THE MODULAR GROUP ACTION ON REAL $SL(2)$-CHARACTERS OF A ONE-HOLED TORUS

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Abstract. The group Γ of automorphisms of the polynomial

$$\kappa(x, y, z) = x^2 + y^2 + z^2 - xyz - 2$$

is isomorphic to

$$\mathrm{PGL}(2, \mathbb{Z}) \ltimes (\mathbb{Z}/2 \oplus \mathbb{Z}/2).$$

For $t \in \mathbb{R}$, the Γ-action on $\kappa^{-1}(t) \cap \mathbb{R}^3$ displays rich and varied dynamics. The Γ-action preserves a Poisson structure defining a Γ-invariant area form on each $\kappa^{-1}(t) \cap \mathbb{R}^3$. For $t < 2$, the action of Γ is properly discontinuous on the four contractible components of $\kappa^{-1}(t) \cap \mathbb{R}^3$ and ergodic on the compact component (which is empty if $t < -2$). The contractible components correspond to Teichmüller spaces of (possibly singular) hyperbolic structures on a torus $\bar{M}$. For $t = 2$, the level set $\kappa^{-1}(t) \cap \mathbb{R}^3$ consists of characters of reducible representations and comprises two ergodic components corresponding to actions of $\mathrm{GL}(2, \mathbb{Z})$ on $(\mathbb{R}/\mathbb{Z})^2$ and $\mathbb{R}^2$ respectively. For $2 < t \leq 18$, the action of Γ on $\kappa^{-1}(t) \cap \mathbb{R}^3$ is ergodic. Corresponding to the Fricke space of a three-holed sphere is a Γ-invariant open subset $\Omega \subset \mathbb{R}^3$ whose components are permuted freely by a subgroup of index 6 in Γ. The level set $\kappa^{-1}(t) \cap \mathbb{R}^3$ intersects Ω if and only if $t > 18$, in which case the Γ-action on the complement $(\kappa^{-1}(t) \cap \mathbb{R}^3) - \Omega$ is ergodic.

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Contents

Introduction

Let M be a compact oriented surface of genus one with one boundary component, a *one-holed torus.* Its fundamental group π is free of rank two. Its mapping class group $\pi_0(\mathrm{Homeo}(M))$ is isomorphic to the outer automorphism group $\mathrm{Out}(\pi)$ of π and acts on the space of equivalence classes of representations $\pi \longrightarrow \mathrm{SL}(2)$. We investigate the dynamics of this action on the set of real points on this moduli space.

Corresponding to the boundary of M is an element $K \in \pi$ which is the commutator of free generators $X, Y \in \pi$. By a theorem of Fricke [8, 9], the moduli space of equivalence classes of $\mathrm{SL}(2,\mathbb{C})$-representations naturally identifies with affine 3-space $\mathbb{C}^3$, via the quotient map

$$\begin{aligned} \mathrm{Hom}(\pi, \mathrm{SL}(2,\mathbb{C})) &\longrightarrow \mathbb{C}^3 \\ \rho &\longmapsto \begin{bmatrix} x \\ y \\ z \end{bmatrix} := \begin{bmatrix} \mathrm{tr}(\rho(X)) \\ \mathrm{tr}(\rho(Y)) \\ \mathrm{tr}(\rho(XY)) \end{bmatrix}. \end{aligned}$$

In terms of these coordinates, the trace $\mathrm{tr}\,\rho(K)$ equals:

$$\kappa(x,y,z) := x^2 + y^2 + z^2 - xyz - 2$$

which is preserved under the action of $\mathrm{Out}(\pi)$. The action of $\mathrm{Out}(\pi)$ on $\mathbb{C}^3$ is commensurable with the action of the group Γ of polynomial automorphisms of $\mathbb{C}^3$ which preserve κ (Horowitz [17]). (Compare also Magnus [22].)

