated to the vector space $\mathrm{End}(V)$ of $(n+1)$-square matrices. If $a = (a^i_j) \in \mathrm{End}(V)$ is such a matrix, let

$$\|a\| = \sqrt{\sum_{i,j=1}^{n+1} |a^i_j|^2}$$

denote its Euclidean norm; projective endomorphisms then correspond to matrices a with $\|a\| = 1$, uniquely determined up to the antipodal map $a \mapsto -a$. The following lemma will be useful in the proof of 2.6:

2.7 LEMMA. *Let V, V' be vector spaces and let $\tilde{f}_n : V \longrightarrow V'$ be a sequence of linear maps converging to $\tilde{f}_\infty : V \longrightarrow V'$. Let $\tilde{K} \subset V$ be a compact subset of $V - \mathrm{Ker}(\tilde{f}_\infty)$ and let f_i be the map defined by*

$$f_i(x) = \frac{\tilde{f}_i(x)}{|\tilde{f}_i(x)|}.$$

Then f_n converges uniformly to f_∞ on $\tilde{K}$ as $n \longrightarrow \infty$.

PROOF: Choose $C > 0$ such that $C \leq |\tilde{f}_\infty(x)| \leq C^{-1}$ for $x \in \tilde{K}$. Let $\epsilon > 0$. There exists N such that if $n > N$, then

$$|\tilde{f}_\infty(x) - \tilde{f}_n(x)| < \frac{C\epsilon}{2},$$
$$|1 - \frac{\tilde{f}_n(x)}{\tilde{f}_\infty(x)}| < \frac{\epsilon}{2}$$

for $x \in \tilde{K}$. It follows that

$$\begin{aligned}
\|f_n(x) - f_\infty(x)\| &= \|\frac{\tilde{f}_n(x)}{\|\tilde{f}_n(x)\|} - \frac{\tilde{f}_\infty(x)}{\|\tilde{f}_\infty(x)\|}\| \\
&= \frac{1}{\|\tilde{f}_\infty(x)\|}\|\frac{\|\tilde{f}_\infty(x)\|}{\|\tilde{f}_n(x)\|}\tilde{f}_n(x) - \tilde{f}_\infty(x)\| \\
&\leq \frac{1}{\|\tilde{f}_\infty(x)\|}(\|\frac{\|\tilde{f}_\infty(x)\|}{\|\tilde{f}_n(x)\|}\tilde{f}_n(x) - \tilde{f}_n(x)\| + \|\tilde{f}_n(x) - \tilde{f}_\infty(x)\|) \\
&= |1 - \frac{\|\tilde{f}_n(x)\|}{\|\tilde{f}_\infty(x)\|}| + \frac{1}{\|\tilde{f}_\infty(x)\|}\|\tilde{f}_n(x) - \tilde{f}_\infty(x)\| \\
&< \frac{\epsilon}{2} + C^{-1}(\frac{C\epsilon}{2}) = \epsilon
\end{aligned}$$

for all $x \in \tilde{K}$ as desired. ∎

The proof of Proposition 2.6 proceeds as follows. If g_m is a sequence of locally projective maps defined on a connected domain $\Omega \subset \mathbf{P}$ converging uniformly on all compact subsets of Ω to a map $g_\infty : \Omega \longrightarrow \mathbf{P}'$, then there exists a lift $\tilde{g}_\infty$ which is a linear transformation of norm 1 and lifts $\tilde{g}_m$, also linear transformations of norm 1, converging to $\tilde{g}_\infty$. It follows that $g_m \longrightarrow g_\infty$ in $\mathrm{End}(\mathbf{P})$. Conversely if $g_m \longrightarrow g_\infty$ in $\mathrm{End}(\mathbf{P})$ and $K \subset \mathbf{P} - N(g_\infty)$, we may choose lifts as above and a compact set $\tilde{K} \subset V$ such that $\Pi(\tilde{K}) = K$. By Lemma 2.7, the normalized linear maps $\frac{\tilde{g}_m}{|\tilde{g}_m|}$ converge uniformly to $\frac{g_\infty}{|g_\infty|}$ on $\tilde{K}$ and hence g_m converges uniformly to g_∞ on K. The proof of Proposition 2.6 is now complete. ∎

Let us consider a few examples of this convergence. Consider the case first when $n = 1$. Let $\lambda_m \in \mathbf{R}$ be a sequence converging to $+\infty$ and consider the projective transformations given by the diagonal matrices

$$g_m = \begin{bmatrix} \lambda_m & 0 \\ 0 & (\lambda_m)^{-1} \end{bmatrix}$$

Then $g_m \longrightarrow g_\infty$ where g_∞ is the singular projective transformation corresponding to the matrix

$$g_\infty = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

— this singular projective transformation is undefined at $N(g_\infty) = \{[0,1]\}$; every point other than $[0,1]$ is sent to $[1,0]$. It is easy to see that a singular projective transformation of $\mathbf{P}^1$ is determined by the ordered pair of points $N(f), \mathrm{Image}(f)$ (which may be coincident).

More interesting phenomena arise when $n = 2$. Let $g_m \in \mathrm{Aut}(\mathbf{P}^2)$ be a sequence of diagonal matrices

$$\begin{bmatrix} \lambda_m & 0 & 0 \\ 0 & \mu_m & 0 \\ 0 & 0 & \nu_m \end{bmatrix}$$

where $0 < \lambda_m < \mu_m < \nu_m$ and $\lambda_m \mu_m \nu_m = 1$. Corresponding to the three eigenvectors (the coordinate axes in $\mathbf{R}^3$) are three fixed points $p_1 = [1,0,0]$, $p_2 = [0,1,0]$, $p_3 = [0,0,1]$ They span three invariant lines $l_1 = \overleftrightarrow{p_2p_3}$, $l_2 = \overleftrightarrow{p_3p_1}$ and $l_3 = \overleftrightarrow{p_3p_1}$. Since $0 < \lambda_m < \mu_m < \nu_m$, the collineation has an repelling fixed point at p_1, a saddle point at p_2 and an attracting fixed point at p_3. Points on l_2 near p_1 are repelled from p_1 faster than points on l_3 and points on l_2 near p_3 are attracted to p_3 more strongly than points on l_1. Suppose that g_m does not converge to a nonsingular matrix; it follows that $\nu_m \longrightarrow +\infty$ and $\lambda_m \longrightarrow 0$ as $m \longrightarrow \infty$. Suppose that $\mu_m/\nu_m \longrightarrow \rho$; then g_m converges to the singular projective transformation g_∞ determined by the matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which, if $\rho > 0$, has undefined set $N(g_\infty) = p_1$ and image l_1; otherwise $N(g_\infty) = l_2$ and $\mathrm{Image}(g_\infty) = p_2$.

Exercise. *Let $U \subset \mathbf{P}$ be a connected open subset of a projective space of dimension greater than 1. Let $f : U \longrightarrow \mathbf{P}$ be a local diffeomorphism. Then f is locally projective $\Longleftrightarrow$ for each line $l \subset \mathbf{P}$, the image $f(l \cap U)$ is a line.*

2.7 Let $H \subset \mathbf{P}$ be a projective hyperplane (projective subspace of codimension one). Then the complement $\mathbf{P} - H$ has a natural affine geometry, i.e. is an affine space in a natural way. Indeed the group of projective automorphisms $\mathbf{P} \to \mathbf{P}$ leaving fixed each $x \in H$ and whose differential $T_x\mathbf{P} \to T_x\mathbf{P}$ is a volume-preserving linear automorphism is a vector group acting simply transitively on $\mathbf{P} - H$. Moreover the group of projective transformations of $\mathbf{P}$ leaving H invariant is the full group of automorphisms of this affine space. In this way affine geometry is "embedded" in projective geometry.

In terms of matrices this appears as follows. Let $E = \mathbf{R}^n$; then the affine subspace of

$$V = \tau_E \oplus \mathbf{R} = \mathbf{R}^{n+1}$$

corresponding to E is $\mathbf{R}^n \times \{1\} \subset \mathbf{R}^{n+1}$, the point of E with *affine* or *inhomogeneous* coordinates $(x^1, \ldots, x^n)$ has homogeneous coordinates $[x^1, \ldots, x^n, 1]$. Let $f \in \mathrm{Aff}(E)$ be the affine transformation with linear part $A \in \mathrm{GL}(n; \mathbf{R})$ and translational part $b \in \mathbf{R}^n$, i.e. $f(x) = Ax + b$, is then represented by the $(n+1)$-square matrix

$$\begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}$$

where b is a column vector and 0 denotes the $1 \times n$ zero row vector.

EXERCISE. *Let $O \in \mathbf{P}^n$ be a point, say $[0, \ldots, 0, 1]$. Show that the group $G_{-1} = G_{-1}(O)$ of projective transformations fixing O and acting trivially on the tangent space $T_O\mathbf{P}^n$ is given by matrices of the form*

$$\begin{bmatrix} I_n & 0 \\ \xi & 1 \end{bmatrix}$$

where I_n is the $n \times n$ identity matrix and $\xi = (\xi_1, \ldots, \xi_n) \in (\mathbf{R}^n)^*$ is a row vector; in affine coordinates such a transformation is given by

$$(x^1, \ldots, x^n) \mapsto (\frac{x^1}{1 + \sum_{i=1}^n \xi_i x^i}, \ldots, \frac{x^n}{1 + \sum_{i=1}^n \xi_i x^i}).$$

Show that this group is isomorphic to a n-dimensional vector group and that its Lie algebra consists of vector fields of the form

$$(\sum_{i=1}^n \xi_i x^i)\rho$$

where

$$\rho = \sum_{i=1}^n x^i \frac{\partial}{\partial x^i}$$

is the radiant vector field radiating from the origin and $\xi \in (\mathbf{R}^n)^*$. Note that such vector fields comprise an n-dimensional abelian Lie algebra of polynomial vector fields of degree 2 in affine coordinates.

Let H be a hyperplane not containing O, e.g.

$$H = \{[x^1, \ldots, x^n, 0] \mid (x^1, \ldots, x^n) \in \mathbf{R}^n\}.$$

Let $G_1 = G_1(H)$ denote the group of translations of the affine space $\mathbf{P} - H$ and let $G_0 = G_0(O, H) \cong \mathrm{GL}(n, \mathbf{R})$ denote the group of collineations of $\mathbf{P}$ fixing O and leaving invariant H. (Alternatively $G_0(O, H)$ is the group of collineations centralizing the radiant vector field $\rho = \rho(O, H)$ above.) Let $\mathfrak{g}$ denote the Lie algebra of $\mathrm{Aut}(\mathbf{P})$ and let $\mathfrak{g}_{-1}, \mathfrak{g}_0, \mathfrak{g}_1$ be the Lie algebras of G_{-1}, G_0, G_1 respectively. Show that there is a vector space decomposition

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

where $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for $i, j = \pm 1, 0$ (where $\mathfrak{g}_i = 0$ for $|i| > 1$). Furthermore show that the stabilizer of O has Lie algebra $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ and the stabilizer of H has Lie algebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1$.

2.8 Let l be a projective line $x, z \in l$ be distinct points. Then there exists a unique reflection (a *harmonic homology* in classical terminology) $\rho_{x,z} : l \longrightarrow l$ whose fixed-point set is $\{x, z\}$. We say that a pair of points y, w are *harmonic* with respect to x, z if $\rho_{x,z}$ interchanges them. In that case one can show that x, z are harmonic with respect to y, w. Furthermore this relation is equivalent to the existence of lines p, q through x and lines r, s through z such that

$$\begin{aligned} y &= \overleftrightarrow{(p \cap r)(q \cap s)} \cap l \\ z &= \overleftrightarrow{(p \cap s)(q \cap r)} \cap l. \end{aligned}$$

This leads to a projective-geometry construction of reflection, as follows. Let $x, y, z \in l$ be fixed; we seek the harmonic conjugate of y with respect to x, z, i.e. the image $R_{x,z}(y)$. Erect arbitrary lines (in general position) p, q through x and a line r through z. Through y draw the line through $r \cap q$; join its intersection with p with z to form line s, i.e.

$$s = \overleftrightarrow{z \quad (p \cap \overleftrightarrow{y\, r \cap q})}.$$

Then $R_{x,z}(y)$ will be the intersection of s with l.

EXERCISE. *Consider the projective line* $\mathbf{P}^1 = \mathbf{R} \cup \{\infty\}$. *Show that for every rational number* $x \in \mathbf{Q}$ *there exists a sequence* $x_0, x_1, x_2, x_3, \ldots, x_n \in \mathbf{P}^1$ *such that* $\{x_0, x_1, x_2\} = \{0, 1, \infty\}$ *and for each* $i \geq 3$, *there is a harmonic quadruple* (x_i, y_i, z_i, w_i) *with* $y_i, z_i, w_i \in \{x_0, x_1, \ldots, x_{i-1}\}$. *If x is written in reduced form p/q then what is the smallest n for which x can be reached in this way?*

2.9 EXERCISE (SYNTHETIC ARITHMETIC). *Using the above synthetic geometry construction of harmonic quadruples, show how to add, subtract, multiply, and divide real numbers by a straightedge-and-pencil construction. In other words, draw a line l on a piece of paper and choose three points to have coordinates* $0, 1, \infty$ *on it. (∞ can be "at infinity" if you like.) Choose arbitrary points corresponding to real numbers x, y. Using only a straightedge (not a ruler!) construct the points corresponding to* $x + y, x - y, xy$, *and* x/y *if* $y \neq 0$.

2.10 If $l \subset \mathbf{P}$ and $l' \subset \mathbf{P}'$ are projective lines, the *Fundamental Theorem of Projective Geometry* asserts that for given triples $x, y, z \in l$ and $x', y', z' \in l'$ of distinct points there exists a unique projective map $f : l \longrightarrow l'$ with $f(x) = x'$, $f(y) = y'$, and $f(z) = z'$. If $w \in l$ then the *cross-ratio* $[x, y, w, z]$ is defined to be the image of w under the unique collineation $f : l \longrightarrow \mathbf{P}^1$ with $f(x) = 0$, $f(y) = 1$, and $f(z) = \infty$. If $l = \mathbf{P}^1$, then the cross-ratio is given by the formula

$$[x, y, w, z] = \frac{w - x}{w - z} / \frac{y - x}{y - z}.$$

The cross-ratio can be extended to quadruples of four points, of which at least three are distinct. A pair y, w is harmonic with respect to x, z (in which case we say that (x, y, w, z) is a *harmonic quadruple*) $\Longleftrightarrow$ the cross-ratio $[x, y, w, z] = -1$.

EXERCISE. *Let σ be a permutation on four symbols. Show that there exists a linear fractional transformation Φ_σ such that*

$$[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}] = \Phi_\sigma([x_1, x_2, x_3, x_4].$$

In particular determine which permutations leave the cross-ratio invariant.

Show that a homeomorphism $f : \mathbf{P}^1 \longrightarrow \mathbf{P}^1$ *is projective* $\Longleftrightarrow$ *f preserves harmonic quadruples* $\Longleftrightarrow$ *f preserves cross-ratios, i.e. for all quadruples (x, y, w, z), the cross-ratios satisfy*

$$[f(x), f(y), f(w), f(z)] = [x, y, w, z].$$

2.11 EXERCISE. *Let $p, p' \in \mathbf{P}$ be distinct points in $\mathbf{P} = \mathbf{P}^2$ and $l, l' \in \mathbf{P}^*$ be distinct lines such that $p \notin l$ and $p' \notin l'$. Let R and R' be the projective reflections of $\mathbf{P}$ (collineations of order two) having fixed-point set $l \cup p$ and $l' \cup p'$ respectively. Let $O = l \cap l'$. Let $\mathbf{P}_O$ denote the projective line whose points are the lines incident to O. Let ρ denote the cross-ratio of the four lines*

$$l, \overleftrightarrow{Op}, \overleftrightarrow{Op'}, l'$$

as elements of $\mathbf{P}_O$. Then RR' fixes O and represents a rotation of angle θ in the tangent space $T_O(\mathbf{P}) \iff$

$$\rho = \frac{1}{2}(1 + \cos\theta)$$

for $0 < \theta < \pi$ and is a rotation of angle $\theta \iff p \in l'$ and $p' \in l$.

§3: DUALITY, NON-EUCLIDEAN GEOMETRY AND PROJECTIVE METRICS

3.1 In an axiomatic development of projective geometry, there is a basic symmetry: A pair of distinct points lie on a unique line and a pair of distinct lines meet in a unique point (in dimension two). As a consequence any statement about the geometry of $\mathbf{P}^2$ can be "dualized" by replacing "point" by "line," "line"by "point," "collinear" with "concurrent," etc. *in a completely consistent fashion.*

Perhaps the oldest nontrivial theorem of projective geometry is Pappus' theorem (300 A.D.), which asserts that if $l, l' \subset \mathbf{P}^2$ are distinct lines and $A, B, C \in l$ and $A', B', C' \in l'$ are triples of distinct points, then the three points

$$\overleftrightarrow{AB'} \cap \overleftrightarrow{A'B}, \quad \overleftrightarrow{BC'} \cap \overleftrightarrow{B'C}, \quad \overleftrightarrow{CA'} \cap \overleftrightarrow{C'A}$$

are collinear. The dual of Pappus' theorem is therefore: if $p, p' \in \mathbf{P}^2$ are distinct points and a, b, c are distinct lines all passing through p and a', b', c' are distinct lines all passing through p', then the three lines

$$\overleftrightarrow{(a \cap b') \quad (a' \cap b)}, \quad \overleftrightarrow{(b \cap c') \quad (b' \cap c)}, \quad \overleftrightarrow{(c \cap a') \quad (c' \cap a)}$$

are concurrent. (According to Coxeter [C1]), Hilbert observed that Pappus' theorem is equivalent to the commutative law of multiplication.)

In terms of our projective geometry/linear algebra dictionary, projective duality translates into duality between vector spaces as follows. Let $\mathbf{P}$ be a projective space and let V be the associated vector space. A nonzero linear functional $\psi : V \longrightarrow \mathbf{R}$ defines a projective hyperplane H_ψ in $\mathbf{P}$; two such functionals define the same hyperplane $\iff$ they differ by a nonzero scalar multiple,i.e. they determine the same line in the vector space V^* dual to V. (Alternately ψ defines the constant projective map $\mathbf{P} - H_\psi \longrightarrow \mathbf{P}^0$ which is completely specified by its undefined set H_ψ.) We thus define the *projective space dual to* $\mathbf{P}$ as follows. The dual projective space $\mathbf{P}^*$ of lines in the dual vector space V^* correspond to hyperplanes in $\mathbf{P}$. The line joining two points in $\mathbf{P}^*$ corresponds to the intersection of the corresponding hyperplanes in $\mathbf{P}$, and a hyperplane in $\mathbf{P}^*$ corresponds to a point in $\mathbf{P}$. In general if $\mathbf{P}$ is an n-dimensional projective space there is a natural correspondence

$$\{k - \text{dimensional subspaces of } \mathbf{P}\} \longleftrightarrow \{l - \text{dimensional subspaces of } \mathbf{P}^*\}$$

where $k + l = n - 1$. In particular we have an isomorphism of $\mathbf{P}$ with the dual of $\mathbf{P}^*$.

Let $f : \mathbf{P} \longrightarrow \mathbf{P}'$ be a projective map between projective spaces. Then for each hyperplane $H' \subset \mathbf{P}'$ the inverse image $f^{-1}(H')$ is a hyperplane in $\mathbf{P}$. There results a map $f^\dagger : (\mathbf{P}')^* \longrightarrow \mathbf{P}^*$, the *transpose* of the projective map f. (Evidently $f^\dagger$ is the projectivization of the transpose of the linear map $\tilde{f} : V \longrightarrow V'$.)

3.2 Let $\mathbf{P}$ be an n-dimensional projective space and $\mathbf{P}^*$ its dual. A *correlation* of $\mathbf{P}$ is a projective isomorphism $\theta : \mathbf{P} \longrightarrow \mathbf{P}^*$. That is, θ associates to each point in $\mathbf{P}$ a hyperplane in $\mathbf{P}$ in such a way that if $x_1, x_2, x_3 \in \mathbf{P}$ are collinear, then the hyperplanes $\theta(x_1), \theta(x_2), \theta(x_3) \subset \mathbf{P}$ are incident, i.e. the intersection $\theta(x_1) \cap \theta(x_2) \cap \theta(x_3)$ is a projective subspace of codimension two (rather than three, as would be the case if they were in general position). The transpose correlation $\theta^\dagger$ is also a projective isomorphism $\mathbf{P} \longrightarrow \mathbf{P}^*$ (using the reflexivity $\mathbf{P}^{**} \cong \mathbf{P}$). A correlation is a *polarity* if it is equal to its transpose.

Using the dictionary between projective geometry and linear algebra, one sees that if V is the vector space corresponding to $\mathbf{P} = \mathbf{P}(V)$, then $\mathbf{P}^* = \mathbf{P}(V^*)$ and a correlation θ is realized as a linear isomorphism $\tilde{\theta} : V \longrightarrow V^*$, which is uniquely determined up to homotheties. Linear maps $\tilde{\theta} : V \longrightarrow V^*$ correspond to bilinear forms

$$\mathbf{B}_{\tilde{\theta}} : V \times V \longrightarrow \mathbf{R}$$

under the correspondence

$$\tilde{\theta}(v)(w) = \mathbf{B}_{\tilde{\theta}}(v, w)$$

and $\tilde{\theta}$ is an isomorphism if and only if $\mathbf{B}_{\tilde{\theta}}$ is nondegenerate. Thus correlations can be interpreted analytically as projective equivalence classes of nondegenerate bilinear forms. Furthermore a correlation θ is self-inverse (i.e. a polarity) $\Longleftrightarrow$ a corresponding bilinear form $\mathbf{B}_\theta$ is symmetric.

Let θ be a polarity on $\mathbf{P}$. A point $p \in \mathbf{P}$ is *conjugate* if it is incident to its polar hyperplane, i.e. if $p \in \theta(p)$. By our dictionary we see that the conjugate points of a polarity correspond to *null vectors* of the associated quadratic form, i.e. to nonzero vectors $v \in V$ such that $\mathbf{B}_\theta(v, v) = 0$. A polarity is said to be *elliptic* if it admits no conjugate points; elliptic polarities correspond to symmetric bilinear forms which are definite. For example here is an elliptic polarity of $\mathbf{P} = \mathbf{P}^2$: a point p in $\mathbf{P}^2$ corresponds to a line $\Pi^{-1}(p)$ in Euclidean 3-space and its orthogonal complement $\Pi^{-1}(p)^\perp$ is a 2-plane corresponding to a line $\theta(p) \in \mathbf{P}^*$. It is easy to check that θ defines an elliptic polarity of $\mathbf{P}$.

