Two papers which changed my life: Milnor's seminal work on flat manifolds

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Two of Milnor's papers on flat manifolds

- *"On the existence of a connection with curvature zero,"* (Commentarii Mathematici Helvetici 1958) began a development of the theory of characteristic classes of flat bundles, foliations, and bounded cohomology.
- *"On fundamental groups of complete affinely flat manifolds,"* (Advances in Mathematics 1977) clarified the theory of complete affine manifolds, and set the stage for startling examples of Margulis of 3-manifold quotients of Euclidean 3-space by free groups of affine transformations.

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Hilbert's Problem 18: Crystallographic groups

- Is there in $\mathbb{R}^n$ only a finite number of essentially different kinds of groups of motions with a compact fundamental domain?
- Such a group is a *crystallographic group* and the quotient is a compact *Euclidean orbifold.*
- Finitely covered by a *Euclidean manifold.*
- Equivalently, a *flat Riemannian manifold.*

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Euclidean manifolds

- When can a group G act on $\mathbb{R}^n$ with quotient $M^n = \mathbb{R}^n/G$ a compact (Hausdorff) manifold?
- (Bieberbach 1912): G acts by Euclidean isometries $\Longrightarrow G$ finite extension of a subgroup of *translations* $G \cap \mathbb{R}^n \cong \mathbb{Z}^n$
- A Euclidean isometry is an *affine transformation*

$$\vec{x} \xmapsto{\gamma} A\vec{x} + \vec{b}$$

$$A \in \mathsf{GL}(n, \mathbb{R}), \vec{b} \in \mathbb{R}^n,$$

where the *linear part* $\mathbb{L}(\gamma) = A$ is *orthogonal.* $(A \in \mathsf{O}(n))$

- Only finitely many topological types in each dimension.

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The Euler Characteristic

- Since closed Euclidean manifold M is finitely covered by a torus, its Euler characteristic vanishes.
  - This also follows from the Chern-Gauss-Bonnet theorem since the Riemannian metric has curvature zero.
  - Chern's integrand is an expression involving the curvature of an *orthogonal* connection.
- An *affine manifold* is a manifold with a distinguished atlas of local coordinate charts mapping to $\mathbb{R}^n$ with locally affine coordinate changes.
  - Equivalently M has flat symmetric *linear* connection.
- **Conjecture:** (Chern 1940's)
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Benzecri's Theorem

- Benzecri (1955): A closed affine 2-manifold has $\chi = 0$.
- Milnor (1958): If ξ is an $\mathbb{R}^2$-bundle over Σ_g^2 with flat connection, then $|\mathrm{Euler}(\xi)| < g$.
- Thus the tangent bundle of Σ_g does not even have a flat connection if $g > 1$.
  - Recall that an X-bundle ξ over M with a flat connection is defined by an action of $\pi_1(\Sigma)$ on X as the quotient $\widetilde{M} \times X$ by diagonal action of $\pi_1(M)$.
- Smillie (1976): For every $n > 1$, there are closed $2n$-manifolds with flat tangent bundle with nonzero Euler characteristic.

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Milnor-Wood inequality

- Wood (1972): Replace $\mathrm{GL}(2,\mathbb{R})$ by $\mathrm{Homeo}^+(S^1)$:
$$|\mathrm{Euler}(\xi)| \leq |\chi(\Sigma)|$$
- A representation $\pi_1(\Sigma) \xrightarrow{\rho} G$, where $G = \mathrm{PSL}(2,\mathbb{R})$, is *maximal* $:\Longleftrightarrow |(\mathrm{Euler}(\xi))| = |\chi(\Sigma)|$.
- Goldman (1980): ρ is maximal if and only if ρ embeds $\pi_1(\Sigma)$ onto a discrete subgroup of G.
  - Equivalence classes of maximal representations form the Fricke space of marked hyperbolic structures on Σ.
  - More generally, connected components of $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ are the $(4g-3)$ preimages
$$\mathrm{Euler}^{-1}(2-2g), \mathrm{Euler}^{-1}(3-2g), \dots, \mathrm{Euler}^{-1}(2g-2).$$
- Ghys: Maximal representations into $\mathrm{Homeo}^+(S^1)$.

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Recent developments: surface group representations

- Rigidity for maximal surface group representations, bounded cohomology (Toledo, Burger-Iozzi-Wienhard)
- Morse theory on space of connections: global topology of representation spaces and Higgs bundles (Hitchin, Bradlow-Garcia-Prada-Gothen, Weitsman-Wentworth-Wilkin)
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Affine manifolds

- In general an affine structure on a *closed* manifold may be *geodesically incomplete* (unlike Riemannian manifolds):
  - *Hopf manifold* $(\mathbb{R}^n \setminus \{0\})/\langle\gamma\rangle$ where $\mathbb{R}^n \xrightarrow{\gamma} \mathbb{R}^n$ is a linear expansion.
  - Discrete holonomy;
  - Homeomorphic to $S^{n-1} \times S^1$
- Geodesics aimed at 0 seem to *speed up* (although their acceleration is zero) and eventually fly off the manifold in finite time.
- Henceforth we restrict to complete manifolds.
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Can a nonabelian free group act properly, freely and discretely by affine transformations on $\mathbb{R}^n$?