Theorem. *Let $\kappa(x,y,z) = x^2 + y^2 + z^2 - xyz - 2$ and let $t \in \mathbb{R}$.*

- *For $t < -2$, the group Γ acts properly on $\kappa^{-1}(t) \cap \mathbb{R}^3$;*
- *For $-2 \leq t < 2$, there is a compact connected component C_t of $\kappa^{-1}(t) \cap \mathbb{R}^3$ upon which Γ acts ergodically; Γ acts properly on the complement $\big(\kappa^{-1}(t) \cap \mathbb{R}^3\big) - C_t$;*
- *For $t = 2$, the action of Γ is ergodic on the compact subset $\kappa^{-1}(2) \cap [-2,2]^3$ and the action is ergodic on the complement $\kappa^{-1}(2) - [-2,2]^3$;*
- *For $18 \geq t > 2$, the group Γ acts ergodically on $\kappa^{-1}(t) \cap \mathbb{R}^3$;*
- *For $t > 18$, the group Γ acts properly and freely on an open subset $\Omega_t \subset \kappa^{-1}(t) \cap \mathbb{R}^3$, permuting its components.The Γ-action on the complement of Ω_t is ergodic.*

The proof uses the interplay between representations of the fundamental group and hyperbolic structures on M. The dynamics breaks up into two strikingly different types: representations corresponding to hyperbolic structures comprise contractible connected components of the the level sets $\kappa^{-1}(t)$, whereas representations which map a simple nonperipheral essential loop to an elliptic element, comprise open subsets of $\kappa^{-1}(t)$ upon which Γ is ergodic. Thus nontrivial dynamics accompanies nontrivial topology of the moduli spaces.

In his doctoral thesis [31], G. Stantchev considers characters corresponding to representations into $\mathrm{PGL}(2,\mathbb{R})$ (that is, actions preserving $\mathbf{H}^2$ but not preserving orientation on $\mathbf{H}^2$). For $t < -14$ (respectively $t > 6$) characters of discrete embeddings representing hyperbolic structures on 2-holed projective planes (respectively 1-holed Klein bottles) give wandering domains in the corresponding level set. For $-14 \leq t < 2$ the Γ-action on the corresponding level set is ergodic, and for $t < -14$, the action is ergodic on the complement of the wandering domains corresponding to Fricke spaces of 2-holed projective planes.