In general the set of conjugate points of a polarity is a *quadric,* which up to a collineation is given in homogeneous coordinates as

$$Q = Q_{p,q} = \{[x^1, \ldots, x^{n+1}] \mid -(x^1)^2 - \cdots - (x^p)^2 + (x^{p+1})^2 + \cdots + (x^{p+q})^2 = 0\}$$

where $p + q = n + 1$ (since the corresponding symmetric bilinear form is given by the diagonal matrix $-I_p \oplus I_q$). We call (p, q) the *signature* of the polarity. The quadric Q

determines the polarity θ as follows. For brevity we consider only the case $p = 1$, in which case the complement $\mathbf{P} - Q$ has two components, a convex component

$$\Omega = \{[x^0, x^1, \ldots, x^n] \mid -(x^0)^2 + (x^1)^2 + \cdots + (x^n)^2 < 0\}$$

and a nonconvex component

$$\Omega^\dagger = \{[x^0, x^1, \ldots, x^n] \mid -(x^0)^2 + (x^1)^2 + \cdots + (x^n)^2 > 0\}$$

diffeomorphic to the total space of the tautological line bundle over $\mathbf{P}^{n-1}$ (for $n = 2$ this is a Möbius band). If $x \in Q$, let $\theta(x)$ denote the hyperplane tangent to Q at x. If $x \in \Omega^\dagger$ the points of Q lying on tangent lines to Q containing x all lie on a hyperplane which is $\theta(x)$. If $H \in \mathbf{P}^*$ is a hyperplane which intersects Q, then either H is tangent to Q (in which case $\theta(H)$ is the point of tangency) or there exists a cone tangent to Q meeting Q in $Q \cap H$ — the vertex of this cone will be $\theta(H)$. If $x \in \Omega$, then there will be no tangents to Q containing x, but by representing x as an intersection $H_1 \cap \ldots H_n$, we obtain $\theta(x)$ as the hyperplane containing $\theta(H_1), \ldots, \theta(H_n)$.

EXERCISE. *Show that $\theta : \mathbf{P} \longrightarrow \mathbf{P}^*$ is indeed a projective map.*

Observe that a polarity on $\mathbf{P}$ of signature (p, q) determines, for each non-conjugate point $x \in \mathbf{P}$ a unique reflection R_x which preserves the polarity. The group of collineations preserving such a polarity is the *projective orthogonal group* $\mathrm{PO}(p, q)$, i.e. the image of the orthogonal group $\mathrm{O}(p, q) \subset \mathrm{GL}(n + 1, \mathbf{R})$ under the projectivization homomorphism $\mathrm{GL}(n+1, \mathbf{R}) \longrightarrow \mathrm{PGL}(n+1, \mathbf{R})$ having kernel the scalar matrices $\mathbf{R}^* \subset \mathrm{GL}(n+1, \mathbf{R})$. Let $\Omega = \{\Pi(v) \in \mathbf{P} \mid \mathbf{B}(v, v) < 0\}$; then by projection from the origin Ω can be identified with the hyperquadric $\{v \in \mathbf{R}^{p,q} \mid \mathbf{B}(v, v) = -1\}$ whose induced pseudo-Riemannian metric has signature $(q, p - 1)$ and constant nonzero curvature. In particular if $(p, q) = (1, n)$ then Ω is a model for hyperbolic n-space $\mathbf{H}^n$ in the sense that the group of isometries of $\mathbf{H}^n$ are represented precisely as the group of collineations of $\mathbf{P}^n$ preserving Ω^n. In this model, geodesics are the intersections of projective lines in $\mathbf{P}$ with Ω; more generally intersections of projective subspaces with Ω define totally geodesic subspaces. Consider the case that $\mathbf{P} = \mathbf{P}^2$. Points "outside" Ω correspond to geodesics in $\mathbf{H}^2$. If $p_1, p_2 \in \Omega^\dagger$, then $\overleftrightarrow{p_1 p_2}$ meets $\Omega \iff$ the geodesics $\theta(p_1), \theta(p_2)$ are ultra-parallel in $\mathbf{H}^2$; in this case $\theta(\overleftrightarrow{p_1 p_2})$ is the geodesic orthogonal to both $\theta(p_1), \theta(p_2)$. (Geodesics $\theta(p)$ and l are orthogonal $\iff p \in l$.) Furthermore $\overleftrightarrow{p_1 p_2}$ is tangent to $Q \iff \theta(p_1)$ and $\theta(p_2)$ are parallel. For more information on this model for hyperbolic geometry, see Coxeter [C1] or Thurston [T,§2]. This model for non-Euclidean geometry seems to have first been discovered by Cayley in 1858.

3.3 We shall discuss the metric on hyperbolic space, however, in the more general setting of the Hilbert-Carathéodory-Kobayashi metric on a convex domain $\mathbf{P} = \mathbf{P}^n$. Let $V = \mathbf{R}^{n+1}$ be the corresponding vector space. A subset $\Omega \subset V$ is a *cone* $\iff$ it is invariant under positive homotheties ($\mathbf{R}^+(\Omega) = \Omega$), i.e. if $x \in \Omega$ and $r > 0$ then $rx \in \Omega$. A subset $\Omega \subset V$ is *convex* if whenever $x, y \in \Omega$, then the line segment $\overline{xy} \subset \Omega$. A convex domain $\Omega \subset V$ is *sharp* $\iff$ there is no entire affine line contained in Ω. For example, V itself and the upper half-space

$$\mathbf{R}^n \times \mathbf{R}^+ = \{(x^0, \ldots, x^n) \in V \mid x^0 > 0$$

are both convex cones, neither of which are sharp. The positive orthant

$$(\mathbf{R}^+)^{n+1} = \{(x^0, \dots, x^n) \in V \mid x^i > 0 \text{ for } i = 0, 1, \dots, n\}$$

and the positive light-cone

$$C_{n+1} = \{(x^0, \dots, x^n) \in V \mid x^0 > 0 \text{ and } -(x^0)^2 + (x^1)^2 + \dots (x^n)^2 < 0\}$$

are both sharp convex cones. Note that the planar region $\{(x, y) \in \mathbf{R}^2 \mid y > x^2\}$ is a convex domain which is sharp but is not affinely equivalent to a cone.

EXERCISE. *Show that the set $\mathfrak{P}_n(\mathbf{R})$ of all positive definite symmetric $n \times n$ real matrices is a sharp convex cone in the $n(n+1)/2$-dimensional vector space V of $n \times n$ symmetric matrices. Are there any affine transformations of V preserving $\mathfrak{P}_n(\mathbf{R})$? What is its group of affine automorphisms?*

We shall say that a subset $\Omega \subset \mathbf{P}$ is *convex* if there is a convex set $\Omega' \subset V$ such that $\Omega = \Pi(\Omega')$. Since $\Omega' \subset V - \{0\}$ is convex, Ω must be disjoint from at least one hyperplane H in $\mathbf{P}$. (In particular we do not allow $\mathbf{P}$ to itself be convex.) Equivalently $\Omega \subset \mathbf{P}$ is convex if there is a hyperplane $H \subset \mathbf{P}$ such that Ω is a convex set in the affine space complementary to H. A domain $\Omega \subset \mathbf{P}$ is *properly convex* $\Longleftrightarrow$ there exists a sharp convex cone $\Omega' \subset V$ such that $\Omega = \Pi(\Omega')$. Equivalently Ω is properly convex $\Longleftrightarrow$ there is a hyperplane $H \subset \mathbf{P}$ such that $\bar{\Omega}$ is a convex subset of the affine space $\mathbf{P} - H$. If Ω is properly convex, then the intersection of Ω with a projective subspace $\mathbf{P}' \subset \mathbf{P}$ is either empty or a properly convex subset $\Omega' \subset \mathbf{P}'$. In particular every line intersecting Ω meets $\partial\Omega$ in exactly two points.

In 1894 Hilbert introduced a projectively invariant metric $d = d_\Omega$ on any properly convex domain $\Omega \subset \mathbf{P}$ as follows. Let $x, y \in \Omega$ be a pair of distinct points; then the line $\overleftrightarrow{xy}$ meets $\partial\Omega$ in two points which we denote by x_∞, y_∞ (the point closest to x will be x_∞, etc.). The *Hilbert distance* $d = d_\Omega^{\mathrm{Hilb}}$ between x and y in Ω will be defined as the logarithm of the cross-ratio of this quadruple:

$$d(x, y) = \log[x_\infty, x, y, y_\infty]$$

It is clear that $d(x, y) \geq 0$, that $d(x, y) = d(y, x)$ and since Ω contains no complete affine line, $x_\infty \neq y_\infty$ so that $d(x, y) > 0$ if $x \neq y$. The same argument shows that this function $d : \Omega \times \Omega \longrightarrow \mathbf{R}$ is *finitely compact*, i.e. for each $x \in \Omega$ and $r > 0$, the "r-neighborhood"

$$B_r(x) = \{y \in \Omega \mid d(x, y) \leq r\}$$

is compact. Once the triangle inequality is established, it will follow that (Ω, d) is a complete metric space. The triangle inequality results from the convexity of Ω, although we shall deduce it by showing that the Hilbert metric agrees with the general intrinsic metric introduced by Kobayashi [Ko], where the triangle inequality is enforced as part of its construction.

To motivate Kobayashi's construction, consider the basic case of intervals in $\mathbf{P}^1$. There are several natural choices to take, e.g. the interval of positive real numbers $\mathbf{R}^+ =$

$(0,\infty)$ or the unit ball $\mathbf{I} = [-1,1]$. They are related by the projective transformation $\tau : \mathbf{I} \longrightarrow \mathbf{R}^+$

$$x = \tau(u) = \frac{1+u}{1-u}$$

mapping $-1 < u < 1$ to $0 < x < \infty$ with $\tau(0) = 1$. The corresponding Hilbert metrics are given by

$$d_{\mathbf{R}^+}(x_1, x_2) = \log|\frac{x_1}{x_2}|$$
$$d_{\mathbf{I}}(u_1, u_2) = 2|\tanh^{-1}(u_1) - \tanh^{-1}(u_2)|$$

which follows from the fact that τ pulls back the parametrization corresponding to Haar measure

$$\frac{|dx|}{x} = |d\log x|$$

on $\mathbf{R}^+$ to the "Poincaré metric"

$$\frac{2|du|}{1-u^2} = 2|d\tanh^{-1} u|$$

on $\mathbf{I}$.

In terms of the Poincaré metric on $\mathbf{I}$ the Hilbert distance $d(x,y)$ can be characterized as an infimum over all projective maps $\mathbf{I} \longrightarrow \Omega$:

$$d(x,y) = \inf\{d_{\mathbf{I}}(a,b) \mid \text{there exists a projective map } f : \mathbf{I} \longrightarrow \Omega \\ \text{with } f(a) = x, f(b) = y\}$$

We now define the Kobayashi pseudo-metric for any domain Ω or more generally any manifold with a projective structure. This proceeds by a general universal construction whereby two properties are "forced:" the triangle inequality and the fact that projective maps are distance-nonincreasing (the projective "Schwarz lemma"). What we must sacrifice in general is positivity of the resulting pseudo-metric.

Let $\Omega \subset \mathbf{P}$ be a domain. If $x, y \in \Omega$, a *chain* from x to y is a sequence C of projective maps $f_1, \dots, f_m \in \mathrm{Proj}(\mathbf{I}, \Omega)$ and pairs $a_i, b_i \in \mathbf{I}$ such that

$$f_1(a_1) = x,\ f_1(b_1) = f_2(a_2),\ \dots,\ f_{m-1}(b_{m-1}) = f_m(a_m),\ f_m(b_m) = y$$

and its length is defined as

$$\ell(C) = \sum_{i=1}^{m} d_{\mathbf{I}}(a_i, b_i).$$

Let $\mathfrak{C}(x,y)$ denote the set of all chains from x to y. The *Kobayashi pseudo-distance* $d^{\mathrm{Kob}}(x,y)$ is then defined as

$$d^{\mathrm{Kob}}(x,y) = \inf\{\ell(C) \mid C \in \mathfrak{C}(x,y)\}.$$

The resulting function enjoys the following obvious properties:

(1) $d^{\mathrm{Kob}}(x,y) \geq 0$;
(2) $d^{\mathrm{Kob}}(x,x) = 0$;
(3) $d^{\mathrm{Kob}}(x,y) = d^{\mathrm{Kob}}(y,x)$;
(4) (The triangle inequality) $d^{\mathrm{Kob}}(x,y) \leq d^{\mathrm{Kob}}(y,z) + d^{\mathrm{Kob}}(z,x)$. (The composition of a chain from x to z with a chain from z to y is a chain from x to y.)
(5) (The Schwarz lemma) If Ω, Ω' are two domains in projective spaces with Kobayashi pseudo-metrics d, d' respectively and $f : \Omega \longrightarrow \Omega'$ is a projective map, then $d'(f(x), f(y)) \leq d(x,y)$. (The composition of projective maps is projective.)
(6) The Kobayashi pseudo-metric on the interval $\mathbf{I}$ equals the Hilbert metric on $\mathbf{I}$.
(7) d^{Kob} is invariant under the group $\operatorname{Aut}\Omega$ consisting of all collineations of $\mathbf{P}$ preserving Ω.

3.4 PROPOSITION (KOBAYASHI [KO]). *Let $\Omega \subset \mathbf{P}$ be properly convex. Then the two functions $d^{\mathrm{Hilb}}, d^{\mathrm{Kob}} : \Omega \times \Omega \longrightarrow \mathbf{R}$ are equal.*

3.5 COROLLARY. *The function $d^{\mathrm{Hilb}} : \Omega \times \Omega \longrightarrow \mathbf{R}$ is a complete metric on Ω.*

PROOF OF PROPOSITION 3.3: Let $x, y \in \Omega$ be distinct points and let $l = \overleftrightarrow{xy}$ be the line incident to them. Now

$$d_{\Omega}^{\mathrm{Hilb}}(x,y) = d_{l\cap\Omega}^{\mathrm{Hilb}}(x,y) = d_{l\cap\Omega}^{\mathrm{Kob}}(x,y) \leq d_{\Omega}^{\mathrm{Kob}}(x,y)$$

by the Schwarz lemma applied to the projective map $l \cap \Omega \hookrightarrow \Omega$. For the opposite inequality, let S be the intersection of a supporting hyperplane to Ω at x_∞ and a supporting hyperplane to Ω at y_∞. Projection from S to l defines a projective map $\Pi_{S,l}\Omega \longrightarrow l \cap \Omega$ which retracts Ω onto $l \cap \Omega$. Thus

$$d_{\Omega}^{\mathrm{Kob}}(x,y) \leq d_{l\cap\Omega}^{\mathrm{Kob}}(x,y) = d_{\Omega}^{\mathrm{Hilb}}(x,y)$$

(again using the Schwarz lemma) as desired. ∎

3.6 COROLLARY. *Line segments in Ω are geodesics. If $\Omega \subset \mathbf{P}$ is properly convex, $x, y \in \Omega$, then the chain consisting of a single projective isomorphism $\mathbf{I} \longrightarrow \overleftrightarrow{xy} \cap \Omega$ minimizes the length among all chains in $\mathfrak{C}(x,y)$.*

3.7 Let $\triangle \subset \mathbf{P}^2$ denote a domain bounded by a triangle. Then the balls in the Hilbert metric are hexagonal regions. (In general if Ω is a convex k-gon in $\mathbf{P}^2$ then the unit balls in the Hilbert metric will be interiors of $2k$-gons.) Note that since $\operatorname{Aut}(\triangle)$ acts transitively on $\triangle$ ($\operatorname{Aut}(\triangle)$ is conjugate to the group of diagonal matrices with positive eigenvalues) all the unit balls are isometric.

Here is a construction which illustrates the Hilbert geometry of $\triangle$. Start with a triangle $\triangle$ and choose line segments l_1, l_2, l_3 from an arbitrary point $p_1 \in \triangle$ to the vertices

v_1, v_2, v_3 of $\triangle$. Choose another point p_2 on l_1, say, and form lines l_4, l_5 joining it to the remaining vertices. Let

$$\rho = \log |[v_1, p_1, p_2, l_1 \cap \overleftrightarrow{v_2 v_3}]|$$

where $[,]$ denotes the cross-ratio of four points on l_1. The lines l_4, l_5 intersect l_2, l_3 in two new points which we call p_3, p_4. Join these two points to the vertices by new lines l_i which intersect the old l_i in new points p_i. In this way one generates infinitely many lines and points inside $\triangle$, forming a configuration of smaller triangles T_j inside $\triangle$. For each p_i, the union of the T_j with vertex p_i is a convex hexagon which is a Hilbert ball in $\triangle$ of radius ρ. Note that this configuration is combinatorially equivalent to the tesselation of the plane by congruent equilateral triangles. Indeed, this tesselation of $\triangle$ arises from an action of a (3,3,3)-triangle group by collineations and converges (in an appropriate sense) to the Euclidean equilateral-triangle tesselation as $\rho \longrightarrow 0$.

EXERCISE. *Let $\triangle$ be the positive quadrant $\{(x,y) \in \mathbf{R}^2 \mid x, y > 0\}$. Then the Hilbert distance is given by*

$$d((x,y),(x',y')) = \log \max\{\frac{x}{x'}, \frac{x'}{x}, \frac{y}{y'}, \frac{y'}{y}, \frac{xy'}{x'y}, \frac{x'y}{xy'}\}.$$

For any two points $p, p' \in \triangle$, show that there are infinitely many geodesics joining p to p'. In fact show that there are even non-smooth polygonal curves from p to p' having minimal length.

Let $Q \subset \mathbf{P}^n$ be a quadric corresponding to a polarity of signature $(1,n)$ and let Ω be the convex region bounded by Q. Indeed, let us take Ω to be the unit ball in $\mathbf{R}^n$ defined by

$$\|x\|^2 = \sum_{i=1}^{n} (x^i)^2 < 1.$$

Then the Hilbert metric is given by the Riemannian metric

$$ds^2 = \frac{-4}{\sqrt{1-\|x\|^2}} d^2 \sqrt{1-\|x\|^2} = \frac{4}{(1-\|x\|^2)^2} \sum_{i=1}^{n} (x^i dx^i)^2 + (1-\|x\|^2)^2 (dx^i)^2$$

which has constant curvature -1. This is the only case when the Hilbert metric is Riemannian; in general the Hilbert metric is *Finsler*, given infinitesimally by a norm on the tangent spaces (not necessarily a norm arising from a quadratic form). By changing $\sqrt{1-\|x\|^2}$ to $\sqrt{1+\|x\|^2}$ in the above formula, one obtains a metric on $\mathbf{P}^n$ of constant curvature $+1$. In 1866 Beltrami showed that the only Riemannian metrics on domains in $\mathbf{P}^n$ where the geodesics are straight line segments are (up to a collineation and change of scale factor) Euclidean metrics and these two metrics. Hilbert's fourth problem was to determine all metric space structures on domains in $\mathbf{P}^n$ whose geodesics are straight line segments. There are many unusual such metrics, see Busemann [Bu] and Pogorelov [Po].

§4: GEOMETRIC STRUCTURES ON MANIFOLDS

If M is a manifold, we wish to impart to it (locally) affine and/or projective geometry. The corresponding global object is a *geometric structure modelled on affine or projective* *geometry*, or simply an *affine structure* or *projective structure* on M. (Such structures are also called "affinely flat structures," "flat affine structures," "flat projective structures," etc. We will not be concerned with the more general "non-flat" structures here and hence refer to such structures as affine or projective structures.) For various reasons, it is useful to approach this subject from the more general point of view of locally homogeneous structures, i.e. geometric structures modelled on a homogeneous space. In what follows X will be a space with a geometry on it and G is the group of transformations of X which preserves this geometry. We shall consider manifolds M having the same dimension as that of X: thus M locally looks like X — topologically — but we wish to model M on X *geometrically*. If (X, G) is affine geometry (so that $X = \mathbf{R}^n$ and $G = \mathrm{Aff}(\mathbf{R}^n)$) then a (X, G)-structure will be called an *affine structure*; if (X, G) is projective geometry (so that $X = \mathbf{P}^n$ and $G = \mathrm{Aut}(\mathbf{P}^n)$ the collineation group of $\mathbf{P}^n$) then an (X, G)-structure will be called a *projective structure*. An affine structure on a manifold is the same thing as a flat torsionfree affine connection, and a projective structure is the same thing as a flat normal projective connection (see Chern-Griffiths [CG], Kobayashi [K1] or Hermann [H] for the theory of projective connections). We shall refer to a projective structure modelled on $\mathbf{RP}^n$ *an* $\mathbf{RP}^n$*-structure;* a manifold with an $\mathbf{RP}^n$-structure will be called an $\mathbf{RP}^n$*-manifold*.

In many cases of interest, there may be a readily identifiable geometric entity on X whose stabilizer is G. In that case the geometry of (X, G) may be considered the geometry centered upon this object. Perhaps the most important such entity is a Riemannian metric. For example if X is a simply-connected Riemannian manifold of constant curvature K and G is its group of isometries, then locally modelling M on (X, G) is equivalent to giving M a Riemannian metric of curvature K. (This idea can be vastly extended, for example to cover indefinite metrics, locally homogeneous metrics whose curvature is not necessarily constant, etc.) In particular Riemannian metrics of constant curvature are special cases of (X, G)-structures on manifolds.

4.1 Let G be a Lie group acting transitively on a manifold X. Let $U \subset X$ be an open set and let $f : U \longrightarrow X$ be a smooth map. We say that f is *locally-*(X, G) if for each component $U_i \subset U$, there exists $g_i \in G$ such that the restriction of g_i to $U_i \subset X$ equals the restriction of f to $U_i \subset U$. (O course f will have to be a local diffeomorphism.) An (X, G)*-atlas* on M is a pair $(\mathcal{U}, \Phi)$ where $\mathcal{U}$ is an open covering of M and $\Phi = \{\phi_\alpha : U_\alpha \longrightarrow X\}_{U_\alpha \in \mathcal{U}}$ is a collection of coordinate charts such that for each pair $(U_\alpha, U_\beta) \in \mathcal{U} \times \mathcal{U}$ the restriction of $\phi_\alpha \circ (\phi_\beta)^{-1}$ to $\phi_\beta(U_\alpha \cap U_\beta))$ is locally-(X, G). An (X, G)*-structure* on M is a maximal (X, G)-atlas and an (X, G)*-manifold* is a manifold together with an (X, G)-structure on it. It is clear that an (X, G)-manifold has an underlying real analytic structure, since the action of G on X is real analytic.

Suppose that M and N are two (X, G)-manifolds and $f : M \longrightarrow N$ is a map. Then f is an (X, G)*-map* if for each pair of charts $\phi_\alpha : U_\alpha \longrightarrow X$ and $\psi_\beta : V_\beta \longrightarrow X$ (for M and N respectively) the composition $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ restricted to $\phi_\alpha(U_\alpha \cap f^{-1}(V_\beta))$ is locally-(X, G). In particular we only consider (X, G)-maps which are local diffeomorphisms. Clearly the set of (X, G)-automorphisms $M \longrightarrow M$ forms a group, which we denote by $\mathrm{Aut}_{(X,G)}(M)$ or just $\mathrm{Aut}(M)$ when the context is clear.

4.2 Exercise. *Let N be an (X, G)-manifold and $f : M \longrightarrow N$ a local diffeomorphism.*

There is a unique (X,G)-structure on M for which f is an (X,G)-map. In particular every covering space of an (X,G)-manifold has a canonical (X,G)-structure. Conversely if M is an (X,G)-manifold upon which a discrete group Γ acts properly and freely by (X,G)-automorphisms, then X/Γ is an (X,G)-manifold.

The fundamental example of an (X,G)-manifold is X itself. Evidently any open subset $\Omega \subset X$ has an (X,G)-structure (with only one chart—the inclusion $\Omega \hookrightarrow X$). Locally-(X,G) maps satisfy the *Unique Extension Property:* If $U \subset X$ is a connected nonempty open subset, and $f : U \longrightarrow X$ is locally-(X,G), then there exists a unique element $g \in G$ whose restriction to U is f. This rigidity property is a distinguishing feature of the kind of geometric structures considered here. It follows that if $\Omega \subset X$ is a domain, an (X,G)-automorphism $f : \Omega \to \Omega$ is the restriction of a unique element $g \in G$ preserving Ω, i.e.:

$$\mathrm{Aut}_{(X,G)}(\Omega) \cong \{g \in G \mid g(\Omega) = \Omega\}$$

4.3 Exercise. *Suppose that $\phi : M \longrightarrow \Omega$ is a local diffeomorphism onto a domain $\Omega \subset X$. Show that there is a homomorphism*

$$\phi_* : \mathrm{Aut}_{(X,G)}(M) \longrightarrow \mathrm{Aut}_{(X,G)}(\Omega)$$

whose kernel consists of all maps $f : M \longrightarrow M$ making the diagram

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d, "f"'] & \Omega \arrow[d, equal] \\ M \arrow[r, "\phi"'] & \Omega \end{tikzcd}$$

commute. Find examples where ϕ_* is: (a) surjective but not injective; (b) injective but not surjective.