- Equivalently (Tits 1971): *"Are there discrete groups other than virtually polycycic groups which act properly, affinely?"*
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- Clearly a geometric problem: free groups act properly by isometries on H^3 hence by diffeomorphisms of $\mathbb{E}^3$
  - These actions are *not* affine.
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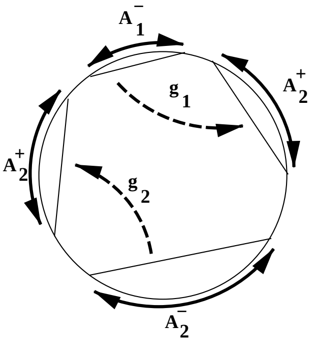

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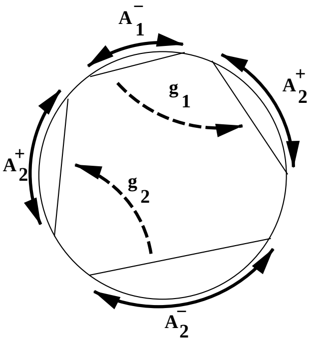

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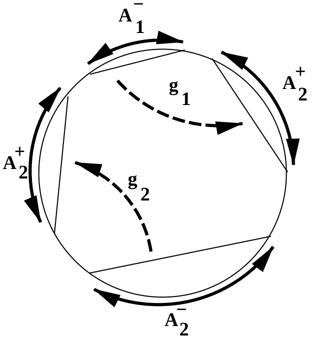

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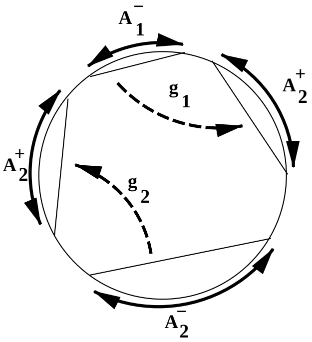

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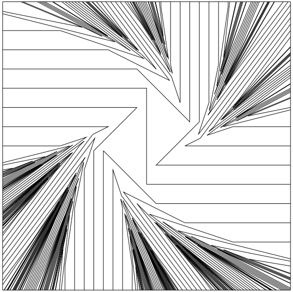

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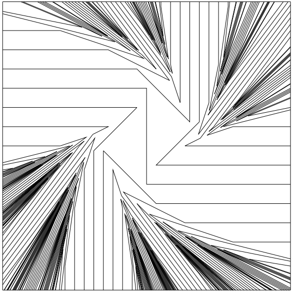

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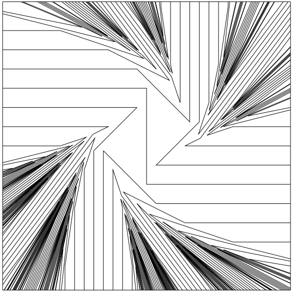

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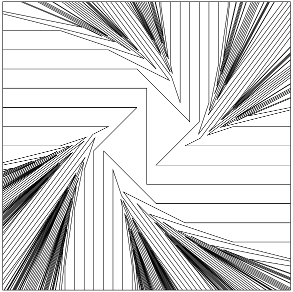

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$$M^3 := \mathbb{E}^{2,1}/\Gamma \longrightarrow \Sigma := \mathrm{H}^2/\mathbb{L}(\Gamma)$$

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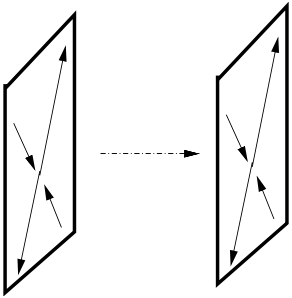
Cyclic groups

- Most elements $\gamma \in \Gamma$ are *boosts*, affine deformations of hyperbolic elements of $O(2, 1)$. A fundamental domain is the *slab* bounded by two parallel planes.