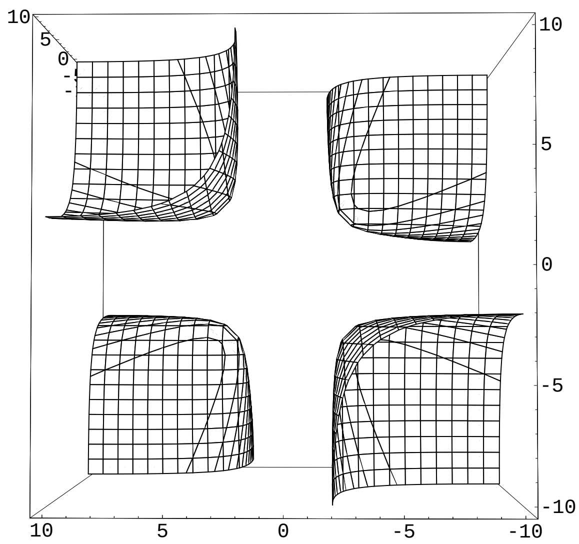

FIGURE 1. Level set $\kappa = -2.1$

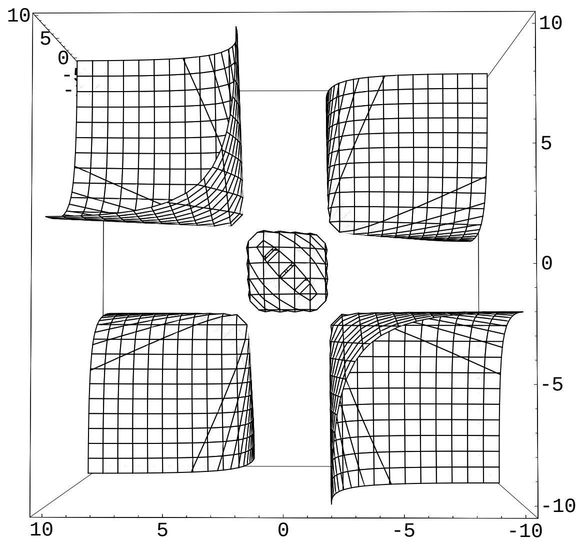

FIGURE 2. Level set $\kappa = 1.9$

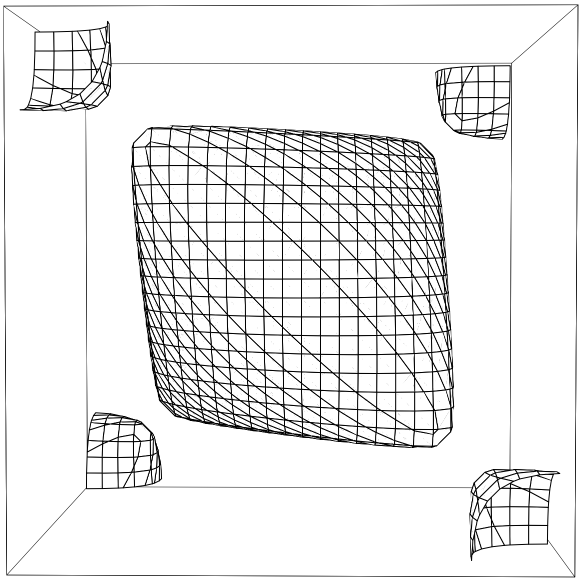

FIGURE 3. Level set $\kappa = 1.9$

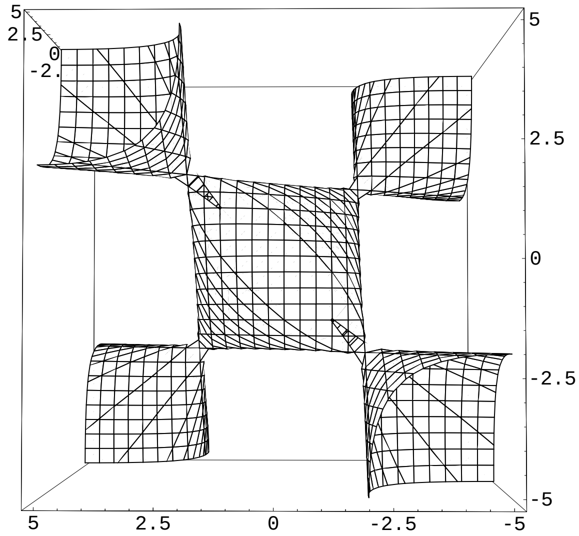

FIGURE 4. Level set $\kappa = 2.1$

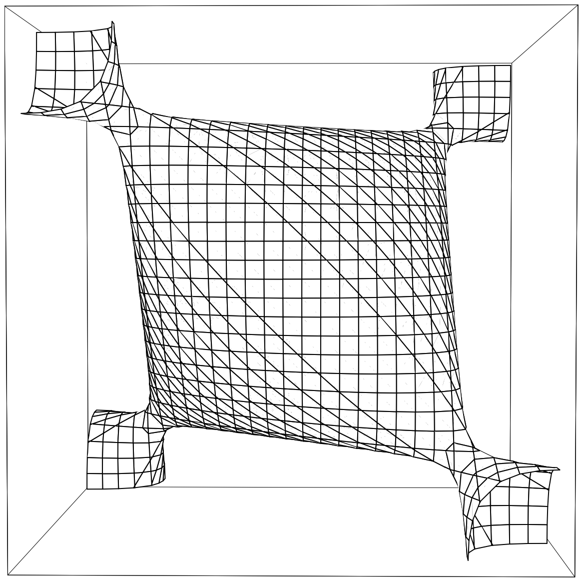

FIGURE 5. Level set $\kappa = 2.1$

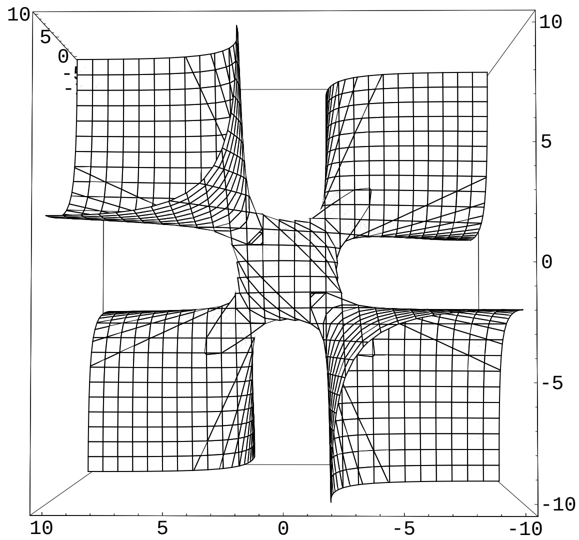

FIGURE 6. Level set $\kappa = 4$

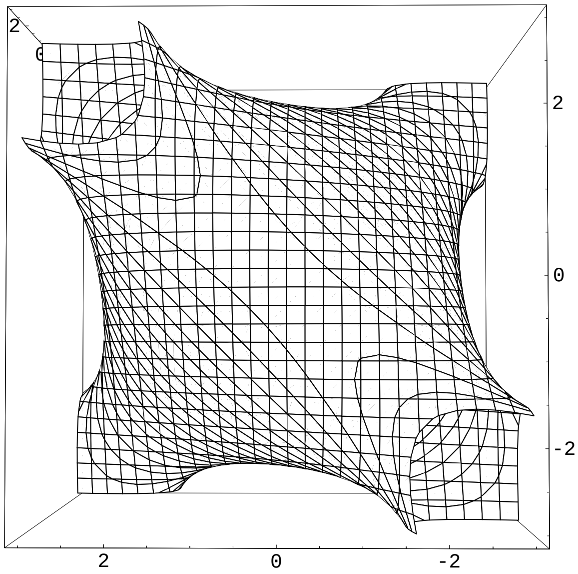

FIGURE 7. Level set $\kappa = 4$

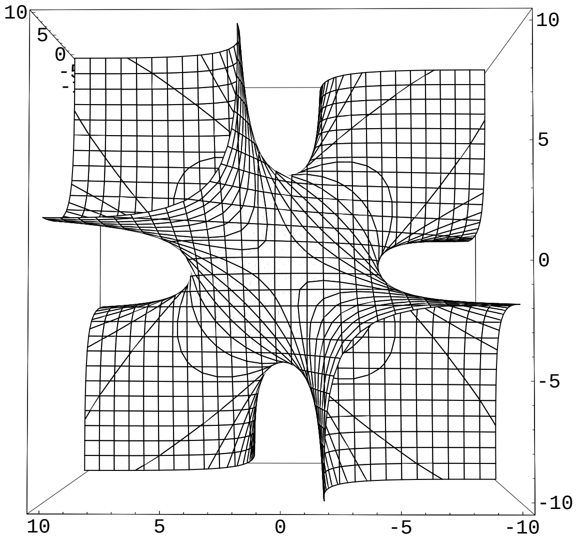

FIGURE 8. Level set $\kappa = 19$

Notation and terminology. We work in Poincaré's model of the hyperbolic plane $\mathbf{H}^2$ as the upper half-plane. We denote the identity map (identity matrix) by $\mathbb{I}$. Commutators are denoted:

$$[A, B] = ABA^{-1}B^{-1}$$

and inner automorphisms are denoted:

$$\iota_g : x \longmapsto gxg^{-1}.$$

For any group π we denote the group of all automorphisms by $\mathrm{Aut}(\pi)$ and the normal subgroup of inner automorphisms by $\mathrm{Inn}(\pi)$. We denote the quotient group $\mathrm{Aut}(\pi)/\mathrm{Inn}(\pi)$ by $\mathrm{Out}(\pi)$. A compact surface S with n boundary components will be called *n-holed.* Thus a *one-holed torus* is the complement of an open disc inside a torus and a *three-holed sphere* (sometimes called a "pair-of-pants") is the complement of three disjoint discs inside a sphere.