4.4 In the first two lectures, we saw how it is possible for one geometry to "contain" or "refine" another one. In this way one can pass from structures modelled on one geometry to structures modelled on a geometry containing it. Let (X,G) and (X',G') be homogeneous spaces and let $\Phi : X \to X'$ be a local diffeomorphism which is equivariant with respect to a homomorphism $\phi : G \to G'$ in the following sense: for each $g \in G$ the diagram

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, "g"'] & X' \arrow[d, "\phi(g)"] \\ X \arrow[r, "\Phi"'] & X' \end{tikzcd}$$

commutes. It follows that locally-(X,G) maps determine locally-(X',G')-maps and an (X,G)-structure on M induces an (X',G')-structure on M in the following way. Let

$\psi_\alpha : U_\alpha \longrightarrow X$ be an (X,G)-chart; the composition $\Phi \circ \psi_\alpha : U_\alpha \longrightarrow X'$ defines an (X',G')-chart.

There are many important examples of this correspondence, most of which occur when Φ is an embedding. For example when Φ is the identity map and $G \subset G'$ is a subgroup, then every (X,G)-structure is a fortiori an (X',G')-structure. Thus every Euclidean structure is a similarity structure which in turn is an affine structure. Similarly every affine structure determines a projective structure, using the embedding $(\mathbf{R}^n, \mathrm{Aff}(\mathbf{R}^n)) \longrightarrow (\mathbf{P}^n, \mathrm{Proj}(\mathbf{P}^n))$ of affine geometry in projective geometry. Using the Klein model of hyperbolic geometry $(\mathbf{H^n}, \mathrm{PO}(n,1)) \longrightarrow (\mathbf{P}^n, \mathrm{Proj}(\mathbf{P}^n))$ every hyperbolic-geometry structure (i.e. Riemannian metric of constant curvature -1) determines a projective structure. Using the inclusion of the projective orthogonal group $\mathrm{PO}(n+1) \subset \mathrm{PGL}(n+1;\mathbf{R})$ one sees that every elliptic-geometry structure (i.e. Riemannian metric of constant curvature $+1$) determines a projective structure. Since every surface admits a metric of constant curvature, we obtain the following:

THEOREM. *Every surface admits an* $\mathbf{RP}^2$*-structure.*

4.5 EXERCISE. *Suppose that* $\Phi : X \longrightarrow X'$ *is a universal covering space and* G *is the group of lifts of transformations* $g' : X' \longrightarrow X'$ *in* G' *to* X. *Let* $\phi : G \longrightarrow G'$ *be the corresponding homomorphism. Show that* (Φ, ϕ) *induces an isomorphism between the categories of* (X,G)*-manifolds/maps and* (X',G')*-manifolds/maps. For this reason we may always assume (when convenient) that our model space* X *is simply-connected.*

Development, Holonomy.

4.6 There is a useful globalization of the coordinate charts of a geometric structure in terms of the universal covering space and the fundamental group. Let M be an (X,G)-manifold. Choose a universal covering space $\mathbf{p} : \tilde{M} \longrightarrow M$ and let $\pi = \pi_1(M)$ be the corresponding fundamental group. The covering projection $\mathbf{p}$ induces an (X,G)-structure on $\tilde{M}$ upon which π acts by (X,G)-automorphisms. The Unique Extension Property has the following important consequence.

PROPOSITION. *Let* M *be a simply connected* (X,G)*-manifold. Then there exists an* (X,G)*-map* $f : M \longrightarrow X$.

It follows that the (X,G)-map f completely determines the (X,G)-structure on M, i.e. the geometric structure on a simply-connected manifold is "pulled back" from the model space X. The (X,G)-map f is called a *developing map* for M and enjoys the following uniqueness property. If $f' : M \longrightarrow X$ is another (X,G)-map, then there exists an (X,G)-automorphism ϕ of M and an element $g \in G$ such that

$$\begin{array}{ccc} M & \xrightarrow{f'} & X \\ \phi \downarrow & & \downarrow g \\ M & \xrightarrow{f} & X \end{array}$$

PROOF OF PROPOSITION: Choose a basepoint $x_0 \in M$ and a coordinate patch U_0 containing x_0. For $x \in M$, we define $f(x)$ as follows. Choose a path $\{x_t\}_{0 \le t \le 1}$ in M from x_0

to $x = x_1$. Cover the path by coordinate patches U_i (where $i = 0, \dots, n$) such that $x_t \in U_i$ for $t \in (a_i, b_i)$ where

$$a_0 < 0 < a_1 < b_0 < a_2 < b_1 < a_3 < b_2 < \cdots < a_{n-1} < b_{n-2} < a_n < b_{n-1} < 1 < b_n$$

Let $\psi_i : U_i \longrightarrow X$ be an (X, G)-chart and let $g_i \in G$ be the unique transformation such that $g_i \circ \psi_i$ and ψ_{i-1} agree on the component of $U_i \cap U_{i-1}$ containing the curve $\{x_t\}_{a_i < t < b_{i-1}}$. Let

$$f(x) = g_1 g_2 \dots g_{n-1} g_n \psi_n(x)$$

and we must show that f is indeed well-defined. The map f does not change if the cover is refined. Suppose that a new coordinate patch U' is "inserted between" U_{i-1} and U_i. Let $\{x_t\}_{a' < t < b'}$ be the portion of the curve lying inside U' so

$$a_{i-1} < a' < a_i < b_{i-1} < b' < b_i.$$

Let $\psi' : U' \longrightarrow X$ be the corresponding coordinate chart and let $h_{i-1}, h_i \in G$ be the unique transformations such that ψ_{i-1} agrees with $h_{i-1} \circ \psi'$ on the component of $U' \cap U_{i-1}$ containing $\{x_t\}_{a' < t < b_{i-1}}$ and ψ' agrees with $h_i \circ \psi_i$ on the component of $U' \cap U_i$ containing $\{x_t\}_{a_i < t < b'}$. By the unique extension property $h_{i-1} h_i = g_i$ and it follows that the corresponding developing map

$$f(x) = g_1 g_2 \dots g_{i-1} h_{i-1} h_i g_{i+1} \dots g_{n-1} g_n \psi_n(x)$$
$$= g_1 g_2 \dots g_{i-1} g_i g_{i+1} \dots g_{n-1} g_n \psi_n(x)$$

is unchanged. Thus the developing map as so defined is independent of the coordinate covering, since any two coordinate coverings have a common refinement.

Next we claim the developing map is independent of the choice of path. Since M is simply connected, any two paths from x_0 to x are homotopic. Every homotopy can be broken up into a succession of "small" homotopies, i.e. homotopies such that there exists a partition $0 = c_0 < c_1 < \cdots < c_{m-1} < c_m = 1$ such that during the course of the homotopy the segment $\{x_t\}_{c_i < t < c_{i+1}}$ lies in a coordinate patch. It follows that the expression defining $f(x)$ is unchanged during each of the small homotopies, and hence during the entire homotopy. Thus f is independent of the choice of path.

Since f is a composition of a coordinate chart with transformations $X \longrightarrow X$ coming from G, it follows that f is an (X, G)-map. The proof of Proposition 4.6 is complete. ∎

4.7 If M is an arbitrary (X, G)-manifold, then we may apply Proposition 4.6 to a universal covering space $\tilde{M}$. We obtain the following basic result:

Development Theorem. *Let M be an (X, G)-manifold with universal covering space $\mathbf{p} : \tilde{M} \longrightarrow M$ and group of deck transformations $\pi = \pi_1(M) \subset \mathrm{Aut}(\mathbf{p} : \tilde{M} \longrightarrow M)$. Then there exists a pair $(\mathbf{dev}, h)$ such that $\mathbf{dev} : \tilde{M} \longrightarrow X$ is an (X, G)-map and $h : \pi \longrightarrow G$ is a homomorphism such that, for each $\gamma \in \pi$,*

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\mathbf{dev}} & X \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle h(\gamma)} \\ \tilde{M} & \xrightarrow[\mathbf{dev}]{} & X \end{array}$$

commutes. Furthermore if $(\mathbf{dev}', h')$ *is another such pair, there exists* $g \in G$ *such that* $\mathbf{dev}' = g \circ \mathbf{dev}$ *and* $h'(\gamma) = gh(\gamma)g^{-1}$ *for each* $\gamma \in \pi$.

We call such a pair $(\mathbf{dev}, h)$ a *development pair*, and the homomorphism h the *holonomy representation*. (It is the holonomy of a flat connection on a principal G-bundle over M associated to the (X, G)-structure.) The developing map is a globalization of the coordinate charts of the manifold and the holonomy representation is a globalization of the coordinate changes. In this generality the Development Theorem seems to be due to C. Ehresmann [Eh] in 1936.

4.8 Exercise. *Let* M *be an* (X, G)*-manifold with development pair* $(\mathbf{dev}, h)$. *Suppose that* $N \longrightarrow M$ *is a covering space. Show that there exists an* (X, G)*-map* $N \longrightarrow X \iff$ *the holonomy representation restricted to* $\pi_1(N) \hookrightarrow \pi_1(M)$ *is trivial. Thus the holonomy covering space* $\hat{M} \longrightarrow M$ *— the covering space of* M *corresponding to the kernel of* h *— is the "smallest" covering space of* M *for which a developing map is "defined."*

4.9 Exercise. *Suppose that* M *is a closed manifold with finite fundamental group. Show that if* X *is noncompact then* M *admits no* (X, G)*-structure. If* X *is compact and simply-connected show that every* (X, G)*-manifold is* (X, G)*-isomorphic to a quotient of* X *by a finite subgroup of* G. *(Hint: if* M *and* N *are manifolds of the same dimension,* $f : M \longrightarrow N$ *is a local diffeomorphism and* M *is closed, show that* f *must be a covering space.)*

As a consequence a closed affine manifold must have infinite fundamental group and every $\mathbf{RP}^n$-manifold with finite fundamental group is a quotient of S^n by a finite group (and hence a spherical space form).

4.10 The process of inducing one geometric structure from another is easily understood in terms of developments:

Exercise. *Suppose that* (X, G) *and* (X', G') *represent a pair of geometries for which there exists a pair* (Φ, ϕ) *as in 4.5. Show that if* M *is an* (X, G)*-manifold with development pair* $(\mathbf{dev}, h)$, *then* $(\Phi \circ \mathbf{dev}, \phi \circ h)$ *is a development pair for the induced* (X', G')*-structure on* M.

Completeness.

4.11 In many important cases the developing map is a diffeomorphism $\tilde{M} \longrightarrow X$, or at least a covering map onto its image. An extremely important case of this occurs when (X, G) is a *Riemannian homogeneous space*, i.e. when X possesses a G-invariant Riemannian metric g_X. Equivalently, $X = G/H$ where the isotropy group H is compact. The Hopf-Rinow theorem from Riemannian geometry has the following important consequence:

Proposition. *Let* (X, G) *be a Riemannian homogeneous space. Suppose that* X *is simply connected and* M *is a compact* (X, G)*-manifold. Let* $\mathbf{p} : \tilde{M} \longrightarrow M$ *be a universal covering space,* π *the associated fundamental group and* $(\mathbf{dev}, h)$ *the corresponding development pair. Then* $\mathbf{dev} : \tilde{M} \longrightarrow X$ *is a diffeomorphism and* $h : \pi \longrightarrow G$ *is an isomorphism of* π *onto a cocompact discrete subgroup* $\Gamma \subset G$.

Proof: The Riemannian metric $g_{\tilde{M}} = \mathbf{dev}^* g_X$ on $\tilde{M}$ is invariant under the group of deck transformations π of $\tilde{M}$ and hence there is a Riemannian metric g_M on M such that

$\mathbf{p}^* g_M = g_{\tilde{M}}$. Since M is compact, the metric g_M on M is complete and so is the metric $g_{\tilde{M}}$ on $\tilde{M}$. By construction, $\mathbf{dev} : (\tilde{M}, g_{\tilde{M}}) \longrightarrow (X, g_X)$ is a local isometry. A local isometry from a complete Riemannian manifold into a Riemannian manifold is necessarily a covering map (Kobayashi-Nomizu [KN,]) so $\mathbf{dev}$ is a covering map of $\tilde{M}$ onto X. Since X is simply connected, it follows that $\mathbf{dev}$ is a diffeomorphism. Let $\Gamma \subset G$ denote the image of h. Since $\mathbf{dev}$ is equivariant respecting h, the action of π on X given by h is equivalent to the action of π by deck transformations on $\tilde{M}$. Thus h is faithful and its image Γ is a discrete subgroup of G acting properly and freely on X. Furthermore $\mathbf{dev}$ defines a diffeomorphism $M = \tilde{M}/\pi \longrightarrow X/\Gamma$. Since M is compact, it follows that X/Γ is compact, and since the fibration $G \longrightarrow G/H = X$ is proper, the homogeneous space $\Gamma\backslash G$ is compact, i.e. Γ is cocompact in G. ∎

One may paraphrase the above result abstractly as follows. Let (X, G) be a Riemannian homogeneous space. Then there is an equivalence of categories between the category of compact (X, G)-manifolds/maps and discrete cocompact subgroups of G which act freely on X (the morphisms being inclusions of subgroups composed with inner automorphisms of G).

We say that an (X, G)-manifold M is *complete* if $\mathbf{dev} : \tilde{M} \longrightarrow X$ is a diffeomorphism (or a covering map if we don't insist that X be simply connected). An (X, G)-manifold M is complete $\iff$ its universal covering $\tilde{M}$ is (X, G)-isomorphic to X, i.e. if M is isomorphic to the quotient X/Γ (at least if X is simply connected). Note that if (X, G) is contained in (X', G') in the sense of 2.5, and $X \neq X'$, then a complete (X, G)-manifold is *never* complete as an (X', G')-manifold.

4.12 Theorem (Auslander-Markus [AM]). *Let M be an affine manifold. Then M is complete in the above sense if and only if M is geodesically complete (in the sense of the affine connection on M corresponding to the affine structure). That is, show that M is a quotient of affine space $\iff$ a particle moving at constant speed in a straight line will continue indefinitely.*

Proof: Clearly it suffices to assume that M is simply connected. If M is a complete affine manifold, then $\mathbf{dev} : M \longrightarrow \mathbf{R}^n$ is an affine isomorphism and since $\mathbf{R}^n$ is geodesically complete, so is M. Conversely, suppose that M is geodesically complete. We must show that $\mathbf{dev} : M \longrightarrow \mathbf{R}^n$ is bijective. Choose a basepoint $u \in M$; we may assume that $\mathbf{dev}(u) = 0 \in \mathbf{R}^n$. Since M is complete, the exponential map is defined on all of $T_u M$. We claim that the composition

$$T_u M \xrightarrow{\exp} M \xrightarrow{\mathbf{dev}} E$$

is an affine isomorphism. In local affine coordinates, $\exp(v) = u + v$ and $\mathbf{dev}(u + v) = \mathbf{dev}(u) + v = v$; it follows that $\mathbf{dev}$ is bijective. ∎

4.13 Exercise. *Suppose that X is simply connected. Let M be a closed (X, G)-manifold with developing pair $(\mathbf{dev}, h)$. Show that M is complete $\iff$ the holonomy representation $h : \pi \longrightarrow G$ is an isomorphism of π onto a discrete subgroup of G which acts properly and freely on X.*

Complete affine structures on the 2-torus. As in §4.??? the compact complete affine 1-manifold $\mathbf{R}/\mathbf{Z}$ is unique up to affine isomorphism. Its Cartesian square $\mathbf{R}/\mathbf{Z} \times \mathbf{R}/\mathbf{Z}$ is a Euclidean structure on the two-torus, unique up to affine isomorphism. In this section we shall describe all other complete affine structures on the two-torus and show that they are parametrized by $\mathbf{RP}^1$. We shall see that affine isomorphism classes are parametrized by the plane $\mathbf{R}^2$ with a (non-Hausdorff) whose open sets are the open subsets of $\mathbf{R}^2 - \{0\}$ as well as $\mathbf{R}^2$ itself.

We begin by considering the one-parameter family of (quadratic) diffeomorphisms of the affine plane $E = \mathbf{R}^2$ defined by

$$\phi_r(x, y) = (x + ry^2, y)$$

It is easy to check that $\phi_r \circ \phi_s = \phi_{r+s}$ and thus ϕ_r and ϕ_{-r} are inverse maps. If $u = (s, t) \in \mathbf{R}^2$ we denote translation by u as $\tau(u) : E \longrightarrow E$. Conjugation of the translation $\tau(u)$ by ϕ_r yields the affine transformation

$$\alpha_r(u) = \phi_r \circ \tau(u) \circ \phi_{-r} = \begin{bmatrix} 1 & 2rt \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s + rt^2 \\ t \end{bmatrix}$$

and $\alpha_r : \mathbf{R}^2 \longrightarrow \mathrm{Aff}(E)$ defines a simply transitive affine action. (Compare [FG,§1.19].) If $\Lambda \subset \mathbf{R}^2$ is a lattice, then $E/\alpha_r(\Lambda)$ is a compact complete affine 2-manifold $M = M(r; \Lambda)$ diffeomorphic to a 2-torus.

The parallel 1-form dy defines a parallel 1-form η on M and its cohomology class $[\eta] \in H^1(M; \mathbf{R})$ is a well-defined invariant of the affine structure up to scalar multiplication. In general, M will have no closed geodesics. If $\gamma \subset M$ is a closed geodesic, then it must be a trajectory of the vector field on M arising from the parallel vector field $\partial/\partial x$ on E; then γ is closed $\iff$ the intersection of the lattice $\Lambda \subset \mathbf{R}^2$ with the line $\mathbf{R} \oplus \{0\} \subset \mathbf{R}^2$ is nonzero.

To classify these manifolds, we observe that the normalizer of $G_r = \alpha_r(\mathbf{R}^2)$ equals

$$\Big\{ \begin{bmatrix} \mu^2 & a \\ 0 & \mu \end{bmatrix} \mid \mu \in \mathbf{R}^*, a \in \mathbf{R} \Big\} \cdot G_r$$

which acts on G_r conjugating

$$\alpha_r(s, t) \mapsto \alpha_r(\mu^2 s + at, \mu t)$$

Let

$$N = \Big\{ \begin{bmatrix} \mu^2 & a \\ 0 & \mu \end{bmatrix} \mid \mu \in \mathbf{R}^*, a \in \mathbf{R} \Big\};$$

then the space of affine isomorphism classes of these tori may be identified with the homogeneous space $\mathrm{GL}(2, \mathbf{R})/N$ which is topologically $\mathbf{R}^2 - \{0\}$. The groups G_r are all conjugate and as $r \longrightarrow 0$, each representation $\alpha_r|_\pi$ converges to an embedding of π as a lattice of translations $\mathbf{R}^2 \longrightarrow \mathbf{R}^2$. It follows that the deformation space of complete affine structures on T^2 form a space which is the union of $\mathbf{R}^2 - \{0\}$ with a point O (representing the Euclidean structure) which is in the closure of every other structure.

Examples of incomplete structures.

4.14 It is quite easy to construct incomplete geometric structures on noncompact manifolds M. Take any immersion $f : M \longrightarrow X$ which is not bijective; then f induces an (X, G)-structure on M. If M is parallelizable, then such an immersion always exists (Hirsch []). More generally, let $h : \pi \longrightarrow G$ be a representation; then as long as the associated flat (X, G)-bundle $E \longrightarrow X$ possesses a section $s : M \longrightarrow E$ whose normal bundle is isomorphic to TM, there exists an (X, G)-structure with holonomy h (see Haefliger []).

It is harder to construct incomplete geometric structures on compact manifolds — indeed for certain geometries (X, G), there exist closed manifolds for which *every* (X, G)-structure on M is complete. As a trivial example, if X is compact and M is a closed manifold with finite fundamental group, then by 4.??? every (X, G)-structure on M is complete. As a less trivial example, if M is a closed manifold whose fundamental group contains a nilpotent subgroup of finite index and whose first Betti number equals one, then every affine structure on M is complete (see Fried-Goldman-Hirsch [FGH]). A simple example arises as follows. Consider the group $\Gamma \subset \mathrm{Euc}(\mathbf{R}^3)$ generated by the three isometries

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and Γ is a discrete group of Euclidean isometries which acts properly and freely on $\mathbf{R}^3$ with quotient a compact 3-manifold M. Furthermore there is a short exact sequence

$$\mathbf{Z}^3 \cong \langle A^2, B^2, C^2 \rangle \hookrightarrow \Gamma \xrightarrow{\mathbf{L}} \mathbf{Z}/2 \oplus \mathbf{Z}/2$$

and it follows that every affine structure on M must be complete.

The basic example of an incomplete affine structure on a closed manifold is a *Hopf manifold*. Consider $\Omega = \mathbf{R}^n - \{0\}$; then the group $\mathbf{R}^*$ of *homotheties* (i.e. scalar multiplications) acts on Ω properly and freely with quotient the projective space $\mathbf{RP}^{n-1}$. Clearly the affine structure on Ω is incomplete. Let $\lambda \in \mathbf{R}$ satisfy $\lambda > 1$; then the cyclic group $\langle \lambda \rangle$ is a discrete subgroup of $\mathbf{R}^*$ and the quotient $\Omega/\langle \lambda \rangle$ is a compact incomplete affine manifold M. We shall denote this manifold by $\mathbf{Hopf}^n_\lambda$. (A geodesic whose tangent vector "points" at the origin will be incomplete; on the manifold M the affinely parametrized geodesic will circle around with shorter and shorter period until in a finite amount of time will "run off" the manifold.) If $n = 1$, then M consists of two disjoint copies of the *Hopf circle* $\mathbf{R}^+/\langle \lambda \rangle$ — this manifold is an incomplete closed geodesic (and every incomplete closed geodesic is isomorphic to a Hopf circle). For $n > 1$, then M is connected and is diffeomorphic to the product $S^1 \times S^{n-1}$. For $n > 2$ both the holonomy homomorphism and the developing map are injective.

If $n = 2$, then M is a torus whose holonomy homomorphism maps $\pi_1(M) \cong \mathbf{Z} \oplus \mathbf{Z}$ onto the cyclic group $\langle\lambda\rangle$. Note that $\mathbf{dev}\tilde{M} \longrightarrow \mathbf{R}^2$ is neither injective nor surjective, although it is a covering map onto its image. For $k \geq 1$ let $\pi^{(k)} \subset \pi$ be the unique subgroup of index k which intersects $\operatorname{Ker} h \cong \mathbf{Z}$ in a subgroup of index k. Let $M^{(k)}$ denote the corresponding covering space of M. Then $M^{(k)}$ is another closed affine manifold diffeomorphic to a torus whose holonomy homomorphism is a surjection of $\mathbf{Z} \oplus \mathbf{Z}$ onto $\langle\lambda\rangle$.