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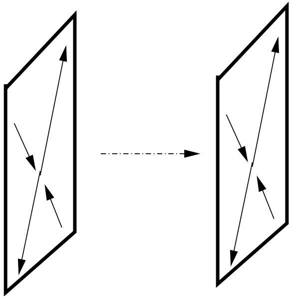

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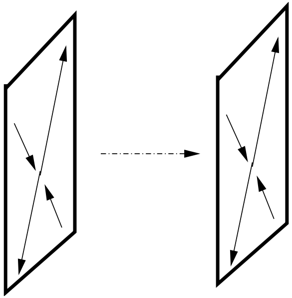

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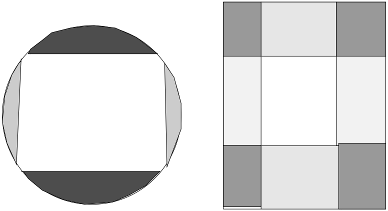

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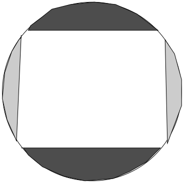

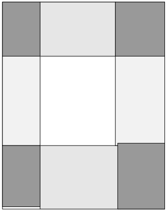

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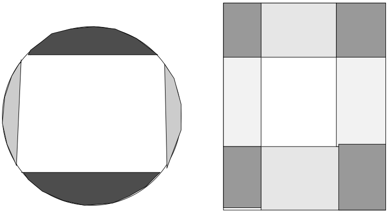

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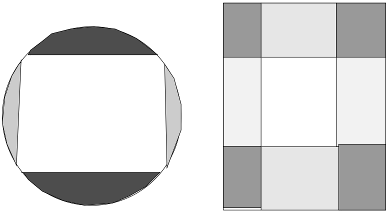

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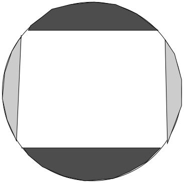

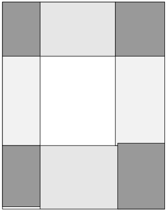

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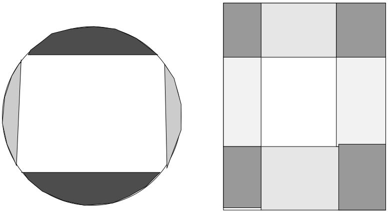

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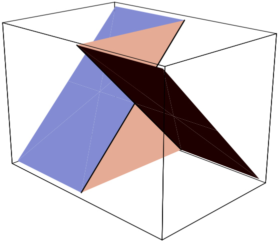

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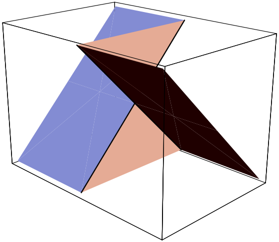

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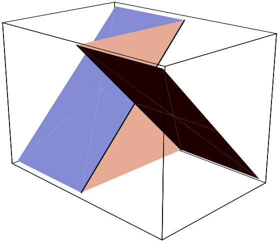

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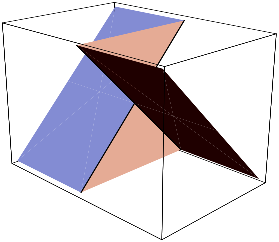

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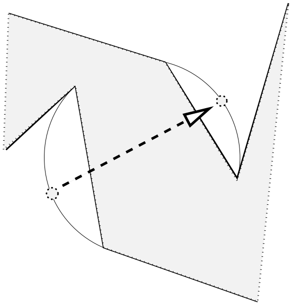

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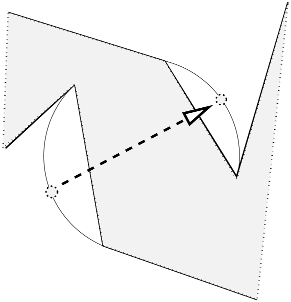

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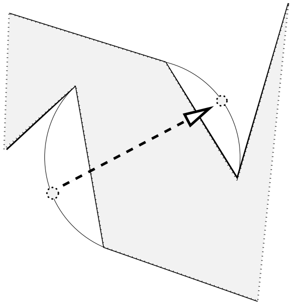

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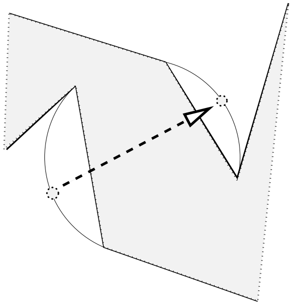

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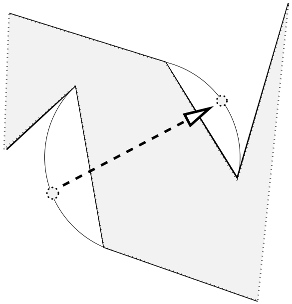

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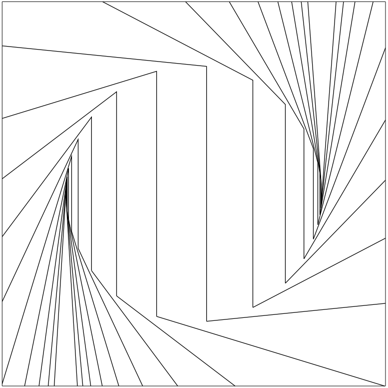

The resulting tessellation for a linear boost.

