(1) A smooth manifold is said to be *parallelizable* if and only if its tangent bundle is trivial. **Prove or disprove:** Every Lie group is parallelizable.

(2) Let $M^3$ be a smooth 3-manifold and let $\omega$ be a *contact form*, that is, a 1-form such that $\omega \wedge d\omega \neq 0$.

(a) Show that there exists a vector field $X$ such that $\omega(X) = 1$ and $\mathcal{L}_X(\omega) = 0$.

(b) Show that every point $x \in M$ has an open neighborhood $U$ and local coordinates $(x, y, z)$ such that $\omega = xdy + dz$.

(3) Consider a regular surface patch for the sphere in spherical coordinates:

$$[0, \pi] \times [0, 2\pi] \xrightarrow{f} \mathbb{R}^3$$

$$(\phi, \theta) \mapsto \begin{bmatrix} \cos(\theta) \cos(\phi) \\ \sin(\theta) \cos(\phi) \\ \sin(\phi) \end{bmatrix}$$

(a) Compute the first fundamental form $f^*(dx^2 + dy^2 + dz^2)$ in this parametrization.

(b) Compute the area 2-form $dA$ in this parametrization. Verify that the area of $S^2$ equals $4\pi$.

(c) **Prove or disprove:** Rotations around the $z$-axis and reflection in the $xy$-plane restrict to orientation-preserving isometries of $S^2$. What is the full isometry group of $S^2$?

(4) Compute the integral curves of the vector fields

$$\partial_x, \ x\partial_x + y\partial_y, \ -y\partial_x + x\partial_y, \ x^2\partial_x, \ f(x)\partial_y$$

where $f(x)$ is an arbitrary function of $x$.
(5) (Unit-speed plane curves) Let \( k(s) \) be a smooth function of \( s \in \mathbb{R} \). Let \( M \) be the smooth manifold \( \mathbb{R} \times SO(2) \times \mathbb{E}^2 \) with global coordinates

\[
\left( s, \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, (x, y) \right).
\]

(a) Compute the integral curve \( \gamma(t) \) of the vector field

\[
\Phi := \partial_s + k(s)\partial_\theta + \cos(\theta)\partial_x + \sin(\theta)\partial_y
\]

with initial condition \((0, \theta_0, x_0, y_0) \in M\).

(b) Prove that through any point in \( \mathbb{E}^2 \) and any initial direction, there exists a unique arclength-parametrized curve with curvature \( k(s) \).

(6) (Unit-speed space curves) Let \( \mathbf{v}(s) := \gamma'(s) \) be a unit-speed curve in \( \mathbb{E}^3 \). Then its \textit{Frenet-Serret frame field} is the positively oriented orthonormal frame field on \( \gamma \) defined by

\[
\mathbf{F}(s) := (\mathbf{v}(s), \mathbf{n}(s), \mathbf{b}(s))
\]

where \( \mathbf{v}(s) \) is the velocity of \( \gamma \), and

\[
\mathbf{n}(s) := k(s)^{-1} \frac{D\mathbf{v}(s)}{ds},
\]

\[
\mathbf{b}(s) := \mathbf{v}(s) \times \mathbf{n}(s)
\]

where

\[
k(s) := \left\| \frac{D\gamma'(s)}{ds} \right\|
\]

is the \textit{curvature} of \( \gamma \).

(a) Defining the \textit{torsion} of \( \gamma \) as:

\[
\tau(s) := \frac{D\mathbf{n}(s)}{ds} \cdot \mathbf{b},
\]

the Frenet-Serret frame satisfies the \textit{structure equation}:

\[
\frac{D\mathbf{F}(s)}{ds} = \begin{bmatrix} 0 & k(s) & 0 \\ -k(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \mathbf{F}(s)
\]

(b) Prove that any two functions \( k(s) \) and \( \tau(s) \) arise as the curvature and torsion of a curve \( \gamma \) in \( \mathbb{E}^3 \). (Hint: find a vector field on \( \mathbb{R} \times SO(3) \times \mathbb{E}^3 \).)

(c) Why is \( \gamma \) parametrized by arc length?

(d) How unique is \( \gamma \)?

(e) What does the structure equation look like for a plane curve parametrized by arc length?