4. 15 Exercise. *Show that for $k \neq l$, the two affine manifolds $M^{(k)}$ and $M^{(l)}$ are not isomorphic. (Hint: consider the invariant defined as the least number of breaks of a broken geodesic representing a simple closed curve on M whose holonomy is trivial.) Thus two different affine structures on the same manifold can have the same holonomy homomorphism.*

4.16 Exercise. *Suppose that $\lambda < -1$. Then $M = (\mathbf{R}^n - \{0\})/\langle\lambda\rangle$ is an incomplete compact affine manifold doubly covered by $\mathbf{Hopf}^n_\lambda$. What is M topologically?*

There is another point of view concerning Hopf manifolds in dimension two. Let M be a two-torus; we may explicitly realize M as a quotient $\mathbf{C}/\Lambda$ where $\Lambda \subset \mathbf{C}$ is a lattice. The complex exponential map $\exp : \mathbf{C} \longrightarrow \mathbf{C}^*$ is a universal covering space having the property that

$$\exp \circ \tau(z) = e^z \cdot \exp$$

where $\tau(z)$ denotes translation by $z \in \mathbf{C}$. For various choices of lattices Λ, the exponential map $\exp : \tilde{M} = \mathbf{C} \longrightarrow \mathbf{C}^*$ is a developing map for a (complex) affine structure on M with holonomy homomorphism

$$\pi \cong \Lambda \xrightarrow{\exp} \exp(\Lambda) \hookrightarrow \mathbf{C}^* \subset \mathrm{Aff}(\mathbf{C})$$

We denote this affine manifold by $\exp(\mathbf{C}/\Lambda)$; it is an incomplete complex affine 1-manifold or equivalently an incomplete similarity 2-manifold. Every compact incomplete orientable similarity manifold is equivalent to an $\exp(\mathbf{C}/\Lambda)$ for a unique lattice $\Lambda \subset \mathbf{C}$. Taking $\Lambda \subset \mathbf{C}$ to be the lattice generated by $\log\lambda$ and $2\pi i$ we obtain the Hopf manifold $\mathbf{Hopf}^2_\lambda$. More generally the lattice generated by $\log\lambda$ and $2k\pi i$ corresponds to the k-fold covering space of $\mathbf{Hopf}^2_\lambda$ described above. There are "fractional" covering spaces of the Hopf manifold obtained from the lattice generated by $\log\lambda$ and $2\pi/n$ for $n > 1$; these manifolds admit n-fold covering spaces by $\mathbf{Hopf}^2_\lambda$. The affine manifold M admits no closed geodesics $\Longleftrightarrow$ $\Lambda \cap \mathbf{R} = \{0\}$. Note that the exponential map defines an isomorphism $\mathbf{C}/\Lambda \longrightarrow M$ which is definitely not an isomorphism of affine manifolds.

4.15 A Hopf manifold is the prototypical example of a *radiant affine manifold*. Many properties of Hopf manifolds are shared by radiant structures. The following theorem characterizes radiant affine structures:

PROPOSITION. *Let M be an affine manifold with development pair $(\mathbf{dev}, h)$. The following conditions are equivalent:*

(1) *$h(\pi)$ fixes a point in E (by conjugation we may assume this fixed point is the origin 0);*
(2) *M is isomorphic to a $(E, \mathrm{GL}(E))$-manifold;*
(3) *M possesses a radiant vector field (see 1.7).*

If M satisfies these conditions, we say the affine structure on M is *radiant*. If ρ_M is a radiant vector field on M, we shall often refer to the pair (M, ρ_M) as well as a *radiant affine structure*. A closed radiant affine manifold M is always incomplete (5.???) and the radiant vector field is always nonsingular so that $\chi(M) = 0$. Furthermore the first Betti number of a closed radiant affine manifold is always positive.

4.16 EXERCISE (PRODUCTS OF AFFINE MANIFOLDS). *Let M^m, N^n be affine manifolds. Show that the Cartesian product $M^m \times N^n$ has a natural affine structure. Show that $M \times N$ is complete $\Longleftrightarrow$ both M and N are complete; $M \times N$ is radiant $\Longleftrightarrow$ both M and N are radiant. On the other hand, find compact manifolds M, N each of which has a projective structure but $M \times N$ does not admit a projective structure. If $M_1, \dots, M_r$ are manifolds with real projective structures, show that the Cartesian product $M_1 \times \cdots \times M_r \times T^{r-1}$ admits a real projective structure (Benzécri [B2]).*

Maps between manifolds with different geometries.

4.17 In many cases, we wish to consider maps between different manifolds with geometric structures modelled on different geometries. To this end we consider the following general situation. Let (X, G) and (X', G') be two homogeneous spaces representing different geometries and consider a family $\mathfrak{M}$ of maps $X \longrightarrow X'$ such that if $f \in \mathfrak{M}, g \in G, g' \in G'$, then the composition $g' \circ f \circ g \in \mathfrak{M}$. If $U \subset X$ is a domain, a map $f : U \longrightarrow X'$ is *locally-$\mathfrak{M}$* if for each component $U_i \subset U$ there exists $f_i \in \mathfrak{M}$ such that the restriction of f to $U_i \subset U$ equals the restriction of f_i to $U_i \subset X$. Let M be an (X, G)-manifold and N an (X', G')-manifold. Suppose that $f : M \longrightarrow N$ is a smooth map. We say that f is *an $\mathfrak{M}$-map* if for each pair of charts $\phi_\alpha : U_\alpha \longrightarrow X$ (for M) and $\psi_\beta : V_\beta \longrightarrow X$ (for N) the composition $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ restricted to $\phi_\alpha(U_\alpha \cap f^{-1}(V_\beta))$ is locally-$\mathfrak{M}$.

The basic examples are affine and projective maps between affine and projective manifolds: For affine maps we take $(X, G) = (\mathbf{R}^m, \mathrm{Aff}(\mathbf{R}^m))$ and $(X', G') = (\mathbf{R}^n, \mathrm{Aff}(\mathbf{R}^n))$ and $\mathfrak{M} = \mathfrak{aff}(\mathbf{R}^m, \mathbf{R}^n)$. For example if M, N are affine manifolds, and $M \times N$ is the product affine manifold (see 4.17), then the projections $M \times N \longrightarrow M$ and $M \times N \longrightarrow N$ are affine. Similarly if $x \in M$ and $y \in N$, the inclusions $\{x\} \times N \hookrightarrow M \times N$ and $M \times \{y\} \hookrightarrow M \times N$ are each affine. For projective maps we take $(X, G) = (\mathbf{P}^m, \mathrm{Proj}(\mathbf{P}^m))$ and $(X', G') = (\mathbf{P}^n, \mathrm{Proj}(\mathbf{P}^n))$ and $\mathfrak{M}$ the set $\mathrm{Proj}(\mathbf{P}^m, \mathbf{P}^n)$ of projective maps $\mathbf{P}^m \to \mathbf{P}^n$ (or more generally the collection of locally projective maps defined on open subsets of $\mathbf{P}^m$). Thus if M is an $\mathbf{RP}^n$-manifold it makes sense to speak of projective maps $\mathbf{I} \longrightarrow M$ and thus the Kobayashi pseudo-metric $d^{\mathrm{Kob}} : M \times M \longrightarrow \mathbf{R}$ is defined. The following theorem combines results of Kobayashi [Kb2] and Vey [V1,V2], and is a kind of converse to 3.4:

4.18 THEOREM. *Let M be a compact $\mathbf{RP}^n$-manifold and let $\tilde{M}$ be its universal covering space. Then d^{Kob} is a metric $\Longleftrightarrow$ $\tilde{M}$ is projectively isomorphic to properly convex domain in $\mathbf{RP}^n$.*

Fibration of geometries.

4.19 One can also "pull back" geometric structures by "fibrations" of geometries as follows. Let (X, G) be a homogeneous space and suppose that $\Phi : X' \longrightarrow X$ is a fibration with fiber F and that $\phi : G' \longrightarrow G$ is a homomorphism such that for each $g' \in G'$ the diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "\Phi"'] & X' \arrow[d, "\Phi"] \\ X' \arrow[r, "\phi(g')"'] & X' \end{tikzcd}$$

commutes.

Suppose that M is an (X, G)-manifold. Let $\mathbf{p} : \tilde{M} \longrightarrow M$ be a universal covering with group of deck transformations π and $(\mathbf{dev}, h)$ a development pair. Then the pullback $\mathbf{dev}^*\Phi$ is an F-fibration $\tilde{M}'$ over $\tilde{M}$ and the induced map $\mathbf{dev}' : M' \longrightarrow X'$ is a local diffeomorphism and thus a developing map for an (X', G')-structure on $\tilde{M}'$. We summarize these maps in the following commutative diagram:

$$\begin{tikzcd} \tilde{M}' \arrow[r, "\mathbf{dev}'"] \arrow[d] & X' \arrow[d, "\Phi"] \\ \tilde{M} \arrow[r, "\mathbf{dev}"'] & X \end{tikzcd}$$

Suppose that the holonomy representation $h : \pi \longrightarrow G$ lifts to $h' : \pi \longrightarrow G'$. (In general the question of whether h lifts will be detected by certain invariants in the cohomology of M.) Then h' defines an extension of the action of π on $\tilde{M}$ to $\tilde{M}'$ by (X', G')-automorphisms. Since the action of π on $\tilde{M}'$ is proper and free, the quotient $M' = \tilde{M}'/\pi$ is an (X', G')-manifold. Moreover the fibration $\tilde{M}' \longrightarrow \tilde{M}$ descends to an F-fibration $M' \longrightarrow M$.

4.20 An important example is the following. Let $G = \mathrm{GL}(n+1; \mathbf{R})$ and $X' = \mathbf{R}^{n+1} - \{0\}$. Let $S^n = \widetilde{\mathbf{RP}^n}$ denote the universal covering space of $\mathbf{RP}^n$; for $n > 1$ this is a two-fold covering space realized geometrically as the *sphere of directions* in $\mathbf{R}^{n+1}$. Furthermore the group of lifts of $\mathrm{PGL}(n+1; \mathbf{R})$ to S^n equals the quotient $\mathrm{GL}(n+1; \mathbf{R})/\mathbf{R}^+ \cong \mathrm{SL}^{\pm}(n+1; \mathbf{R})$. The quotient map $\Phi : \mathbf{R}^{n+1} - \{0\} \longrightarrow S^n$ is a principal $\mathbf{R}^+$-bundle.

Let M be an $\mathbf{RP}^n$-manifold with development pair $(\mathbf{dev}, h)$; then there exists a lift of $h : \pi \longrightarrow \mathrm{PGL}(n+1, \mathbf{R})$ to $\tilde{h} : \pi \longrightarrow \mathrm{GL}(n+1; \mathbf{R})$. The preceding construction then applies and we obtain a radiant affine structure on the total space M' of a principal $\mathbf{R}^+$-bundle over M with holonomy representation $\tilde{h}$. The radiant vector field $\rho_{M'}$ generates the (fiberwise) action of $\mathbf{R}^+$; this action of $\mathbf{R}^*$ on M' is affine, given locally in affine coordinates by homotheties. (This construction is due to Benzécri [B2] where the affine manifolds are called *variétés coniques affines.* He observes there that this construction defines an embedding of the category of $\mathbf{RP}^n$-manifolds into the category of $(n+1)$-dimensional affine manifolds.)

Since $\mathbf{R}^+$ is contractible, every principal $\mathbf{R}^+$-bundle is trivial (although there is in general no preferred trivialization). Choose any $\lambda > 1$; then the cyclic group $\langle\lambda\rangle \subset \mathbf{R}^+$ acts properly and freely on M' by affine transformations. We denote the resulting affine manifold by M'_λ and observe that it is homeomorphic to $M \times S^1$. (Alternatively, one may work directly with the Hopf manifold $\mathbf{Hopf}^{n+1}_\lambda$ and its $\mathbf{R}^*$-fibration $\mathbf{Hopf}^{n+1}_\lambda \longrightarrow \mathbf{RP}^n$.) We thus obtain:

4.21 Proposition (Benzécri [B2, §2.3.1]). *Suppose that M is an $\mathbf{RP}^n$-manifold. Let $\lambda > 1$. Then $M \times S^1$ admits a radiant affine structure for which the trajectories of the radiant vector field are all closed geodesics each affinely isomorphic to the Hopf circle $\mathbf{R}^+/\langle\lambda\rangle$.*

Since every (closed) surface admits an $\mathbf{RP}^2$-structure, we obtain:

4.21 Corollary (Benzécri). *Let Σ be a closed surface. Then $\Sigma \times S^1$ admits an affine structure.*

If Σ is a closed hyperbolic surface, the affine structure on $M = \Sigma \times S^1$ can be described as follows. A developing map maps the universal covering of M onto the convex cone $\Omega = \{(x,y,z) \in \mathbf{R}^3 \mid x^2+y^2-z^2 < 0, z > 0\}$ which is invariant under the identity component G of $\mathrm{SO}(2,1)$. The group $G \times \mathbf{R}^+$ acts transitively on Ω with isotropy group $\mathrm{SO}(2)$. Choosing a hyperbolic structure on Σ determines an isomorphism of $\pi_1(\Sigma)$ onto a discrete subgroup Γ of G; then for each $\lambda > 1$, the group $\Gamma \times \langle\lambda\rangle$ acts properly and freely on Ω with quotient the compact affine 3-manifold M.

4.22 Exercise. *A $\mathbf{CP}^n$-structure is a geometric structure modelled on complex projective space $\mathbf{CP}^n$ with coordinate changes locally from the projective group $\mathrm{PGL}(n+1;\mathbf{C})$). If M is a $\mathbf{CP}^n$-manifold, show that there is a T^2-bundle over M which admits a complex affine structure and an S^1-bundle over M which admits an $\mathbf{RP}^{2n+1}$-structure.*

The classification of $\mathbf{RP}^1$-manifolds.

4.23 The basic general question concerning geometric structures on manifolds is, given a topological manifold M and a geometry (X,G) , whether an (X,G)-structure on M exists, and if so, to classify all (X,G)-structures on M. Ideally, one would like a *deformation space,* a topological space whose points correspond to isomorphism classes of (X,G)-manifolds.

As an exercise to illustrate these general ideas, we classify $\mathbf{RP}^1$-manifolds (Compare Kuiper [Kp3], Goldman [G2]). To simplify matters, we pass to the universal covering $X = \widetilde{\mathbf{RP}^1}$, which is homeomorphic to $\mathbf{R}$ and the corresponding covering group $G = \widetilde{\mathrm{PGL}(2,\mathbf{R})}$ which acts on X. Suppose that M is a connected noncompact $\mathbf{RP}^1$-manifold (and thus diffeomorphic to an open interval). Then a developing map $\mathbf{dev} : M \approx \mathbf{R} \longrightarrow \mathbf{R} \approx X$ is necessarily an embedding of M onto an open interval in X. Given two such embeddings $f, f' : M \longrightarrow X$ whose images are equal, there exists a diffeomorphism $j : M \longrightarrow M$ such that $f' = j \circ f$. Thus two $\mathbf{RP}^1$-structures on M which have equal developing images are isomorphic. Thus the classification of $\mathbf{RP}^1$-structures on M is reduced to the classification of G-equivalence classes of intervals $J \subset X$. Choose a diffeomorphism $X \approx \mathbf{R} \approx (-\infty, \infty)$; an interval in X is determined by its pair of endpoints in $[-\infty, \infty]$. Since G acts transitively on X, an interval J is either bounded in X or projectively equivalent to X itself or one component of the complement of a point in X. Suppose that J is bounded. Then either the endpoints of J project to the same point in $\mathbf{RP}^1$ or to different points. In the first case, let $N > 0$ denote the degree of the map $J/\partial J \longrightarrow \mathbf{RP}^1$ induced by $\mathbf{dev}$; in the latter case choose an interval J^+ such that the the restriction of the covering projection $X \longrightarrow \mathbf{RP}^1$ to J^+ is injective and the union $J \cup J^+$ is an interval in X whose endpoints project to the same point in $\mathbf{RP}^1$. Let $N > 0$ denote the degree of the restriction of the covering projection to $J \cup J^+$. Since G acts transitively on pairs of distinct points in $\mathbf{RP}^1$, it follows easily that bounded intervals in X are determined up to equivalence by G by the two discrete invariants: whether the endpoints project to the same point in $\mathbf{RP}^1$ and the positive integer N. It follows that every (X, G)-structure on M is (X, G)-equivalent to one of the following types. We shall identify X with the real line and group of deck transformations of $X \longrightarrow \mathbf{RP}^1$ with the group of integer translations.

(1) A complete (X, G)-manifold (i.e. $\mathbf{dev} : M \longrightarrow X$ is a diffeomorphism);
(2) $\mathbf{dev} : M \longrightarrow X$ is a diffeomorphism onto one of two components of the complement of a point in X,e.g. $(0, \infty)$.
(3) $\mathbf{dev}$ is a diffeomorphism onto an interval $(0, N)$ where $N > 0$ is a positive integer;
(4) $\mathbf{dev}$ is a diffeomorphism onto an interval $(0, N + \frac{1}{2})$.

Next consider the case that M is a compact 1-manifold; choose a basepoint $x_0 \in M$. Let $\pi = \pi(M, x_0)$ be the corresponding fundamental group of M and let $\gamma \in \pi$ be a generator. We claim that the conjugacy class of $h(\gamma) \in G$ completely determines the structure. Choose a lift J of $M - \{x_0\}$ to $\tilde{M}$ which will be a fundamental domain for π. Then J is an open interval in $\tilde{M}$ with endpoints y_0 and y_1. Choose a developing map $\mathbf{dev} : \tilde{M} \longrightarrow X$ and a holonomy representation $h : \pi \longrightarrow G$; then $\mathbf{dev}(y_1) = h(\gamma)\mathbf{dev}(y_0)$. If $\mathbf{dev}'$ is a developing map for another structure with the same holonomy, then by applying an element of G we may assume that $\mathbf{dev}(y_0) = \mathbf{dev}'(y_0)$ and that $\mathbf{dev}(y_1) = \mathbf{dev}'(y_1)$. Furthermore there exists a diffeomorphism $\phi : J \longrightarrow J$ such that $\mathbf{dev}' = \phi \circ \mathbf{dev}$; this diffeomorphism lifts to a diffeomorphism $\tilde{M} \longrightarrow \tilde{M}$ taking $\mathbf{dev}$ to $\mathbf{dev}'$. Conversely suppose that $\eta \in G$ is orientation-preserving (this means simply that η lies in the identity component of G) and is not the identity. Then there exists $x_0 \in X$ which is not fixed by η; let $x_1 = \eta x_0$. There exists a diffeomorphism $J \longrightarrow X$ taking the endpoints y_i of J to x_i for $i = 0, 1$. This diffeomorphism extends to a developing map $\mathbf{dev} : \tilde{M} \longrightarrow X$. In summary:

4.24 THEOREM. *A compact $\mathbf{RP}^1$-manifold is either projectively equivalent to a Hopf circle $\mathbf{R}^+/\langle\lambda\rangle$, the complete affine manifold $\mathbf{R}/\mathbf{Z}$ or is complete as an $\mathbf{RP}^1$-manifold. Let G^0 denote the identity component of the universal covering group G of* $\mathrm{PGL}(2, \mathbf{R})$. *Let M be a closed 1-manifold. Then the set of isomorphism classes of $\mathbf{RP}^1$-structures on M is in bijective correspondence with the set of G-conjugacy classes in the set $G^0 - \{1\}$ of elements* *of G^0 not equal to the identity.*

EXERCISE. *Determine all automorphisms of each of the above list of $\mathbf{RP}^1$-manifolds.*

§5: Affine structures on surfaces

In this section we give the classification of affine structures on closed 2-manifolds. This classification falls into two steps: first is the basic result of Benzécri that a closed surface admits an affine structure if and only if its Euler characteristic vanishes. From this it follows that the affine holonomy group of a closed affine 2-manifold is abelian and the second step uses simple algebraic methods to classify affine structures.

We observe that affine structures on noncompact surfaces have a much different theory. First of all, every orientable noncompact surface admits an immersion into $\mathbf{R}^2$ and such an immersion determines an affine structure with trivial holonomy. Immersions can be classified up to crude relation of regular homotopy, although the isotopy classification of immersions of noncompact surfaces seems forbiddingly complicated. Furthermore if $h : \pi \longrightarrow \mathrm{Aff}(E)$ is a homomorphism such that the character $\det \circ \mathbf{L} \circ h : \pi \longrightarrow \mathbf{Z}/2$ equals the first Stiefel-Whitney class (i.e. its kernel is the subgroup of π corresponding to the orientable double covering of M), then it can be shown that there is an affine structure on M with holonomy h. In general it seems hopeless to try to classify general geometric structures (i.e. not satisfying some extra geometric hypothesis) on noncompact manifolds under anything but the crudest equivalence relations.

Suspensions.

5.2 Before discussing Benzécri's theorem and the classification of 2-dimensional affine manifolds, we describe several constructions for affine structures from affine structures and projective structures of lower dimension. Namely, let Σ be a smooth manifold and $f : \Sigma \to \Sigma$ a diffeomorphism. The mapping torus of f is defined to be the quotient $M = \mathbf{M}_f(\Sigma)$ of the product $\Sigma \times \mathbf{R}$ by the $\mathbf{Z}$-action defined by

$$n : (x, t) \mapsto (f^{-n}x, t + n)$$

It follows that dt defines a nonsingular closed 1-form ω on M tangent to the fibration $t : M \longrightarrow S^1 = \mathbf{R}/\mathbf{Z}$. Furthermore the vector field $\frac{\partial}{\partial t}$ on $\Sigma \times \mathbf{R}$ defines a vector field S_f on M, the *suspension* of the diffeomorphism $f : \Sigma \to \Sigma$. The dynamics of f is mirrored in the dynamics of S_f: there is a natural correspondence between the orbits of f and the trajectories of S_f. The embedding $\Sigma \hookrightarrow \Sigma \times \{t\}$ is transverse to the vector field S_f and each trajectory of S_f meets Σ. Such a hypersurface is called a *cross-section* to the vector field. Given a cross-section Σ to a flow $\{\xi_t\}_{t\in\mathbf{R}}$, then (after possibly reparametrizing $\{\xi_t\}_{t\in\mathbf{R}}$), the flow can be recovered as a suspension. Namely, given $x \in \Sigma$, let $f(x)$ equal $\xi_t(x)$ for the smallest $t > 0$ such that $\xi_t(x) \in \Sigma$, i.e. the first-return map or Poincaré map for $\{\xi_t\}_{t\in\mathbf{R}}$ on Σ. For the theory of cross-sections to flows we refer to Fried [F1].

Suppose that $\mathfrak{F}$ is a foliation of a manifold M; then $\mathfrak{F}$ is locally defined by an atlas of smooth submersions $U \longrightarrow \mathbf{R}^q$ for coordinate patches U. An (X, G)-*atlas transverse to* $\mathfrak{F}$

is defined to be a collection of coordinate patches U_α and coordinate charts $\psi_\alpha : U_\alpha \longrightarrow X$ such that for each pair (U_α, U_β) and each component $C \subset U_\alpha \cap U_\beta$ there exists an element $g_C \in G$ such that $g_C \circ \psi_\alpha = \psi_\beta$ on C. An *(X, G)-structure transverse to $\mathfrak{F}$* is a maximal *(X, G)-atlas transverse to $\mathfrak{F}$*. Consider an (X, G)-structure transverse to $\mathfrak{F}$; then an immersion $f : \Sigma \longrightarrow M$ which is transverse to $\mathfrak{F}$ induces an (X, G)-structure on Σ.

A foliation $\mathfrak{F}$ of an affine manifold is said to be *affine* if its leaves are parallel affine subspaces (i.e. totally geodesic subspaces). It is easy to see that transverse to an affine foliation of an affine manifold is a natural affine structure. In particular if M is an affine manifold and ζ is a parallel vector field on M, then ζ determines a one-dimensional affine foliation which thus has a transverse affine structure. Moreover if Σ is a cross-section to ζ, then Σ has a natural affine structure for which the Poincaré map $\Sigma \longrightarrow \Sigma$ is affine.