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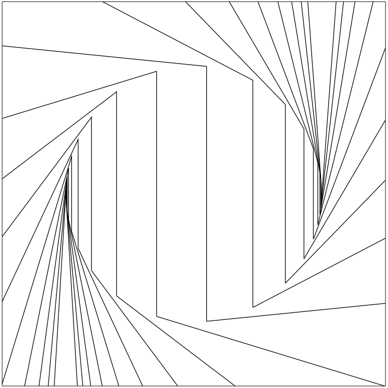

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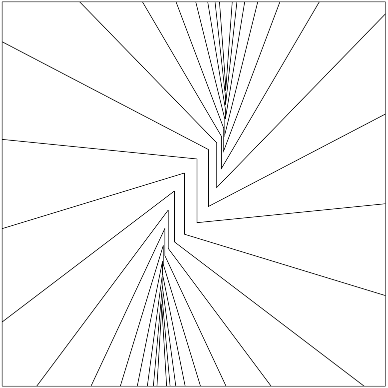

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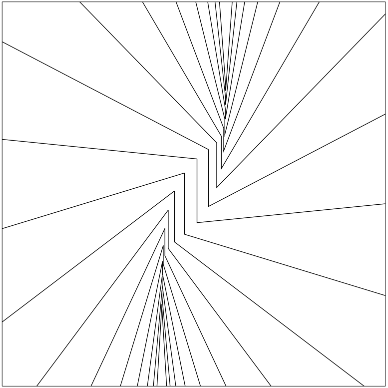

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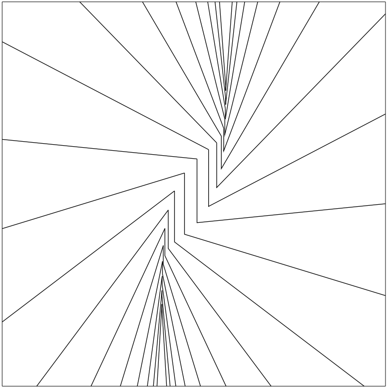

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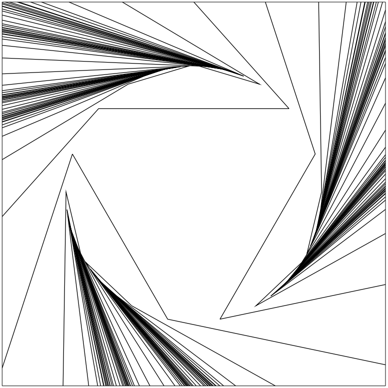

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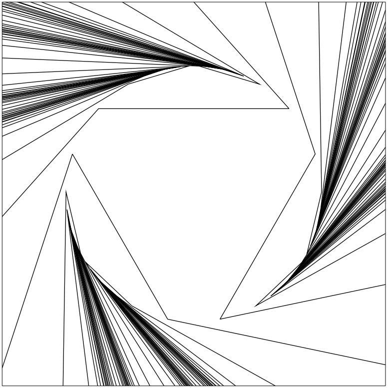

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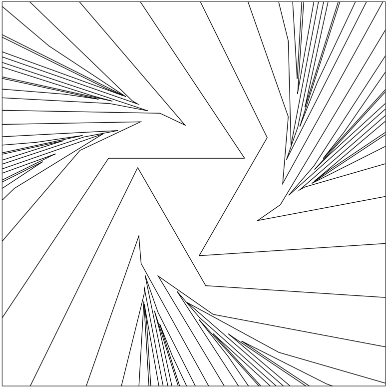

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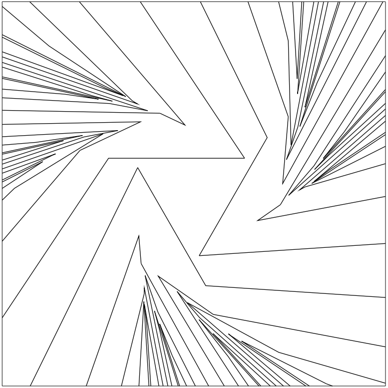

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- **Conjecture:** Every Margulis spacetime M^3 admits a fundamental polyhedron bounded by disjoint crooked planes.
  - Corollary: (Tameness) $M^3 \approx$ open solid handlebody.
- Proved when $\chi(\Sigma) = -1$ (that is, $rank(\pi_1(\Sigma)) = 2$). (Charette-Drumm-Goldman 2010)
- Four possible topologies for Σ:
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