5.3 Exercise. *Show that the Hopf manifold $\mathbf{Hopf}^n_\lambda$ has an affine foliation with one closed leaf if $n > 1$ (two if $n = 1$) and its complement consists of two Reeb components.*

Let Σ be an affine manifold and $f \in \mathrm{Aff}(M)$ an automorphism. We shall define an affine manifold M with a parallel vector field S_f and cross-section $\Sigma \hookrightarrow M$ such that the corresponding Poincaré map is f. We proceed as follows. Let $\Sigma \times \mathbf{R}$ be the Cartesian product with the product affine structure and let $t : \Sigma \times \mathbf{R} \longrightarrow \mathbf{R}$ be an affine coordinate on the second factor. Then the map $\tilde{f} : \Sigma \times \mathbf{R} \longrightarrow \Sigma \times \mathbf{R}$ given by $(x, t) \mapsto (f^{-1}(x), t + 1)$ is affine and generates a free proper $\mathbf{Z}$-action on $\Sigma \times \mathbf{R}$. Let M be the corresponding quotient affine manifold. Then $\frac{\partial}{\partial t}$ is a parallel vector field on $\Sigma \times \mathbf{R}$ invariant under $\tilde{f}$ and thus defines a parallel vector field S_f on M. Similarly the parallel 1-form dt on $\Sigma \times \mathbf{R}$ defines a parallel 1-form ω_f on M for which $\omega_f(S_f) = 1$. For each $t \in \mathbf{R}/\mathbf{Z}$, the inclusion $\Sigma \times \{t\} \hookrightarrow M$ defines a cross-section to S_f. We call (M, S_f) the *parallel suspension* or *affine mapping torus* of the affine automorphism (Σ, f).

5.4 Exercise. *Suppose that N and Σ are affine manifolds and that $\phi : \pi_1(\Sigma) \longrightarrow \mathrm{Aff}(N)$ is an action of $\pi_1(\Sigma)$ on N by affine automorphisms. The flat N-bundle over Σ with holonomy ϕ is defined as the quotient of $\tilde{\Sigma} \times N$ by the diagonal action of $\pi_1(\Sigma)$ given by deck transformations on $\tilde{\Sigma}$ and by ϕ on N. Show that the total space M is an affine manifold such that the fibration $M \longrightarrow \Sigma$ is an affine map and the the flat structure (the foliation of M induced by the foliation of $\tilde{\Sigma} \times N$ by leaves $\tilde{\Sigma} \times \{y\}$, for $y \in N$) is an affine foliation.*

5.5 Now let (M, ρ_M) be a radiant affine manifold of dimension $n + 1$. Then there is an $\mathbf{RP}^n$-structure transverse to ρ_M. For in local affine coordinates the trajectories of ρ_M are rays through the origin in $\mathbf{R}^{n+1}$ and the quotient projection maps coordinate patches submersively into $\mathbf{RP}^n$. In particular if Σ is an n-manifold and $f : \Sigma \longrightarrow M$ is transverse to ρ_M, then f determines an $\mathbf{RP}^n$-structure on Σ.

Proposition. *Let Σ be a compact $\mathbf{RP}^n$-manifold and $f \in \mathrm{Aut}(\Sigma)$ a projective automorphism. Then there exists a radiant affine manifold (M, ρ_M) and a cross-section $\iota : \Sigma \hookrightarrow M$ to ρ_M such that the Poincaré map for ι equals $\iota^{-1} \circ f \circ \iota$. In other words, the mapping torus of a projective automorphism of an compact $\mathbf{RP}^n$-manifold admits a radiant affine structure.*

PROOF: Let S^n be the double covering of $\mathbf{RP}^n$ (realized as the sphere of directions in $\mathbf{R}^{n+1}$) and let $\Phi : \mathbf{R}^{n+1} \longrightarrow S^n$ be the corresponding principal $\mathbf{R}^+$-fibration. Let M' be the principal $\mathbf{R}^+$-bundle over M constructed in 4.20 and choose a section $\sigma : M \longrightarrow M'$ and let $\{\xi'_t\}_{t\in\mathbf{R}}$ be the radiant flow on M'; let $\{\tilde{\xi}'_t\}_{t\in\mathbf{R}}$ be the radiant flow on $\tilde{M}'$. Let $(\mathbf{dev}, h)$ be a development pair; then there exists a lift of f to an affine automorphism $\tilde{f}$ of $\tilde{M}$; there exists a projective automorphism $g \in \mathrm{GL}(n+1;\mathbf{R})/\mathbf{R}^+$ of the sphere of directions S^n such that

$$\begin{array}{ccc} \tilde{M}' & \xrightarrow{\mathbf{dev}'} & \mathbf{R}^{n+1} - \{0\} \\ \tilde{f}\Big\downarrow & & \Big\downarrow g \\ \tilde{M}' & \xrightarrow{\mathbf{dev}'} & \mathbf{R}^{n+1} - \{0\} \end{array}$$

Choose a compact set $K \subset \tilde{M}$ such that $\pi_1(M)\cdot K = \tilde{M}$. Let $\tilde{K} \subset \tilde{M}'$ be the image of K under a lift of σ to a section $\tilde{M} \longrightarrow \tilde{M}'$. Then there exists $t_0 > 0$ such that

$$\tilde{K} \cap \tilde{f}\tilde{\xi}'_t(\tilde{K}) = \emptyset$$

for $t > t_0$. It follows that the affine automorphism $\xi_t\tilde{f}$ generates a free and proper affine $\mathbf{Z}$-action on M' for $t > t_0$. We denote the quotient by M. In terms of the trivialization of $M' \longrightarrow M$ arising from σ, it is clear that the quotient of M' by this $\mathbf{Z}$-action is diffeomorphic to the mapping torus of f. Furthermore the setion σ defines a cross-section $\Sigma \hookrightarrow M$ to ρ_M whose Poincaré map corresponds to f. ∎

We call the radiant affine manifold (M, ρ_M) the *radiant suspension* of (Σ, f).

In general affine automorphisms of affine manifolds can display quite complicated dynamics and thus the flows of parallel vector fields and radiant vector fields can be similarly complicated. For example, any element of $\mathrm{GL}(2;\mathbf{Z})$ acts affinely on the flat torus $\mathbf{R}^2/\mathbf{Z}^2$; the most interesting of these are the hyperbolic elements of $\mathrm{GL}(2;\mathbf{Z})$ which determine Anosov diffeomorphisms on the torus. Their suspensions thus determine Anosov flows on affine 3-manifolds which are generated by parallel or radiant vector fields. Indeed, it can be shown (Fried [F4]) that every Anosov automorphism of a nilmanifold M can be made affine for some complete affine structure on M.

As a simple example of this we consider the linear diffeomorphism of the two-torus $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ defined by a hyperbolic element $A \in \mathrm{GL}(2;\mathbf{Z})$. The parallel suspension of A is the complete affine 3-manifold $\mathbf{R}^3/\Gamma$ where $\Gamma \subset \mathrm{Aff}(\mathbf{R}^3)$ is consists of the affine transformations

$$\begin{bmatrix} A^n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ n \end{bmatrix}$$

where $n \in \mathbf{Z}$ and $p \in \mathbf{Z}^2$. Since A is conjugate in $\mathrm{SL}(2;\mathbf{R})$ to a diagonal matrix with reciprocal eigenvalues, Γ is conjugate to a discrete cocompact subgroup of the subgroup of $\mathrm{Aff}(\mathbf{R}^3)$

$$G = \left\{ \begin{bmatrix} e^u & 0 & 0 \\ 0 & e^{-u} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix} \mid s,t,u \in \mathbf{R} \right\}$$

which acts simply transitively. Since there are infinitely many conjugacy classes of hyperbolic elements in $\mathrm{SL}(2;\mathbf{Z})$ (for example the matrices

$$\begin{bmatrix} n+1 & n \\ 1 & 1 \end{bmatrix}$$

for $n > 1, n \in \mathbf{Z}$ are all non-conjugate), there are infinitely many isomorphism classes of discrete groups Γ. It follows (L. Auslander) that there are infinitely many homotopy classes of compact complete affine 3-manifolds — in contrast to the theorem of Bieberbach that in each dimension there are only finitely many homotopy classes of compact flat Riemannian manifolds. Notice that each of these affine manifolds possesses a parallel Lorentz metric and hence is a flat Lorentz manifold.

Existence of affine structures on 2-manifolds.

5.6 The following result is proved in [B1]; a more algebraic generalization/clarification may be found in Milnor [Mi1]; for generalizations of Milnor's result, see Benzécri [B3], Gromov [], Sullivan [Su]. For an interpretation of this inequality in terms of hyperbolic geometry, see Goldman [G2].

THEOREM (BENZÉCRI 1955). *Let M be a closed 2-dimensional affine manifold. Then $\chi(M) = 0$.*

PROOF: By replacing M by its orientable double covering, we assume that M is orientable. By the classification of surfaces, M is diffeomorphic to a closed surface of genus $g \geq 0$. Since a simply connected closed manifold admits no affine structure (§4.?), M cannot be a 2-sphere and hence $g \neq 0$. We assume that $g > 1$ and obtain a contradiction.

There exists a decomposition of M along $2g$ simple closed curves $a_1, b_1, \dots, a_g, b_g$ which intersect in a single point $x_0 \in M$ such that the complement $M - \bigcup_{i=1}^{g}(a_i \cup b_i)$ is the interior of a $4g$-gon F with edges $a_1^+, a_1^-, b_1^+, b_1^-, \dots, a_g^+, a_g^-, b_g^+, b_g^-$. There are generators $A_1, B_1, \dots, A_g, B_g \in \pi$ such that $A_i(b_i^+) = b_i^-$ and $B_i(a_i^+) = a_i^-$ define identifications for a quotient map $F \longrightarrow M$. A universal covering space is the quotient space of the product $\pi \times F$ by identifications defined by the generators $A_1, B_1, \dots, A_g, B_g$. Fix a development pair $(\mathbf{dev}, h)$. For convenience we assume that the curves $a_1, b_1, \dots, a_g, b_g$ all share the same tangent vector at x_0. Thus F is a polygon with $4g$ vertices, one of which has angle 2π and all others have angle 0.

Let $\mathbf{I} = [a, b]$ be a closed interval. If $f : \mathbf{I} \longrightarrow \mathbf{R}^2$ is a smooth immersion, then its *turning number* $\tau(f)$ is defined as the total angular displacement of its tangent vector. If $f(t) = (x(t), y(t))$, then

$$\tau(f) = \int_a^b d\tan^{-1}(y'(t)/x'(t)) = \int_a^b \frac{x'(t)y''(t) - x''(t)y'(t)}{x'(t)^2 + y'(t)^2} dt$$

is an analytic formula for the turning number. We can extend this invariant to piecewise smooth immersions as follows. Suppose that $f : [a, b] \longrightarrow \mathbf{R}^2$ is an immersion which is smooth on subintervals $[a_i, a_{i+1}]$ where $a = a_0 < a_1 < \cdots < a_m < a_{m+1} = b$. Let

$f'_+(a_i) = \lim_{t\to a_i+} f'(t)$ and $f'_-(a_i) = \lim_{t\to a_i-} f'(t)$ be the two tangent vectors to f at a_i; then the total turning number of f is defined as

$$\tau(f) = \sum_{i=0}^{m} (\tau(f|_{[a_i,a_{i+1}]}) + \theta(f'_-(a_{i+1}), f'_+(a_{i+1}))$$

where $\theta(v_1, v_2)$ represents the positively measured angle between the vectors v_1, v_2. Clearly reversing the orientation multiplies the turning number by -1.

5.7 If $f : S^1 \longrightarrow \mathbf{R}^2$ is an immersion, then $\tau(f)$ is an integer. The *Whitney-Graustein theorem* asserts that two immersions $f_1, f_2 : S^1 \longrightarrow \mathbf{R}^2$ are regularly homotopic $\Longleftrightarrow$ $\tau(f_1) = \tau(f_2)$. In particular if $f : S^1 \longrightarrow \mathbf{R}^2$ is the restriction to the boundary of an orientation-preserving immersion $D^2 \longrightarrow \mathbf{R}^2$, then $\tau(f) = 1$.

An elementary property relating turning number to affine transformations is the following:

LEMMA. *Suppose that $f : [a, b] \longrightarrow \mathbf{R}^2$ is a smooth immersion and $\phi \in \mathrm{Aff}^+(\mathbf{R}^2)$ is an orientation-preserving affine automorphism. Then*

$$|\tau(f) - \tau(\phi \circ f)| < \pi$$

PROOF: If ψ is an orientation-preserving Euclidean isometry, then $\tau(f) = \tau(\psi \circ f)$; by composing ϕ with an isometry we may assume that $f(a) = (\phi\circ f)(a)$ and $f'(a) = \lambda(\phi\circ f)'(a)$ for $\lambda > 0$.

Suppose that $|\tau(f) - \tau(\phi \circ f)| \geq \pi$. Since for $a \leq t \leq b$, the function

$$|\tau(f|_{[a,t]}) - \tau(\phi \circ f|_{[a,t]})|$$

is a continuous function of t and equals 0 for $t = a$, there exists $0 < t_0 \leq b$ such that

$$|\tau(f|_{[a,t_0]}) - \tau(\phi \circ f|_{[a,t_0]})| = \pi.$$

Then the tangent vectors $f'(t_0)$ and $(\phi \circ f)'(t_0)$ have opposite direction, i.e. there exists $\mu > 0$ such that

$$\mathbf{L}(\phi)(f'(t_0)) = (\phi \circ f)'(t_0) = -\mu f'(t_0).$$

Thus the linear part $\mathbf{L}(\phi)$ has two eigenvalues $\lambda, -\mu$ contradicting ϕ being orientation-preserving. ▮

We apply these ideas to the restriction of the developing map $\mathbf{dev}$ to ∂F. Since $\mathbf{dev}|_{\partial F}$ is the restriction of the immersion $\mathbf{dev}|_F$ of the 2-disc,

$$\begin{aligned}
2\pi =& \tau(\mathbf{dev}|_{\partial F}) \\
&= \sum_{i=1}^{g} \tau(\mathbf{dev}|_{a_i^+}) + \tau(\mathbf{dev}|_{a_i^-}) + \tau(\mathbf{dev}|_{b_i^+}) + \tau(\mathbf{dev}|_{b_i^-}) + (4g-2)\pi \\
&= \sum_{i=1}^{g} \tau(\mathbf{dev}|_{a_i^+}) - \tau(h(B_i) \circ \mathbf{dev}|_{a_i^+}) + \tau(\mathbf{dev}|_{b_i^+}) - \tau(h(A_i) \circ \mathbf{dev}|_{b_i^+}) + (4g-2)\pi
\end{aligned}$$

(The $4g-1$ contributions of π arise from the $4g-1$ vertices of F having interior angle 0; the single vertex of F having interior angle 2π contributes $-\pi$.) Thus

$$(4-4g)\pi = \sum_{i=1}^{g} \tau(\mathbf{dev}|_{a_i^+}) - \tau(h(B_i)\circ\mathbf{dev}|_{a_i^+}) + \tau(\mathbf{dev}|_{b_i^+}) - \tau(h(A_i)\circ\mathbf{dev}|_{b_i^+})$$

and

$$(4g-4)\pi < \sum_{i=1}^{g} |\tau(\mathbf{dev}|_{a_i^+}) - \tau(h(B_i)\circ\mathbf{dev}|_{a_i^+})| + |\tau(\mathbf{dev}|_{b_i^+}) - \tau(h(A_i)\circ\mathbf{dev}|_{b_i^+})| < 2g\pi$$

from which it follows $g = 1$. ∎

5.8 Shortly after Benzécri proved the above theorem, Milnor observed that this result follows from a more general theorem on flat vector bundles. Let E be the 2-dimensional oriented vector bundle over M whose total space is the quotient of $\tilde{M}\times\mathbf{R}^2$ by the diagonal action of π by deck transformations on $\tilde{M}$ and via $\mathbf{L}\circ h$ on $\mathbf{R}^2$, (i.e. the *flat vector bundle over M associated to the linear holonomy representation.)* This bundle has a natural flat structure, since the coordinate changes for this bundle are (locally) constant linear maps. Now an oriented $\mathbf{R}^2$-bundle over a space M is classified by its *Euler class* which lies in $H^2(M;\mathbf{Z})$. For M a closed oriented surface $H^2(M;\mathbf{Z})\cong\mathbf{Z}$ and if ξ is an oriented $\mathbf{R}^2$-bundle over M which admits a flat structure, Milnor [Mi1] showed that

$$|e(\xi)| < g.$$

If M is an affine manifold, then the bundle E is isomorphic to the tangent bundle of M and hence has Euler number $e(TM) = 2-2g$. Thus the only closed orientable surface whose tangent bundle has a flat structure is a torus. Furthermore Milnor showed that any $\mathbf{R}^2$-bundle whose Euler number satisfies the above inequality has a flat connection.

In the early 1950's Chern suggested that in general the Euler characteristic of a compact affine manifold must vanish. Based on the Chern-Weil theory of representing characteristic classes by curvature, several special cases of this conjecture can be solved: if M is a compact *complex* affine manifold, then the Euler characteristic is the top Chern number and hence can be expressed in terms of curvature of the complex linear connection (which is zero). However, in general, for a real vector bundle, only the Pontrjagin classes are polynomials in the curvature — indeed Milnor's examples show that the Euler class *cannot* be expressed as a polynomial in the curvature of a linear connection (although it can be expressed as a polynomial in the curvature of an *orthogonal* connection. This difficulty was overcome by a clever trick by Kostant and Sullivan [KS] who showed that the Euler characteristic of a compact complete affine manifold vanishes.

Nonexistence of affine structures on certain connected sums.

5.9 In 1961, L. Markus posed the following "research problem" among the exercises in lecture notes for a class in cosmology at the University of Minnesota:

QUESTION. *Let M be a closed affine manifold. Then M is geodesically complete $\iff$ M has parallel volume.*

An affine manifold has *parallel volume* $\iff$ it admits a parallel volume form $\iff$ the affine coordinate changes are volume-preserving $\iff$ the linear holonomy group lies in $\mathrm{SL}(E)$. If $h : \pi \longrightarrow \mathrm{Aff}(E)$ is the affine holonomy homomorphism and $\mathbf{L} \circ h : \pi \longrightarrow \mathrm{GL}(E)$ is the linear holonomy then M has parallel volume $\iff$ the composition

$$\det \circ \mathbf{L} \circ h : \pi \longrightarrow \mathbf{R}^*$$

is the trivial homomorphism. Thus every affine structure on a manifold with zero first Betti number has parallel volume.

This somewhat surprising conjecture seems to be one of the main barriers in constructing examples of affine manifolds. A purely topological consequence of this conjecture is that a compact affine manifold M with zero first Betti number is covered by Euclidean space (in particular all of its higher homotopy groups vanish). Thus there should be no such structure on a nontrivial connected sum in dimensions greater than two. (In fact no affine structure is presently known on a nontrivial connected sum.) Since the fundamental group of a connected sum is a free product the following result is relevant in this connection:

5.10 THEOREM (SMILLIE [SM3]). *Let M be a closed affine manifold with parallel volume. Then the affine holonomy homomorphism cannot factor through a free group.*

This theorem can be generalized much further — see Smillie [Sm3] and Goldman-Hirsch [GH3].

5.11 COROLLARY(SMILLIE [SM3]). *Let M be a closed manifold whose fundamental group is a free product of finite groups (e.g. a connected sum of manifolds with finite fundamental group). Then M admits no affine structure.*

PROOF OF 5.11 ASSUMING 5.10: Suppose M has an affine structure. Since $\pi_1(M)$ is a free product of finite groups, the first Betti number of M is zero. Thus M has parallel volume. Furthermore if $\pi_1(M)$ is a free product of finite groups, there exists a free subgroup $\Gamma \subset \pi_1(M)$ of finite index. Let $\hat{M}$ be the covering space with $\pi_1(\hat{M}) = \Gamma$. Then the induced affine structure on $\hat{M}$ also has parallel volume contradicting Theorem. ∎

PROOF OF 5.10: Let M be a closed affine manifold modelled on an affine space E, $\mathbf{p} : \tilde{M} \longrightarrow M$ a universal covering, and $(\mathbf{dev} : \tilde{M} \longrightarrow E, h : \pi \longrightarrow \mathrm{Aff}(E))$ a development pair. Suppose that M has parallel volume and that there is a free group Π through which the affine holonomy homomorphism h factors:

$$\pi \xrightarrow{\phi} \Pi \xrightarrow{\bar{h}} \mathrm{Aff}(E)$$

Choose a graph G with fundamental group Π; then there exists a map $f : M \longrightarrow G$ inducing the homomorphism $\phi : \pi = \pi_1(M) \longrightarrow \pi_1(G) = \Pi$. By general position, there exist points $s_1, \ldots s_k \in G$ such that f is transverse to s_i and the complement $G - \{s_1, \ldots, s_k\}$ is connected and simply connected. Let H_i denote the inverse image $f^{-1}(s_i)$ and let

$H = \cup_i H_i$ denote their disjoint union. Then H is an oriented closed smooth hypersurface such that the complement $M - H \subset M$ has trivial holonomy. Let $M|H$ denote the manifold with boundary obtained by *splitting* M along H; i.e. $M|H$ has two boundary components H_i^+, H_i^- for each H_i and there exist diffeomorphisms $g_i : H_i^+ \longrightarrow H_i^-$ (generating Π) such that M is the quotient of $M|H$ by the identifications g_i. There is a canonical diffeomorphism of $M - H$ with the interior of $M|H$.

Let ω_E be a parallel volume form on E; then there exists a parallel volume form ω_M on M such that $\mathbf{p}^*\omega_M = \mathbf{dev}^*\omega_E$. Since $H^n(E) = 0$, there exists an $(n-1)$-form η on E such that $d\eta = \omega_E$. For any immersion $f : S \longrightarrow E$ of an oriented closed $(n-1)$-manifold S, the integral

$$\int_S f^*\eta$$

is independent of the choice of η satisfying $d\eta = \omega_E$. Since $H^{n-1}(E)$, any other η' must satisfy $\eta' = \eta + d\theta$ and

$$\int_S f^*\eta' - \int_S f^*\eta = \int_S d(f^*\theta) = 0.$$

Since $M - H$ has trivial holonomy there is a developing map $\mathbf{dev} : M - H \longrightarrow E$ and its restriction to $M - H$ extends to a developing map $\mathbf{dev} : M|H \longrightarrow E$ such that

$$\mathbf{dev}|_{H_i^+} = \bar{h}(g_i) \circ \mathbf{dev}|_{H_i^-}$$

and the normal orientations of H_i^+, H_i^- induced from $M|H$ are opposite. Since $h(g_i)$ preserves the volume form ω_E, $d(h(g_i)^*\eta) = d(\eta) = \omega_E$ and we have

$$\int_{H_i^+} \mathbf{dev}^*\eta = \int_{H_i^+} \mathbf{dev}^* h(g_i)^*\eta = -\int_{H_i^-} \mathbf{dev}^*\eta$$

since the normal orientations of $H_i^\pm$ are opposite. We now compute the ω_M-volume of M:

$$\mathrm{vol}(M) = \int_M \omega_M = \int_{M|H} \mathbf{dev}^*\omega_E = \int_{\partial(M|H)} \eta = \sum_{i=1}^{k} \int_{H_i^+} \eta + \int_{H_i^-} \eta = 0$$

a contradiction. ∎

One basic method of finding a primitive η for ω_E is by a radiant vector field ρ. Since ρ expands volume, we have $d\iota_\rho\omega_E = n\omega_E$ and $\eta = \frac{1}{n}\iota_\rho\omega_E$ is a primitive for ω_E. An affine manifold is *radiant* $\Longleftrightarrow$ it possesses a radiant vector field $\Longleftrightarrow$ the affine structure comes from an $(E, \mathrm{GL}(E))$-structure $\Longleftrightarrow$ its affine holonomy has a fixed point in E. The following result generalizes the above theorem:

5.12 THEOREM(SMILLIE). *Let M be a closed affine manifold with a parallel exterior differential k-form which has nontrivial de Rham cohomology class. Suppose $\mathcal{U}$ is an open covering of M such that for each $U \in \mathcal{U}$, the affine structure induced on U is radiant. Then $\dim\mathcal{U} \geq k$; i.e. there exist $k+1$ distinct open sets $U_1, \ldots, U_{k+1} \in \mathcal{U}$ such that the intersection $U_1 \cap \cdots \cap U_{k+1} \neq \emptyset$. (Equivalently the nerve of $\mathcal{U}$ has dimension at least k.)*

A published proof of this theorem can be found in Goldman-Hirsch [GH3].

Radiant affine structures.

5.13 Radiant affine manifolds have many special properties, derived from the existence of a radiant vector field. If M is a manifold with radiant affine structure modelled on an affine space E, let $(\mathbf{dev}, h)$ be a development pair and ρ_E a radiant vector field on E invariant under $h(\pi)$, then there exists a (radiant) vector field ρ_M on M such that

$$\mathbf{p}^*\rho_M = \mathbf{dev}^*\rho_E.$$

THEOREM. *Let M be a compact radiant manifold.*

(1) *Then M cannot have parallel volume. (In other words a compact manifold cannot support a $(\mathbf{R}^n, \mathrm{SL}(n;\mathbf{R}))$-structure.) In particular the first Betti number of a closed radiant manifold is positive.*

(2) *The developing image $\mathbf{dev}(\tilde{M})$ does not contain any stationary points of the affine holonomy. (Thus M is incomplete.) In particular the radiant vector field ρ_M is nonsingular and the Euler characteristic $\chi(M) = 0$.*

PROOF OF (1): Let $\omega_E = dx^1 \wedge \cdots \wedge dx^n$ be a parallel volume form on E and let ω_M be the corresponding parallel volume form on M, i.e. $\mathbf{p}^*\omega_M = \mathbf{dev}^*\omega_E$. Let η_M denote the interior product

$$\eta_M = \frac{1}{n}\iota_{\rho_M}\omega_M$$

and it follows from

$$d\iota_{\rho_E}\omega_E = n\omega_E$$

that $d\eta_M = \omega_M$. But ω_M is a volume form on M and

$$\mathrm{vol}(M) = \int_M \omega_M = \int_M d\eta_M = 0$$

a contradiction. (Intuitively, the main idea in the proof above is that the radiant flow on M expands the parallel volume uniformly. Thus by "conservation of volume"a compact manifold cannot support both a radiant vector field and a parallel volume form.) ∎

PROOF OF (2): We may assume that

$$\rho_E = \sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i}$$

and it will suffice to prove that $0 \notin \mathbf{dev}(\tilde{M})$. Since the only zero of ρ_E is the origin $0 \in E$, ρ_M is nonsingular on the complement of $F = \mathbf{p}(\mathbf{dev}^{-1}(0))$. Since $\mathbf{p}$ and $\mathbf{dev}$ are local diffeomorphisms and $0 \in E$ is discrete, it follows that $F \subset M$ is a discrete set; since $\mathbf{dev}$ is continuous and 0 is $h(\pi)$-invariant, $F \subset M$ is closed. Hence F is a finite subset of M.

Since M is a closed manifold, ρ_M is completely integrable and thus there is a flow $\{R_t : M \longrightarrow M\}_{t\in\mathbf{R}}$ whose infinitesimally generated by $-\rho_M$. The flow lifts to a flow $\{\tilde{R}_t : M \longrightarrow M\}_{t\in\mathbf{R}}$ on $\tilde{M}$ which satisfies

$$\mathbf{dev}(\tilde{R}_t x) = e^{-t}\mathbf{dev}(x)$$

for $x \in \tilde{M}, t \in \mathbf{R}$. Choose a neighborhood U of F, each component of which develops to a small ball B about 0 in E. Let $K \subset \tilde{M}$ be a compact set such that the saturation $\pi(K) = \tilde{M}$; then there exists $N >> 0$ such that if $e^{-t}(\mathbf{dev}(K)) \subset B$ for $t \geq N$ and thus $\tilde{R}_t(K) \subset B$ for $t \geq N$. It follows that U is an attractor for the flow of ρ_M and that $R_N : M \longrightarrow U$ is a deformation retraction of the closed manifold M onto U. Since a closed manifold is not homotopy-equivalent to a finite set, this contradiction shows that $0 \notin \mathbf{dev}(\tilde{M})$ as desired. ∎

There is a large class of discrete groups Γ for which every affine representation $\Gamma \longrightarrow \mathrm{Aff}(E)$ is conjugate to a representation factoring through $\mathrm{SL}(E)$, i.e. $\Gamma \longrightarrow \mathrm{SL}(E) \subset \mathrm{Aff}(E)$. For example finite groups have this property, and the above theorem gives an alternate proof that the holonomy of a compact affine manifold must be infinite. Another class of groups having this property are the *Margulis groups,* i.e. irreducible lattices in semisimple Lie groups of $\mathbf{R}$-rank greater than one (e.g. $SL(n, \mathbf{Z})$ for $n > 2$). It follows that the affine holonomy of a compact affine manifold cannot factor through a Margulis group. However, since $\mathrm{SL}(n; \mathbf{R})$ does admit a left-invariant $\mathbf{RP}^{n^2-1}$-structure, it follows that if $\Gamma \subset \mathrm{SL}(n; \mathbf{R})$ is a torsionfree cocompact lattice, then there exists a compact affine manifold with holonomy group $\Gamma \times \mathbf{Z}$ although Γ itself is not the holonomy group of an affine structure.

Associative algebras: the group objects in the category of affine manifolds.

5.14 Let $\mathfrak{a}$ be an associative algebra over the field of real numbers. We shall associate to $\mathfrak{a}$ a Lie group $G = G(\mathfrak{a})$ with a bi-invariant affine structure. Conversely, if G is a Lie group with a bi-invariant affine structure, then we show that its Lie algebra $\mathfrak{g}$ supports an associative multiplication $\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying

$$(A1) \qquad [X, Y] = XY - YX$$

and that the corresponding Lie group with with bi-invariant affine structure is locally isomorphic to G.

We begin by discussing invariant affine structures on Lie groups. If G is a Lie group and $a \in G$, then the operations of left- and right- multiplication are defined by

$$L_a b = R_b a = ab$$

Suppose that G is a Lie group with an affine structure. The affine structure is *left-invariant* (resp. *right-invariant*) $\iff$ the operations $L_a : G \longrightarrow G$ (resp. $R_a : G \longrightarrow G$) are affine. An affine structure is *bi-invariant* $\iff$ it is both left-invariant and right-invariant.

Suppose that G is a Lie group with a left-invariant (resp. right-invariant, bi-invariant) affine structure. Let $\tilde{G}$ be its universal covering group and

$$\pi_1(G) \hookrightarrow \tilde{G} \longrightarrow G$$

the corresponding central extension. Then the induced affine structure on $\tilde{G}$ is also left-invariant (resp. right-invariant, bi-invariant). Conversely, since $\pi_1(G)$ is central in $\tilde{G}$, every left-invariant (resp. right-invariant, bi-invariant) affine structure on $\tilde{G}$ determines a left-invariant (resp. right-invariant, bi-invariant) affine structure on G. Thus there is a bijection between left-invariant (resp. right-invariant, bi-invariant) affine structures on a Lie group and left-invariant (resp. right-invariant, bi-invariant) affine structures on any covering group. For this reason we shall for the most part only consider simply connected Lie groups.

Suppose that G is a simply connected Lie group with a left-invariant affine structure and let $\mathbf{dev} : G \longrightarrow E$ be a developing map. Then corresponding to the affine action of G on itself by left-multiplications there is a homomorphism $\alpha : G \longrightarrow \mathrm{Aff}(E)$ such that the diagram

(A2)
$$\begin{tikzcd} G \arrow[r, "\mathbf{dev}"] \arrow[d, "L_g"'] & E \arrow[d, "\alpha(g)"] \\ G \arrow[r, "\mathbf{dev}"'] & E \end{tikzcd}$$

commutes for each $g \in G$. We may assume that $\mathbf{dev}$ maps the identity element $e \in G$ to the origin $0 \in E$; it follows from (A2) that $\mathbf{dev}(g) = \alpha(g)\mathbf{dev}(e) = \alpha(g) \cdot 0$ is the translational part of the affine representation $\alpha : G \longrightarrow \mathrm{Aff}(E)$ for each $g \in G$. Furthermore since $\mathbf{dev}$ is open, it follows that the orbit $\alpha(G)(0)$ equals the developing image and is open. Indeed the translational part, which is the differential of the evaluation map

$$T_eG = \mathfrak{g} \longrightarrow E = T_0E$$

is a linear isomorphism. Such an action will be called *locally simply transitive*.

Conversely suppose that $\alpha : G \longrightarrow \mathrm{Aff}(E)$ is an affine representation and $O \subset E$ is an open orbit. Then for any point $x_0 \in O$, the evaluation map $g \mapsto \alpha(g)(x_0)$ defines a developing map for an affine structure on G. Since $\mathbf{dev}(L_gh) = \alpha(gh)(x_0) = \alpha(g)\alpha(h)(x_0) = \alpha(g)\mathbf{dev}(h)$ for $g, h \in G$, this affine structure is left-invariant. Thus *there is an isomorphism between the category of Lie groups G with left-invariant affine structure and open orbits of locally simply transitive affine representations $G \longrightarrow \mathrm{Aff}(E)$.*

Now suppose that $\mathfrak{a}$ is an associative algebra; we shall associate to $\mathfrak{a}$ a Lie group with bi-invariant affine structure as follows. We formally adjoin to $\mathfrak{a}$ a two-sided identity element 1 to construct an associative algebra $\mathfrak{a} \oplus \mathbf{R}1$; then the affine hyperplane $E = \mathfrak{a} \times \{1\}$ in $\mathfrak{a} \oplus \mathbf{R}1$ is a multiplicatively closed subset; the multiplication is given by the *Jacobson product*

$$(a \oplus 1)(b \oplus 1) = (a + b + ab) \oplus 1$$

and in particular left-multiplications and right-multiplications are affine maps. Let $G = G(\mathfrak{a})$ be the set of all $a \oplus 1$ which have left-inverses (necessarily also in $\mathfrak{a} \oplus \{1\}$); it follows from associativity that $a \oplus 1$ is left-invertible $\iff$ it is right-invertible as well. It is easy to see that G is an open subset of $\mathfrak{a} \oplus \{1\}$ and forms a group. Furthermore, the associative property in $\mathfrak{a}$ implies that the actions of G by both left- and right- multiplication on E are affine. In this way we define a bi-invariant affine structure on G.

The semiassociative property.

5.15 We seek the converse construction, namely to associate to an bi-invariant affine structure on a Lie group G an associative algebra. This can be accomplished neatly as follows. Let $\mathfrak{g}$ be the Lie algebra of left-invariant vector fields on G and let ∇ be the flat torsionfree affine connection on G corresponding to a left-invariant affine structure. Since the connection is left-invariant, for any two left-invariant vector fields $X, Y \in \mathfrak{g}$, the covariant derivative $\nabla_X Y \in \mathfrak{g}$ is left-invariant. It follows that covariant differentiation

$$(X, Y) \mapsto \nabla_X Y$$

defines a bilinear multiplication $\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ which we denote $(X, Y) \mapsto XY$. Now the condition that ∇ has zero torsion is

$$XY - YX = [X, Y] \tag{A3}$$

and the condition that ∇ has zero curvature (using (A3)) is

$$X(YZ) - Y(XZ) = (XY - YX)Z$$

which is equivalent to the semi-associative property

$$(XY)Z - X(YZ) = (YX)Z - Y(XZ). \tag{A4}$$

Now suppose that ∇ is bi-invariant. Thus the right-multiplications are affine maps; it follows that the infinitesimal right-multiplications — the left-invariant vector fields — are affine vector fields. For a flat torsionfree affine connection a vector field Z is affine $\Longleftrightarrow$ the second covariant differential $\nabla\nabla Z$ vanishes. Now $\nabla\nabla Z$ is the tensor field which associates to a pair of vector fields X, Y the vector field

$$\nabla\nabla Z(X, Y) = \nabla_X(\nabla Z(Y)) - \nabla Z(\nabla_X Y) = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z$$

and if $X, Y, Z \in \mathfrak{g}$ we obtain the associative law $X(YZ) - (XY)Z = 0$. One can check that these two constructions

$$\{\text{Associative algebras}\} \Longleftrightarrow \{\text{Bi-invariant affine structures on Lie groups}\}$$

are mutually inverse.

A (not necessarily associative) algebra whose multiplication satisfies (A2) is said to be a *left-symmetric algebra*, (algèbre symétrique à gauche) or a *Koszul-Vinberg algebra*. We propose the name "semi-associative algebra." Of course every associative algebra satisfies this property. If $\mathfrak{a}$ is a semi-associative algebra, then the operation

$$[X, Y] = XY - YX \tag{A5}$$

is skew-symmetric and satisfies the Jacobi identity. Thus every semi-associative algebra has an underlying Lie algebra structure. We denote this Lie algebra by $\mathfrak{g}$. If $\mathfrak{g}$ is a Lie algebra, then a semi-associative operation satisfying (2) will be called an *affine structure* on $\mathfrak{g}$.

Let $L : \mathfrak{a} \longrightarrow \operatorname{End} \mathfrak{a}$ be the operation of left-multiplication defined by

$$L(X) : Y \mapsto XY$$

In terms of left-multiplication and the commutator operation defined in (2), a condition equivalent to (1) is

$$L([X,Y]) = [L(X), L(Y)] \tag{A6}$$

i.e. that $L : \mathfrak{g} \longrightarrow \operatorname{End}(\mathfrak{a})$ is a Lie algebra homomorphism. We denote by $\mathfrak{a}_L$ the corresponding $\mathfrak{g}$-module. Furthermore the identity map $I : \mathfrak{g} \longrightarrow \mathfrak{a}_L$ defines a cocycle of the Lie algebra $\mathfrak{g}$ with coefficients in the $\mathfrak{g}$-module $\mathfrak{a}_L$:

$$L(X)Y - L(Y)X = [X,Y] \tag{A7}$$

Let E denote an affine space with associated vector space $\mathfrak{a}$; then it follows from (3) and (4) that the map $\alpha : \mathfrak{g} \longrightarrow \mathfrak{aff}(E)$ defined by

$$\alpha(X) : Y \mapsto L(X)Y + X \tag{A8}$$

is a Lie algebra homomorphism.

5.16 THEOREM. *There is an isomorphism between the categories of semi-associative algebras and simply connected Lie groups with left-invariant affine structure. Under this isomorphism the associative algebras correspond to bi-invariant affine structures.*

There is a large literature on semi-associative algebras; we refer to Helmstetter [He], Auslander [A1], Boyom [], Kim [], Medina [], Koszul [], Vey [V1], Vinberg [Vb] and the references cited there for more information.

5.17 One can translate geometric properties of a left-invariant affine structure on a Lie group G into algebraic properties of the corresponding semi-associative algebra $\mathfrak{a}$. For example, the following theorem is proved in Helmstetter [He] and indicates a kind of infinitesimal version of Markus' conjecture relating geodesic completeness to parallel volume. For more discussion of this result and proofs, see Helmstetter [He] and also Goldman-Hirsch [GH4].

THEOREM. *Let G be a simply connected Lie group with left-invariant affine structure. Let $\alpha : G \longrightarrow \operatorname{Aff}(E)$ be the corresponding locally simply transitive affine action and $\mathfrak{a}$ the corresponding semi-associative algebra. Then the following conditions are equivalent:*

(1) *G is a complete affine manifold;*
(2) *α is simply transitive;*
(3) *A volume form on G is parallel $\Longleftrightarrow$ it is right-invariant;*
(4) *For each $g \in G$, $\det \mathbf{L}\alpha(g) = \det \operatorname{Ad}(g)^{-1}$, i.e. the distortion of parallel volume by α equals the modular function of G;*
(5) *The subalgebra of $\operatorname{End}(\mathfrak{a})$ generated by right-multiplications $R_a : x \mapsto xa$ is nilpotent.*

In a different direction, we may say that a left-invariant affine structure is *radiant* $\iff$ the affine representation α corresponding to left-multiplication has a fixed point, i.e. is conjugate to a representation $G \longrightarrow \mathrm{GL}(E)$. Equivalently, $\alpha(G)$ preserves a radiant vector field on E. A left-invariant affine structure on G is radiant $\iff$ the corresponding semi-associative algebra has a right-identity, i.e. an element $e \in \mathfrak{a}$ satisfying $ae = a$ for all $a \in \mathfrak{a}$.

Since the affine representation $\alpha : G \longrightarrow \mathrm{Aff}(E)$ corresponds to left-multiplication, the associated Lie algebra representation $\alpha : \mathfrak{g} \longrightarrow \mathfrak{aff}(E)$ maps $\mathfrak{g}$ into affine vector fields which correspond to the infinitesimal generators of left-multiplications, i.e. to *right-invariant vector fields.*. Thus with respect to a left-invariant affine structure on a Lie group G, the right-invariant vector fields are affine. Let $X_1, \dots, X_n$ be a basis for the right-invariant vector fields; it follows that the exterior product

$$\alpha(X_1) \wedge \cdots \wedge \alpha(X_n) = f(x) dx^1 \wedge \cdots \wedge dx^n$$

for a polynomial $f \in \mathbf{R}[x^1, \dots, x^n]$, called the *characteristic polynomial* of the left-invariant affine structure. In terms of the algebra $\mathfrak{a}$, we have

$$f(X) = \det(R_{X \oplus 1})$$

where $R_{X\oplus 1}$ denotes right-multiplication by $X \oplus 1$. In [He] and [GH4] it is shown that the developing map is a covering map of G onto a connected component of the set where $f(x) \neq 0$. In particular the nonvanishing of f is equivalent to completeness of the affine structure.

2-dimensional commutative associative algebras.

5.17 One obtains many examples of affine structures on closed 2-manifolds from commutative associative algebras as follows. Let $\mathfrak{a}$ be such an algebra and let $\Lambda \subset \mathfrak{a}$ be a lattice. Then the universal covering group G of the group of invertible elements $a \oplus 1 \in \mathfrak{a} \oplus \mathbf{R}1$ acts locally simply transitively and affinely on the affine space $E = \mathfrak{a} \oplus \{1\}$. The Lie algebra of G is naturally identified with the algebra $\mathfrak{a}$ and there is an exponential map $\exp : \mathfrak{a} \longrightarrow G$ defined by the usual power series (in $\mathfrak{a}$). The corresponding evaluation map at 1 defines a developing map for an invariant affine structure on the vector group $\mathfrak{a}$ and thus the quotient $\mathfrak{a}/\Lambda$ is a torus with an invariant affine structure. Some of these affine structures we have seen previously in other contexts. It is a simple algebraic exercise to classify 2-dimensional commutative associative algebras:

(1) $\mathfrak{a} = \mathbf{R}[x, y]$ where $x^2 = y^2 = xy = 0$. The corresponding affine representation is the action of $\mathbf{R}^2$ on the plane by translation and the corresponding affine structures on the torus are the Euclidean structure.

(2) $\mathfrak{a} = \mathbf{R}[x]$ where $x^3 = 0$. The corresponding affine representation is the simply transitive action discussed in 4.14. The corresponding affine structures are complete but non-Riemannian.

(3) $\mathfrak{a} = \mathbf{R}[x, y]$ where $x^2 = xy = 0$ and $y^2 = y$. The algebra $\mathfrak{a}$ is the product of two 1-dimensional algebras, one corresponding to the complete structure

and the other corresponding to the radiant structure. For various choices of Λ one obtains parallel suspensions of Hopf circles. In these cases the developing image is a half-plane.

(4) $\mathfrak{a} = \mathbf{R}[x, y]$ where $x^2 = 0$ and $xy = x$, $y^2 = y$. Since y is an identity element, the corresponding affine structure is radiant. For various choices of Λ one obtains radiant suspensions of the complete affine 1-manifold $\mathbf{R}/\mathbf{Z}$. The developing image is a half-plane.

(5) $\mathfrak{a} = \mathbf{R}[x, y]$ where $x^2 = x$, $y^2 = y$ and $xy = 0$. This algebra is the product of two algebras corresponding to radiant structures; this structure is radiant since $x + y$ is an identity element. Radiant suspensions of Hopf circles are examples of affine manifolds constructed in this way. The developing image is a quadrant in $\mathbf{R}^2$.

(6) $\mathfrak{a} = \mathbf{R}[x, y]$ where $x^2 = -y^2 = x$ and $xy = y$. In this case $\mathfrak{a} \cong \mathbf{C}$ and we obtain the complex affine 1-manifolds, in particular the Hopf manifolds are all obtained from this algebra. Clearly x is the identity and these structures are all radiant. The developing image is the complement of a point in the plane.

Convex affine and projective structures

These are the lecture notes from the last two weeks of the spring semester 1988. Most of this material is taken from Benzecri [B2] and Vinberg [Vb], although some of the proofs of Benzecri's results are simplified. The main goal was a description of which kinds of convex sets arise as covering spaces of compact manifolds with real projective structures. In dimension two, the universal covering space of a closed surface of negative Euler characteristic with a convex projective structure is bounded by either a conic or a C^1 convex curve which is nowhere twice differentiable. This statement is given in C.18. I am grateful to William Thurston for explaining to me in 1981 why the boundary of such a domain is differentiable and to Sid Frankel, Karsten Grove, John Millson and Ser Peow Tan for several clarifying conversations on the proofs given here.

The geometry of convex cones in affine space

C.1 Let V be a real vector space of dimension n. A *convex cone* in V is a subset $\Omega \subset V$ such that if $t_1, t_2 > 0$ and $x_1, x_2 \in \Omega$, then $t_1x_1 + t_2x_2 \in \Omega$. A convex cone Ω is *sharp* if it contains no complete affine line.

Lemma. *Let $\Omega \subset V$ be an open convex cone in a vector space. Then there exists a unique linear subspace $W \subset V$ such that:*

(1) *Ω is invariant under translation by vectors in W (i.e. Ω is W-invariant;)*

(2) *There exists a sharp convex cone $\Omega_0 \subset V/W$ such that $\Omega = \pi_W^{-1}(\Omega_0)$ where $\pi_W : V \longrightarrow V/W$ denotes linear projection with kernel W.*

PROOF: Let

$$W = \{w \in V \mid x + tw \in \Omega, \forall x \in \Omega, t \in \mathbf{R}\}.$$

Then W is a linear subspace of V and Ω is W-invariant. Let $\Omega_0 = \pi_W(\Omega) \subset V/W$; then $\Omega = \pi_W^{-1}(\Omega_0)$. It remains to show that Ω_0 is sharp and to this end we can immediately reduce to the case $W = 0$. Suppose that Ω contains a complete affine line $\{y + tw \mid t \in \mathbf{R}\}$ where $y \in \Omega$ and $w \in V$. Then for each $s, t \in \mathbf{R}$

$$x_{s,t} = \frac{s}{s+1}x + \frac{1}{s+1}(y + stw) \in \Omega$$

whence

$$\lim_{s \to \infty} x_{s,t} = x + tw \in \bar{\Omega}.$$

Thus $x + tw \in \bar{\Omega}$ for all $t \in \mathbf{R}$. Since $x \in \Omega$ and Ω is open and convex, it follows that $x + tw \in \Omega$ for all $t \in \mathbf{R}$ and $w \in W$ as claimed. ∎

C.2 Suppose that $\Omega \subset V$ is a sharp convex cone. Its *dual cone* is defined to be the set

$$\Omega^* = \{\psi \in V^* \mid \psi(x) > 0, \forall x \in \bar{\Omega}\}$$

where V^* is the vector space dual to V.

LEMMA. *Let $\Omega \subset V$ be a sharp convex cone. Then its dual cone Ω^* is a sharp convex cone.*

PROOF: Clearly Ω^* is a convex cone. We must show that Ω^* is sharp and open. Suppose first that Ω^* contains a line; then there exists $\psi_0, \lambda \in V^*$ such that $\lambda \neq 0$ and $\psi_0 + t\lambda \in \Omega^*$ for all $t \in \mathbf{R}$, i.e. for each $x \in \Omega$,

$$\psi_0(x) + t\lambda(x) > 0$$

for each $t \in \mathbf{R}$. Let $x \in \Omega$; then necessarily $\lambda(x) = 0$. For if $\lambda(x) \neq 0$, there exists $t \in \mathbf{R}$ with $\psi_0(x) + t\lambda(x) \leq 0$, a contradiction. Thus Ω^* is sharp. The openness of Ω^* follows from the sharpness of Ω. Since Ω is sharp, its projectivization $\mathbf{P}(\Omega)$ is a properly convex domain; in particular its closure lies in an open ball in an affine subspace E of $\mathbf{P}$ and thus the set of hyperplanes in $\mathbf{P}$ disjoint from $\mathbf{P}(\Omega)$ is open. It follows that $\mathbf{P}(\Omega^*)$, and hence Ω^*, is open. ∎

C.3. LEMMA. *The canonical isomorphism $V \longrightarrow V^{**}$ maps Ω onto Ω^{**}.*

PROOF: We shall identify V^{**} with V; then clearly $\Omega \subset \Omega^{**}$. Since both Ω and and Ω^{**} are open convex cones, either $\Omega = \Omega^{**}$ or there exists $y \in \partial\Omega \cap \Omega^{**}$. Let $H \subset V$ be a supporting hyperplane for Ω at y. Then the linear functional $\psi \in V^*$ defining H satisfies $\psi(y) = 0$ and $\psi(x) > 0$ for all $x \in \Omega$. Thus $\psi \in \Omega^*$. But $y \in \Omega^{**}$ implies that $\psi(y) > 0$, a contradiction. ∎

C.4. THEOREM. *Let $\Omega \subset V$ be a sharp convex cone. Then there exists a real analytic* $\mathrm{Aff}(\Omega)$*-invariant closed 1-form α on Ω such that its covariant differential $\nabla\alpha$ is an* $\mathrm{Aff}(\Omega)$*-invariant Riemannian metric on Ω. Furthermore $\alpha(\rho_V) = -n < 0$ where ρ_V is the radiant vector field on V.*

Let $d\psi$ denote a parallel volume form on V^*. The *characteristic function* $f : \Omega \longrightarrow \mathbf{R}$ of the sharp convex cone Ω is defined by the integral

$$f(x) = \int_{\Omega^*} e^{-\psi(x)} d\psi \tag{C-1}$$

for $x \in \Omega$. This function will define a canonical Riemannian geometry on Ω which is invariant under the automorphism group $\mathrm{Aff}(\Omega)$ as well as a canonical diffeomorphism $\Omega \longrightarrow \Omega^*$. (Note that replacing the parallel volume form $d\psi$ by another one $c d\psi$ changes replaces the characteristic function f by a constant multiple cf. Thus $f : \Omega \longrightarrow \mathbf{R}$ is well-defined only up to scaling.) For example in the one-dimensional case, where $\Omega = \mathbf{R}_+ \subset V = \mathbf{R}$ we have $\Omega^* = \mathbf{R}_+$ and

$$f(x) = \int_0^\infty e^{-\psi x} d\psi = \frac{1}{x}.$$

We begin by showing the integral (C-1) converges for $x \in \Omega$. For $x \in V$ and $t \in \mathbf{R}$ consider the hyperplane cross-section

$$V_x^*(t) = \{\psi \in V^* \mid \psi(x) = t\}$$

and let

$$\Omega_x^*(t) = \Omega^* \cap V_x^*(t).$$

For each $x \in \Omega$ we obtain a decomposition

$$\Omega^* = \bigcup_{t>0} \Omega_x^*(t)$$

and for each $s > 0$ there is a diffeomorphism

$$h_s : \Omega_x^*(t) \longrightarrow \Omega_x^*(st)$$
$$h_s(\psi) = s\psi.$$

We decompose the volume form $d\psi$ on Ω^* as

$$d\psi = d\psi_t \wedge dt$$

where $d\psi_t$ is an $(n-1)$-form on $V_x^*(t)$. Now the volume form $(h_s)^* d\psi_{st}$ on $\Omega_x^*(t)$ is a parallel translate of $t^{n-1} d\psi_t$. Thus

$$\begin{aligned} f(x) &= \int_0^\infty \left(e^{-t} \int_{\Omega_x^*(t)} d\psi_t \right) dt \\ &= \int_0^\infty e^{-t} t^{n-1} \left(\int_{\Omega_x^*(1)} d\psi_1 \right) dt \\ &= (n-1)!\ \mathrm{area}(\Omega_x^*(1)) < \infty \end{aligned}$$

since $\Omega_x^*(1)$ is a bounded subset of $V_x^*(1)$. Since $\mathrm{area}(\Omega_x^*(n)) = n^{n-1}\,\mathrm{area}(\Omega_x^*(1))$, we have

$$f(x) = \frac{n!}{n^n}\,\mathrm{area}(\Omega_x^*(n)) \tag{C-2}$$

Let $\Omega_{\mathbf{C}}$ denote the *tube domain* $\Omega + \sqrt{-1}\,V \subset V \otimes \mathbf{C}$. Then the integral defining $f(x)$ converges absolutely for $x \in \Omega_{\mathbf{C}}$. It follows that $f : \Omega \longrightarrow \mathbf{R}$ extends to a holomorphic function $\Omega_{\mathbf{C}} \longrightarrow \mathbf{C}$ from which it follows that f is real analytic on Ω.

C.5 LEMMA. *The function $f(x) \longrightarrow +\infty$ as $x \longrightarrow \partial\Omega$.*

PROOF: Consider a sequence $\{x_n\}_{n>0}$ in Ω converging to $x_\infty \in \partial\Omega$. Then the functions

$$\begin{aligned} F_k : \Omega^* &\longrightarrow \mathbf{R} \\ \psi &\mapsto e^{-\psi(x_k)} \end{aligned}$$

are nonnegative functions converging uniformly to F_∞ on every compact subset of Ω^* so that

$$\liminf f(x_k) = \liminf \int_{\Omega^*} F_k(\psi) d\psi \geq \int_{\Omega^*} F_\infty(\psi) d\psi.$$

Suppose that $\psi_0 \in V^*$ defines a supporting hyperplane to Ω at x_∞; then $\psi_0(x_\infty) = 0$. Let $K \subset \Omega^*$ be a closed ball; then $K + \mathbf{R}_+\psi_0$ is a cylinder in Ω^* with cross-section $K_1 = K \cap \psi_0^{-1}(c)$ for some $c > 0$.

$$\begin{aligned} \int_{\Omega^*} F_\infty(\psi) d\psi &\geq \int_{K+\mathbf{R}_+\psi_0} e^{-\psi(x_\infty)} d\psi \\ &\geq \int_{K_1} \Big(\int_0^\infty dt\Big) e^{-\psi(x_\infty)}\, d\psi_1 = \infty \end{aligned}$$

where $d\psi_1$ is a volume form on K_1. ∎

C.6 LEMMA. *If $\gamma \in \mathrm{Aff}(\Omega) \subset \mathrm{GL}(V)$ is an automorphism of Ω, then*

$$f \circ \gamma = \det(\gamma)^{-1} \cdot f. \tag{C-3}$$

In other words, if dx is a parallel volume form on E, then $f(x)\,dx$ defines an $\mathrm{Aff}(\Omega)$-invariant volume form on Ω.

PROOF:

$$\begin{aligned} f(\gamma x) &= \int_{\Omega^*} e^{-\psi(\gamma x)} d\psi \\ &= \int_{\gamma^{-1}\Omega^*} e^{-\psi(x)} \gamma^* d\psi \\ &= \int_{\Omega^*} e^{-\psi(x)} (\det \gamma)^{-1} d\psi \\ &= (\det \gamma)^{-1} f(x) \end{aligned}$$ ∎

C.7 Since $\det(\gamma)$ is a constant, it follows from (C-3) that $\log f$ transforms under γ by the additive constant $\log\det(\gamma)^{-1}$ and thus

$$\alpha = d\log f = f^{-1}df$$

is an $\mathrm{Aff}(\Omega)$-invariant closed 1-form on Ω. Furthermore taking γ to be the homothety $h_s : x \mapsto sx$ we see that $f \circ h_s = s^{-n} \cdot f$ and by differentiating we have

$$\alpha(\rho_V) = -n.$$

Let $X \in T_x\Omega \cong V$ be a tangent vector; then $df(x) \in T_x^*\Omega$ maps

$$X \mapsto -\int_{\Omega^*} \psi(X) e^{-\psi(x)} d\psi.$$

Using the identification $T_x^*\Omega \cong V^*$ we obtain a map

$$\begin{aligned} \Phi : \Omega &\longrightarrow V^* \\ x &\mapsto -d\log f(x). \end{aligned}$$

As a linear functional, $\Phi(x)$ maps $X \in V$ to

$$\frac{\int_{\Omega^*} \psi(X) e^{-\psi(x)} d\psi}{\int_{\Omega^*} e^{-\psi(x)} d\psi}$$

so if $X \in \Omega$, the numerator is positive and $\Phi(x) > 0$ on Ω. Thus $\Phi : \Omega \longrightarrow \Omega^*$. Furthermore by decomposing the volume form on Ω^* we obtain

$$\begin{aligned} \Phi(x) &= \frac{\int_0^\infty e^{-t} t^n \left(\int_{\Omega_x^*(1)} \psi_1 d\psi_1 \right) dt}{\int_0^\infty e^{-t} t^{n-1} \left(\int_{\Omega_x^*(1)} d\psi_1 \right) dt} \\ &= n \frac{\int_{\Omega_x^*(1)} \psi_1 d\psi_1 dt}{\int_{\Omega_x^*(1)} d\psi_1 dt} \\ &= n \,\mathrm{centroid}(\Omega_x^*(1)). \end{aligned}$$

Since

$$\Phi(x) \in n \cdot \Omega_x^*(1) = \Omega_x^*(n),$$

i.e. $\Phi(x) : x \mapsto n$, we have

$$\Phi(x) = \mathrm{centroid}(\Omega_x^*(n)). \tag{C-4}$$

C.8 The logarithmic Hessian $d^2 \log f = \nabla d \log f = \nabla \alpha$ is an $\mathrm{Aff}(\Omega)$-invariant symmetric 2-form on Ω. Now for any function $f : \Omega \longrightarrow \mathbf{R}$ we have

$$d^2(\log f) = \nabla(f^{-1}df) = f^{-1}d^2 f - (f^{-1}df)^2$$

and $d^2 f(x) \in S^2 T_x^* \Omega$ assigns to a pair $(X, Y) \in T_x\Omega \times T_x\Omega = V \times V$

$$\int_{\Omega^*} \psi(X)\psi(Y)e^{-\psi(x)}d\psi$$

We claim that $d^2 \log f$ is positive definite:

$$\begin{aligned} f(x)^2 d^2 \log f(x)(X,X) &= \int_{\Omega^*} e^{-\psi(x)}d\psi \int_{\Omega^*} \psi(X)^2 e^{-\psi(x)}d\psi \\ &\qquad - \Big(\int_{\Omega^*} \psi(X)e^{-\psi(x)}d\psi\Big)^2 \\ &= \|e^{-\psi(x)/2}\|_2^2 \, \|\psi(X)e^{-\psi(x)/2}\|_2^2 \\ &\qquad - \langle e^{-\psi(x)/2}, \psi(X)e^{-\psi(x)/2}\rangle_2^2 > 0 \end{aligned}$$

by the Schwartz inequality, since the functions

$$\psi \mapsto e^{-\psi(x)/2}, \qquad \psi \mapsto \psi(X)e^{-\psi(x)/2}$$

on Ω^* are not proportional. (Here $\langle , \rangle_2$ and $\| \, \|_2$ respectively denote the usual L^2 inner product and norm on $(\Omega^*, d\psi)$.) It follows that $d^2 \log f$ is positive definite and hence defines an $\mathrm{Aff}(\Omega)$-invariant Riemannian metric on Ω.

C.9 We can characterize the linear functional $\Phi(x) \in \Omega^*$ quite simply. Since $\Phi(x)$ is parallel to $df(x)$, each of its level hyperplanes is parallel to the tangent plane of the level set S_x of $f : \Omega \longrightarrow \mathbf{R}$ containing x. Since $\Phi(x)(x) = n$, we obtain:

Proposition. *The tangent space to the level set S_x of $f : \Omega \longrightarrow \mathbf{R}$ at x equals $\Phi(x)^{-1}(n)$.*

This characterization yields the following result:

C.10 Theorem. *$\Phi : \Omega \longrightarrow \Omega^*$ is bijective.*

Proof: Let $\psi_0 \in \Omega^*$ and let $Q_0 = \{z \in V \mid \psi_0(z) = n\}$. Then the restriction of $\log f$ to the affine hyperplane Q_0 is a convex function which approaches $+\infty$ on $\partial(Q_0 \cap \Omega)$. It follows that $f|_{Q_0\cap\Omega}$ has a unique critical point x_0, which is necessarily a minimum. Then $T_{x_0}S_{x_0} = Q_0$ from which it follows from the above proposition that $\Phi(x_0) = \psi_0$. Furthermore if $\Phi(x) = \psi_0$, then $f|_{Q_0\cap\Omega}$ has a critical point at x so $x = x_0$. It follows that $\Phi : \Omega \longrightarrow \Omega^*$ is bijective as claimed. ∎

If $\Omega \subset V$ is a sharp convex cone and Ω^* is its dual, then let $\Phi_{\Omega^*} : \Omega^* \longrightarrow \Omega$ be the diffeomorphism $\Omega^* \longrightarrow \Omega^{**} = \Omega$ defined above. If $x \in \Omega$, then $\psi = (\Phi^*)^{-1}(x)$ is the unique $\psi \in V^*$ such that:

(1) $\psi(x) = n$;
(2) The centroid of $\Omega \cap \psi^{-1}(n)$ equals x.

C.11 The duality isomorphism $\mathrm{GL}(V) \longrightarrow \mathrm{GL}(V^*)$ (given by inverse transpose of matrices) defines an isomorphism $\mathrm{Aff}(\Omega) \longrightarrow \mathrm{Aff}(\Omega^*)$. Let $\Phi_\Omega : \Omega \longrightarrow \Omega^*$ and $\Phi_{\Omega^*} : \Omega^* \longrightarrow \Omega^{**} = \Omega$ be the duality maps for Ω and Ω^* respectively. Vinberg points out in [Vb] that in general the composition $\Phi_{\Omega^*} \circ \Phi_\Omega : \Omega \longrightarrow \Omega$ is not the identity, although if Ω is *homogeneous*, i.e. $\mathrm{Aff}(\Omega) \subset \mathrm{GL}(V)$ acts transitively on Ω, then $\Omega_{\Omega^*} \circ \Phi_\Omega = \mathrm{id}_\Omega$:

PROPOSITION (VINBERG [VB]). *Let $\Omega \subset V$ be a homogeneous sharp convex cone. Then $\Phi_{\Omega^*} : \Omega^* \longrightarrow \Omega$ and $\Phi_\Omega : \Omega \longrightarrow \Omega^*$ are inverse maps.*

PROOF: Let $x \in \Omega$ and $Y \in V \cong T_x\Omega$ be a tangent vector. Denote by $g_x : T_x\Omega \times T_x\Omega \longrightarrow \mathbf{R}$ the canonical Riemannian metric $\nabla\alpha = d^2 \log f$ at x. Then the differential of $\Phi_\Omega : \Omega \longrightarrow \Omega^*$ at x is the composition

$$T_x\Omega \xrightarrow{\tilde{g}_x} T_x^*\Omega \cong V^* \cong T_{\Phi(x)}\Omega^*$$

where $\tilde{g}_x : T_x\Omega \longrightarrow T_x^*\Omega$ is the linear isomorphism corresponding to g_x and the isomorphisms $T_x^*\Omega \cong V^* \cong T_{\Phi(x)}\Omega^*$ are induced by parallel translation. Taking the directional derivative of the equation

$$\alpha_x(\rho_x) = -n$$

with respect to $Y \in V \cong T_x\Omega$ we obtain

$$\text{(C-C)} \qquad 0 = (\nabla_Y\alpha)(\rho) + \alpha(\nabla_Y\rho) = g_x(\rho_x, Y) + \alpha_x(Y) = g_x(x, Y) - \Phi(x)(Y).$$

Now let $f_\Omega : \Omega \longrightarrow \mathbf{R}$ and $f_{\Omega^*} : \Omega^* \longrightarrow \mathbf{R}$ be the characteristic functions for Ω and Ω^* respectively. Then $f_\Omega(x)\,dx$ is a volume form on Ω invariant under $\mathrm{Aff}(\Omega)$ and $f_{\Omega^*}(\psi)\,d\psi$ is a volume form on Ω^* invariant under the induced action of $\mathrm{Aff}(\Omega)$ on Ω^*. Moreover $\Phi : \Omega \longrightarrow \Omega^*$ is equivariant with respect to the isomorphism $\mathrm{Aff}(\Omega) \longrightarrow \mathrm{Aff}(\Omega^*)$ and thus the tensor field

$$x \mapsto f_\Omega(x)\,dx \otimes (f_{\Omega^*} \circ \Phi(x)\,d\psi) \in \wedge^n T_x\Omega\omega \otimes \wedge^n T_{\Phi(x)}\Omega^* \cong \wedge^n V \otimes \wedge^n V^*$$

is invariant under $\mathrm{Aff}(\Omega)$. But the tensor field $dx \otimes d\psi \in \wedge^n V \otimes \wedge^n V^*$ is invariant under all of $\mathrm{Aff}(V)$ and thus the coefficient

$$h(x) = f_\Omega(x)\,dx \otimes (f_{\Omega^*} \circ \Phi(x)\,d\psi)$$

is a function on Ω which is invariant under $\mathrm{Aff}(\Omega)$. Since Ω is homogeneous, it follows that h is constant.

Differentiating $\log h$ we obtain

$$0 = d\log f_\Omega(x) + d\log(f_{\Omega^*} \circ \Phi)(x)$$

which, since $d\log f_{\Omega^*}(\psi) = \Phi_{\Omega^*}(\psi)$,

$$0 = -\Phi(x)(Y) + \Phi_{\Omega^*}(d\Phi(Y)) = -\Phi(x)(Y) + g_x(Y, \Phi_{\Omega^*} \circ \Phi_\Omega(x))$$

Combining this equation with (C-C) we obtain

$$\Phi_{\Omega^*} \circ \Phi_\Omega(x) = x$$

as desired. ∎

It follows that if Ω is a homogeneous cone, then $\Phi(x) \in \Omega^*$ is the centroid of the cross-section $\Omega^*_x(n) \subset \Omega^*$ in V^*.

Convex bodies in projective space.

C.12 Let $\mathbf{P} = \mathbf{P}(V)$ and $\mathbf{P}^* = \mathbf{P}(V^*)$ be the associated projective spaces. Then the projectivization $\mathbf{P}(\Omega) \subset \mathbf{P}$ of Ω is by definition a *properly convex domain* and its closure $K = \overline{\mathbf{P}(\Omega)}$ a *convex body*. Then the *dual convex body* K^* equals the closure of the projectivization $\mathbf{P}(\Omega^*)$ consisting of all hyperplanes $H \subset \mathbf{P}$ such that $\bar{\Omega} \cap H = \emptyset$. A *pointed convex body* consists of a pair (K, x) where K is a convex body and $x \in \mathrm{int}(K)$ is an interior point of K. Let $H \subset \mathbf{P}$ be a hyperplane and $E = \mathbf{P} - H$ its complementary affine space. We say that the pointed convex body (K, u) is *centered relative to* E (or H) if u is the centroid of K in the affine geometry of E. By projectivization we obtain from Theorem C.10:

PROPOSITION. *Let (K, u) be a pointed convex body in a projective space $\mathbf{P}$. Then there exists a hyperplane $H \subset \mathbf{P}$ disjoint from K such that in the affine space $E = \mathbf{P} - H$, the centroid of $K \subset E$ equals u.*

PROOF: Let $V = V(\mathbf{P})$ be the vector space corresponding to the projective space $\mathbf{P}$ and let $\Omega \subset V$ be a sharp convex cone whose projectivization is the interior of K. Let $x \in \Omega$ be a point corresponding to $u \in \mathrm{int}(K)$. Let $\Phi_{\Omega^*} : \Omega^* \longrightarrow \Omega$ be the duality map for Ω^* and let $\psi = (\Phi_{\Omega^*})^{-1}(y)$. Then the centroid of the cross-section

$$\Omega_\psi(n) = \{x \in \Omega \mid \psi(x) = n\}$$

in the affine hyperplane $\psi^{-1}(n) \subset V$ equals y. Let $H = \mathbf{P}(\mathrm{Ker}(\psi))$ be the projective hyperplane in $\mathbf{P}$ corresponding to ψ; then projectivization defines an affine isomorphism $\psi^{-1}(n) \longrightarrow \mathbf{P} - H$ mapping $\Omega_\psi(n) \longrightarrow K$. Since affine maps preserve centroids, it follows that (K, u) is centered relative to H. ∎

Thus every pointed convex body (K, u) is centered relative to a unique affine space containing K. In dimension one, this means the following: let $K \subset \mathbf{RP}^1$ be a closed interval $[a, b] \subset \mathbf{R}$ and let $a < x < b$ be an interior point. Then x is the midpoint of $[a, b]$ relative to the "hyperplane" H obtained by projectively reflecting x with respect to the pair $\{a, b\}$:

$$H = R_{[a,b]}(x) = \frac{(a+b)x - 2ab}{2x - (a+b)}$$

C.13 An equivalent version of C.12 involves using collineations to "move a pointed convex body" inside affine space to center it:

PROPOSITION. *Let $K \subset E$ be a convex body in an affine space and let $x \in \mathrm{int}(E)$ be an interior point. Let $\mathbf{P} \supset E$ be the projective space containing E. Then there exists a collineation $g : \mathbf{P} \longrightarrow \mathbf{P}$ such that:*

(1) *$g(K) \subset E$;*
(2) *$(g(K), g(x))$ is centered relative to E.*

The one-dimensional version of this is just the fundamental theorem of projective geometry: if $[a,b]$ is a closed interval with interior point x, then there is a unique collineation taking

$$a \mapsto -1, \qquad x \mapsto 0, \qquad b \mapsto 1$$

thereby centering $([a,b],x) \in \mathfrak{C}_*(\mathbf{P})$.

C.14 We also have the following uniqueness statement:

PROPOSITION. *Let $K_i \subset E$ be convex bodies ($i = 1,2$) in an affine space E with centroids u_i, and suppose that $g : \mathbf{P} \longrightarrow \mathbf{P}$ is a collineation such that $g(K_1) = K_2$ and $g(u_1) = u_2$. Then g is an affine automorphism of E, i.e. $g(E) = E$.*

PROOF: Let V be a vector space containing E as an affine hyperplane and let Ω_i be the sharp convex cones in V whose projective images are K_i. By assumption there exists a linear map $\tilde{g} : V \longrightarrow V$ and points $x_i \in \Omega_i$ mapping to $u_i \in K_i$ such that $\tilde{g}(\Omega_1) = \Omega_2$ and $\tilde{g}(x_1) = x_2$. Let $S_i \subset \Omega_i$ be the level set of the characteristic function $f_i : \Omega_i \longrightarrow \mathbf{R}$ containing x_i. Since (K_i, u_i) is centered relative to E, it follows that the tangent plane $T_{x_i} S_i = E \subset V$. Since the construction of the characteristic function is linearly invariant, it follows that $\tilde{g}(S_1) = S_2$. Moreover $\tilde{g}(T_{x_1} S_1) = T_{x_2} S_2$, i.e. $\tilde{g}(E) = E$ and $g \in \mathrm{Aff}(E)$ as desired. ∎

Spaces of convex bodies in projective space.

C.15 Let $\mathfrak{C}(\mathbf{P})$ denote the set of all convex bodies in $\mathbf{P}$, with the topology induced from the Hausdorff metric on the set of all closed subsets of $\mathbf{P}$ (which is induced from the Fubini-Study metric on $\mathbf{P}$). Let

$$\mathfrak{C}_*(\mathbf{P}) = \{(K,x) \in \mathfrak{C}(\mathbf{P}) \times \mathbf{P} \mid x \in \mathrm{int}(K)\}$$

be the corresponding set of pointed convex bodies, with a topology induced from the product topology on $\mathfrak{C}(\mathbf{P}) \times \mathbf{P}$. The collineation group G acts continuously on $\mathfrak{C}(\mathbf{P})$ and on $\mathfrak{C}_*(\mathbf{P})$. Recall that an action of a group Γ on a space X is *syndetic* if there exists a compact subset $K \subset X$ such that $\Gamma K = X$.

THEOREM (BENZÉCRI). *The collineation group G acts properly and syndetically on $\mathfrak{C}_*(\mathbf{P})$. quotient. In particular the quotient $\mathfrak{C}_*(\mathbf{P})/G$ is a compact Hausdorff space.*

C.16 While the quotient $\mathfrak{C}_*(\mathbf{P})/G$ is Hausdorff, the space of equivalence classes of convex bodies $\mathfrak{C}(\mathbf{P})/G$ is generally *not* Hausdorff. Some basic examples are the following. Suppose that Ω is a properly convex domain whose boundary is not C^1 at a point x_1. Then $\partial\Omega$ has a "corner" at x_1 and we may choose homogeneous coordinates so that $x_1 = [1,0,0]$ and $\bar{\Omega}$ lies in the domain

$$\triangle = \{[x,y,z] \in \mathbf{RP}^2 \mid x,y,z > 0\}$$

in such a way that $\partial\Omega$ is tangent to $\partial\triangle$ at x_1. Under the one-parameter group of collineations defined by

$$g_t = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^t \end{bmatrix}$$

as $t \longrightarrow +\infty$, the domains $g_t\Omega$ converge to $\triangle$. It follows that the G-orbit of $\bar{\Omega}$ in $\mathfrak{C}(\mathbf{P})$ is not closed and the equivalence class of $\bar{\Omega}$ is not a closed point in $\mathfrak{C}(\mathbf{P})/G$ unless Ω was already a triangle.

Similarly suppose that Ω is a properly convex domain which is not *strictly convex,* i.e. its boundary contains a nontrivial line segment σ. (We suppose that σ is a maximal line segment contained in $\partial\Omega$.) As above, we may choose homogeneous coordinates so that $\Omega \subset \triangle$ and such that $\bar{\Omega} \cap \bar{\triangle} = \bar{\sigma}$ and σ lies on the line $\{[x, y, 0] \mid x, y \in \mathbf{R}\}$. As $t \longrightarrow +\infty$ the image of Ω under the collineation

$$g_t = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}$$

converges to a triangle region with vertices $\{[0, 0, 1]\} \cup \partial\sigma$. As above, the equivalence class of $\bar{\Omega}$ in $\mathfrak{C}(\mathbf{P})/G$ is not a closed point in $\mathfrak{C}(\mathbf{P})/G$ unless Ω is a triangle.

As a final example, consider a properly convex domain Ω with C^1 boundary such that there exists a point $u \in \partial\Omega$ such that $\partial\Omega$ is C^2 at u. In that case there is an osculating conic C to $\partial\Omega$ at u. Choose homogeneous coordinates such that $u = [1, 0, 0]$ and $C = \{[x, y, z] \mid xy + z^2 = 0\}$. Then as $t \longrightarrow +\infty$ the image of Ω under the collineation

$$g_t = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

converges to the convex region $\{[x, y, z] \mid xy + z^2 < 0\}$ bounded by C. As above, the equivalence class of $\bar{\Omega}$ in $\mathfrak{C}(\mathbf{P})/G$ is not a closed point in $\mathfrak{C}(\mathbf{P})/G$ unless $\partial\Omega$ is a conic.

In summary:

PROPOSITION. *Suppose $\bar{\Omega} \subset \mathbf{RP}^2$ is a convex body whose equivalence class $[\bar{\Omega}]$ is a closed point in $\mathfrak{C}(\mathbf{P})/G$. Suppose that $\partial\Omega$ is neither a triangle nor a conic. Then $\partial\Omega$ is a C^1 strictly convex curve which is nowhere C^2.*

C.17 Let $\Pi : \mathfrak{C}_*(\mathbf{P}) \longrightarrow \mathfrak{C}(\mathbf{P})$ denote the map which forgets the point of a pointed convex body; it is induced from the Cartesian projection $\mathfrak{C}(\mathbf{P}) \times \mathbf{P} \longrightarrow \mathfrak{C}(\mathbf{P})$.

THEOREM (BENZÉCRI). *Let $\Omega \subset \mathbf{P}$ is a properly convex domain such that there exists a subgroup $\Gamma \subset \mathrm{Aut}(\Omega)$ which acts syndetically on Ω. Then the corresponding point $[\bar{\Omega}] \in \mathfrak{C}(\mathbf{P})/G$ is closed.*

In the following result, all but the continuous differentiability of the boundary in the following result was originally proved in Kuiper [Kp2] using a somewhat different technique; the C^1 statement is due to Benzécri [B2] as well as the proof given here.

C.18 COROLLARY. *Suppose that $M = \Omega/\Gamma$ is a convex $\mathbf{RP}^2$-manifold such that $\chi(M) < 0$. Then either the $\mathbf{RP}^2$-structure on M is a hyperbolic structure or the boundary $\partial\Omega$ of its universal covering is a C^1 strictly convex curve which is nowhere C^2.*

PROOF: Apply Proposition C.16 to Theorem C.17. ∎

PROOF OF THEOREM C.17 ASSUMING THEOREM C.15: Let Ω be a properly convex domain with an automorphism group $\Gamma \subset \mathrm{Aff}(\Omega)$ acting syndetically on Ω. It suffices to show that the G-orbit of $\{\bar{\Omega}\}$ in $\mathfrak{C}(\mathbf{P})$ is closed, which is equivalent to showing that the G-orbit of $\Pi^{-1}(\{\bar{\Omega}\}) = \{\bar{\Omega}\} \times \Omega$ in $\mathfrak{C}_*(\mathbf{P})$ is closed. This is equivalent to showing that the image of $\{\bar{\Omega}\} \times \Omega \subset \mathfrak{C}_*(\mathbf{P})$ under the quotient map $\mathfrak{C}_*(\mathbf{P}) \longrightarrow \mathfrak{C}_*(\mathbf{P})/G$ is closed. Let $K \subset \Omega$ be a compact subset such that $\Gamma K = \Omega$; then $\{\bar{\Omega}\} \times K$ and $\{\bar{\Omega}\} \times \Omega$ have the same image in $\mathfrak{C}_*(\mathbf{P})/\Gamma$ and hence in $\mathfrak{C}_*(\mathbf{P})/G$. Hence it suffices to show that the image of $\{\bar{\Omega}\} \times K$ in $\mathfrak{C}_*(\mathbf{P})/G$ is closed. Since K is compact and the composition

$$K \longrightarrow \{\bar{\Omega}\} \times K \hookrightarrow \{\bar{\Omega}\} \times \Omega \subset \mathfrak{C}_*(\mathbf{P}) \longrightarrow \mathfrak{C}_*(\mathbf{P})/G$$

is continuous, it follows that the image of K in $\mathfrak{C}_*(\mathbf{P})/G$ is compact. By Theorem A, $\mathfrak{C}_*(\mathbf{P})/G$ is Hausdorff and hence the image of K in $\mathfrak{C}_*(\mathbf{P})/G$ is closed, as desired. The proof of Theorem C.15 is now complete. ∎

C.19 Now we begin the proof of Theorem C.15. Choose a fixed hyperplane $H_\infty \subset \mathbf{P}$ and let $E = \mathbf{P} - H_\infty$ be the corresponding affine patch and $\mathrm{Aff}(E)$ the group of affine automorphisms of E. Let $\mathfrak{C}(E) \subset \mathfrak{C}(\mathbf{P})$ denote the set of convex bodies $K \subset E$, with the induced topology. (Note that the $\mathfrak{C}(E)$ is a complete metric space with respect to the Hausdorff metric induced from the Euclidean metric on E and we may use this metric to define the topology on $\mathfrak{C}(E)$. The inclusion map $\mathfrak{C}(E) \hookrightarrow \mathfrak{C}(\mathbf{P})$ is continuous, although not uniformly continuous.) We define a map $\iota : \mathfrak{C}(E) \longrightarrow \mathfrak{C}_*(\mathbf{P})$ as follows. Let $K \in \mathfrak{C}(E)$ be a convex body in affine space E; let $\iota(K)$ to be the pointed convex body

$$\iota(K) = (K, \mathrm{centroid}(K)) \in \mathfrak{C}_*(\mathbf{P});$$

clearly this map is equivariant with respect to the embedding $\mathrm{Aff}(E) \longrightarrow G$.

We must relate the actions of $\mathrm{Aff}(E)$ on $\mathfrak{C}(E)$ and G on $\mathfrak{C}(\mathbf{P})$. Recall that a *topological transformation groupoid* consists of a small category $\mathfrak{G}$ whose objects form a topological space X upon which a topological group G acts such that the morphisms $a \to b$ consist of all $g \in G$ such that $g(a) = b$. We write $\mathfrak{G} = (X, G)$. A *homomorphism of topological transformation groupoids* is a functor $(f, F) : (X, G) \longrightarrow (X', G')$ arising from a continuous map $f : X \longrightarrow X'$ which is equivariant with respect to a continuous homomorphism $F : G \longrightarrow G'$.

The space of isomorphism classes of objects in a category $\mathfrak{G}$ will be denoted $\mathbf{Iso}(\mathfrak{G})$. We shall say that $\mathfrak{G}$ is *proper (resp. syndetic)* if the corresponding action of G on X is proper (resp. syndetic). If $\mathfrak{G}$ and $\mathfrak{G}'$ are topological categories, a functor $F : \mathfrak{G} \longrightarrow \mathfrak{G}'$ is an *equivalence of topological categories* if the induced map $\mathbf{Iso}(F) : \mathbf{Iso}(\mathfrak{G}) \longrightarrow \mathbf{Iso}(G')$ is a homeomorphism and F is *fully faithful,* i.e. for each pair of objects a, b of $\mathfrak{G}$, the induced map $F_* : \mathrm{Hom}(a, b) \longrightarrow \mathrm{Hom}(F(a), F(b))$ is a homeomorphism. If F is fully faithful it is enough to prove that $\mathbf{Iso}(F)$ is surjective. (Compare Jacobson [].) We have the following general proposition:

C.20 LEMMA. *Suppose that* $(f,F):(X,G)\longrightarrow(X',G')$ *is a homomorphism of topological transformation groupoids which is an equivalence of groupoids and such that* f *is an open map. If* (X,G) *is proper, so is* (X',G'). *If* (X,G) *is syndetic, so is* (X',G').

PROOF: An equivalence of topological groupoids induces a homeomorphism of quotient spaces

$$X/G\longrightarrow X'/G'$$

so if X'/G' is compact (resp. Hausdorff) so is X/G. Since (X,G) is syndetic if and only if X/G is compact, this proves the assertion about syndeticity. By Koszul [p.3,Remark 2] (X,G) is proper if and only if X/G is Hausdorff and the action (X,G) is *wandering (or locally proper):* each point $x\in X$ has a neighborhood U such that $G(U,U)=\{g\in G\mid g(U)\cap U\neq\emptyset\}$ is precompact. Since (f,F) is fully faithful, F maps $G(U,U)$ isomorphically onto $G'(f(U),f(U))$. Suppose that (X,G) is proper. Then X/G is Hausdorff and so is X'/G'. We claim that G' acts locally properly on X'. Let $x'\in X'$. Then there exists $g'\in G'$ and $x\in X$ such that $g'f(x)=x'$. Since G acts locally properly on X, there exists a neighborhood U of $x\in X$ such that $G(U,U)$ is precompact. It follows that $U'=g'f(U)$ is a neighborhood of $x'\in X'$ such that $G'(U',U')\cong G(U,U)$ is precompact, as claimed. Thus G' acts properly on X'. ∎

C.21 THEOREM. *Let* $E\subset\mathbf{P}$ *be an affine patch in projective space. Then the map*

$$\begin{aligned}\iota:\mathfrak{C}(E)&\longrightarrow\mathfrak{C}_*(\mathbf{P})\\ K&\mapsto(K,\operatorname{centroid}(K))\end{aligned}$$

is equivariant with respect to the inclusion $\operatorname{Aff}(E)\longrightarrow G$ *and the corresponding homomorphism of topological transformation groupoids*

$$\iota:(\mathfrak{C}(E),\operatorname{Aff}(E))\longrightarrow(\mathfrak{C}_*(\mathbf{P}),G)$$

is an equivalence of groupoids.

PROOF: The surjectivity of $\iota_*:\mathfrak{C}(E)/\operatorname{Aff}(E)\longrightarrow\mathfrak{C}_*(\mathbf{P})/G$ follows immediately from C.11 and the bijectivity of $\iota_*:\operatorname{Hom}(a,b)\longrightarrow\operatorname{Hom}(\iota(a),\iota(b))$ follows immediately from C.13. ∎

Thus the proof of C.14 is reduced (via C.20 and C.21) to the following:

C.22 THEOREM. $\operatorname{Aff}(E)$ *acts properly and syndetically on* $\mathfrak{C}(E)$.

Let $\mathbf{Ell}\subset\mathfrak{C}(E)$ denote the subspace of ellipsoids in E; the affine group $\operatorname{Aff}(E)$ acts transitively on $\mathbf{Ell}$ with isotropy group the orthogonal group — in particular this action is proper. If $K\in\mathfrak{C}(E)$ is a convex body, then there exists a unique ellipsoid $\operatorname{ell}(K)\in\mathbf{Ell}$ (*the ellipsoid of inertia of* K such that for each affine map $\psi:E\longrightarrow\mathbf{R}$ such that $\psi(\operatorname{centroid}(K))=0$ the moments of inertia satisfy:

$$\int_K\psi^2\,dx=\int_{\mathbf{ell}(K)}\psi^2\,dx$$

C.23 PROPOSITION. *Taking the ellipsoid-of-inertia of a convex body*

$$\mathbf{ell} : \mathfrak{C}(E) \longrightarrow \mathbf{Ell}$$

defines an $\mathrm{Aff}(E)$*-invariant proper retraction of* $\mathfrak{C}(E)$ *onto* $\mathbf{Ell}$.

PROOF OF C.22 ASSUMING C.23: Since $\mathrm{Aff}(E)$ acts properly and syndetically on $\mathbf{Ell}$ and $\mathbf{ell}$ is a proper map, it follows that $\mathrm{Aff}(E)$ acts properly and syndetically on $\mathfrak{C}(E)$. ∎

PROOF OF C.23: $\mathbf{ell}$ is clearly affinely invariant and continuous. Since $\mathrm{Aff}(E)$ acts transitively on $\mathbf{Ell}$, it suffices to show that a single fiber $\mathbf{ell}^{-1}(e)$ is compact for $e \in \mathbf{Ell}$. We may assume that e is the unit sphere in E centered at the origin 0. Since the collection of compact subsets of E which lie between two compact balls is compact subset of $\mathfrak{C}(E)$, Theorem C.23 will follow from:

C.24 PROPOSITION. *For each* n *there exist constants* $0 < r(n) < R(n)$ *such that every convex body* $K \subset \mathbf{R}^n$ *whose centroid is the origin and whose ellipsoid-of-inertia is the unit sphere satisfies*

$$B_{r(n)}(O) \subset K \subset B_{R(n)}(O).$$

The proof of C.24 is based on:

LEMMA. *Let* $K \subset E$ *be a convex body with centroid* O. *Suppose that* l *is a line through* O *which intersects* ∂K *in the points* X, X'. *Then*

$$\frac{1}{n} \leq \frac{OX}{OX'} \leq n. \tag{C-5}$$

PROOF: Let $\psi \in E^*$ be a linear functional such that $\psi(X) = 0$ and $\psi^{-1}(1)$ is a supporting hyperplane for K at X'; then necessarily $0 \leq \psi(x) \leq 1$ for all $x \in K$. We claim that

$$\psi(O) \leq \frac{n}{n+1}. \tag{C-6}$$

For $t \in \mathbf{R}$ let $h_t : E \longrightarrow E$ be the homothety fixing X having strength t,i.e.

$$h_t(x) = t(x - X) + X.$$

We shall compare the linear functional ψ with the "polar coordinates" on K defined by the map

$$\begin{aligned} F : [0,1] \times \partial K &\longrightarrow K \\ (t, \mathbf{s}) &\mapsto h_t \mathbf{s} \end{aligned}$$

which is bijective on $(0,1] \times \partial K$ and collapses $\{0\} \times \partial K$ onto X. Thus there is a well-defined function $\mathbf{t} : K \longrightarrow \mathbf{R}$ such that for each $x \in K$, there exists $x' \in \partial K$ such that $x = F(\mathbf{t}, x')$. Since $0 \leq \psi(F(t, \mathbf{s})) \leq 1$, it follows that for $x \in K$,

$$0 \leq \psi(x) \leq \mathbf{t}(x)$$

Let $\mu = \mu_K$ denote the probability measure supported on K defined by

$$\mu(S) = \frac{\int_{S\cap K} dx}{\int_K dx}.$$

There exists a measure ν on ∂K such that for each measurable function $f : E \longrightarrow \mathbf{R}$

$$\int f(x) d\mu(x) = \int_{t=0}^{1} \int_{\mathbf{s}\in\partial K} f(t\mathbf{s}) t^{n-1} d\nu(\mathbf{s}) dt,$$

i.e. $F^* d\mu = t^{n-1} d\nu \wedge dt$.

The first moment of $\mathbf{t} : K \longrightarrow [0,1]$ is then given by

$$\bar{\mathbf{t}}(K) = \int_K \mathbf{t}\, d\mu = \frac{\int_K \mathbf{t}\, d\mu}{\int_K d\mu} = \frac{\int_0^1 t^n \int_{\partial K} d\nu\, dt}{\int_0^1 t^{n-1} \int_{\partial K} d\nu\, dt} = \frac{n}{n+1}$$

and since the value of the affine function ψ on the centroid equals the first moment of ψ on K, we have

$$0 < \psi(O) = \int_K \psi(x)\, d\mu(x) < \int_K \mathbf{t}(x)\, d\mu = \frac{n}{n+1}.$$

Now the distance function on the line $\overleftrightarrow{XX'}$ is affinely related to the linear functional ψ, i.e. there exists a constant $c > 0$ such that for $x \in \overleftrightarrow{XX'}$ the distance $Xx = c|\psi(x)|$; since $\psi(X') = 1$ it follows that

$$\psi(x) = \frac{Xx}{XX'}$$

and since $OX + OX' = XX'$ it follows that

$$\frac{OX'}{OX} = \frac{XX'}{OX} - 1 \geq \frac{n+1}{n} - 1 = \frac{1}{n}.$$

This gives the second inequality of (C-5). The first inequality follows by reversing the roles of X, X'. ∎

Proof of C.24: Let $X \in \partial K$ be a point at minimum distance from the centroid O; then there exists a supporting hyperplane H at x which is orthogonal to $\overleftrightarrow{OX}$ and let $\psi : E \longrightarrow \mathbf{R}$ be the corresponding linear functional of unit length. Let $a = \psi(X) > 0$ and $b = \psi(X') < 0$; by C.23 we have $-b < na$.

We claim that $0 < |\psi(x)| < na$ for all $x \in K$. To this end let $x \in K$; we may assume that $\psi(x) > 0$ since $-na < \psi(X') \leq \psi(x)$. Furthermore we may assume that $x \in \partial K$. Let $z \in \partial K$ be the other point of intersection of $\overleftrightarrow{Ox}$ with ∂K; then $\psi(z) < 0$. Now

$$\frac{1}{n} \leq \frac{Oz}{Ox} \leq n$$

implies that

$$\frac{1}{n} \leq \frac{|\psi(z)|}{|\psi(x)|} \leq n$$

(since the linear functional ψ is affinely related to signed distance along $\overleftrightarrow{Ox}$). Since $0 > \psi(z) \geq -a$, it follows that $|\psi(x)| \leq na$ as claimed.

Let w_n denote the moment of inertia of ψ for the unit sphere; then we have

$$w_n = \int_K \psi^2 \, d\mu \leq \int_K n^2 a^2 \, d\mu = n^2 a^2$$

whence $a \geq \sqrt{w_n}/n$. Taking $r(n) = \sqrt{w_n}/n$ we see that K contains the $r(n)$-ball centered at O.

To obtain the upper bound, observe that if C is a right circular cone with vertex X, altitude h and base a sphere of radius ρ and $t : C \longrightarrow [0, h]$ is the altitudinal distance from the base, then the integral

$$\int_C t^2 \, d\mu = \frac{2h^3 \rho^{n-1} v_{n-1}}{(n+2)(n+1)n}$$

where v_{n-1} denotes the $(n-1)$-dimensional volume of the unit $(n-1)$-ball. Let $X \in \partial K$ and C be a right circular cone with vertex X and base an $(n-1)$-dimensional ball of radius $r(n)$. We have just seen that K contains $B_{r(n)}(O)$; it follows that $K \supset C$. Let $t : K \longrightarrow \mathbf{R}$ be the unit-length linear functional vanishing on the base of C; then $t(X) = h = OX$. Its second moment is

$$w_n = \int_K t^2 \, d\mu \geq \int_C t^2 \, d\mu = \frac{2h^3 r(n)^{n-1} v_{n-1}}{(n+2)(n+1)n}$$

and thus it follows that

$$OX = h \leq R(n)$$

where

$$R(n) = \left(\frac{(n+2)(n+1)n w_n}{2r(n)^{n-1} v_{n-1}} \right)^{\frac{1}{3}}$$

as desired. The proof is now complete. ∎

The volume of the unit ball in $\mathbf{R}^n$ is given by

$$v_n = \begin{cases} \dfrac{\pi^{n/2}}{(n/2)!} & \text{for } n \text{ even} \\[2ex] \dfrac{2^{(n+1)/2} \pi^{(n-1)/2}}{1 \cdot 3 \cdot 5 \cdots n} & \text{for } n \text{ odd} \end{cases}$$

and its moments of inertia are

$$w_n = \begin{cases} \dfrac{v_n}{n+2} & \text{for } n \text{ even} \\[2ex] \dfrac{2v_n}{n+2} & \text{for } n \text{ odd} \end{cases}$$

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