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1. The modular group and the moduli space

In this section we define the modular group and the moduli space for the one-holed torus. The modular group is isomorphic to

$$\pi_0(\mathrm{Homeo}(M)) \cong \mathrm{Out}(\pi) \cong \mathrm{GL}(2, \mathbb{Z})$$

and the moduli space is affine space $\mathbb{C}^3$. We explain how the invariant function κ originates. We describe various elements of the modular group and how they act on the moduli space.

1.1. **The mapping class group.**

Let M denote a compact connected orientable surface of genus one with one boundary component. Since attaching a disc to M yields a torus, we refer to M as a *one-holed torus.* The *mapping class group* of M is the group $\pi_0(\mathrm{Homeo}(M, \partial M))$ of isotopy classes of homeomorphisms of M. We investigate the action of $\pi_0(\mathrm{Homeo}(M, \partial M))$ on the moduli space of flat $\mathrm{SL}(2)$-connections on M.

1.1.1. *Relation to* $\pi_1(M)$. Choose a basepoint $x_0 \in M$ and let $\pi := \pi_1(M; x_0)$. Any homeomorphism of M is isotopic to one which fixes x_0 and hence defines an automorphism of π. Two such isotopic homeomorphisms determine automorphisms of π differing by an inner automorphism, producing a well-defined homomorphism

$$N : \pi_0(\mathrm{Homeo}(M)) \longrightarrow \mathrm{Out}(\pi) \cong \mathrm{Aut}(\pi)/\,\mathrm{Inn}(\pi). \tag{1.1.1}$$

If M is a closed surface, then Dehn (unpublished) and Nielsen [30] proved that N is an isomorphism. (See Stillwell [32] for a proof of the Dehn-Nielsen theorem.) When $\partial M \neq \emptyset$, then each component $\partial_i M$ determines a conjugacy class $\mathcal{C}_i$ of elements of $\pi_1(M)$. The image of N consists of elements of $\mathrm{Out}(\pi)$ represented by automorphisms which preserve each $\mathcal{C}_i$.

Let M be a one-holed torus. Its fundamental group admits a geometric redundant presentation

$$\pi = \langle X, Y, K \mid [X, Y] = K \rangle \tag{1.1.2}$$

where K corresponds to the generator of $\pi_1(\partial M)$. Of course, π is freely generated by X, Y.

1.1.2. *Nielsen's theorem.* The following remarkable property of M is due to Jakob Nielsen [29] and does not generalize to other hyperbolic surfaces with boundary. For a proof see Magnus-Karrass-Solitar [23], Theorem 3.9 or Lyndon-Schupp [21], Proposition 5.1.

Proposition 1.1.1. *Any automorphism of the rank two group*

$$\pi = \langle X, Y, K \mid K = [X, Y] \rangle$$

takes K to a conjugate of either K itself or its inverse K^{-1}.

An equivalent geometric formulation is:

Proposition 1.1.2. *Every homotopy-equivalence $M \to M$ is homotopic to a homeomorphism of M.*

Thus the homomorphism (1.1.1) defines an isomorphism

$$N : \pi_0(\mathrm{Homeo}(M)) \longrightarrow \mathrm{Out}(\pi).$$

1.2. **The structure of the modular group.** We say that an automorphism of π which takes K to either K or K^{-1} is *normalized.* The normalized automorphisms form a subgroup $\mathrm{Aut}(\pi, K)$ of $\mathrm{Aut}(\pi)$.

Let ϕ be an automorphism of π. Proposition 1.1.1 implies that $\phi(K)$ is conjugate to $K^{\pm 1}$. Thus there exists an inner automorphism ι_g such that

$$\phi(K) = g K^{\epsilon} g^{-1},$$

that is

$$\iota_g^{-1} \circ \phi \in \mathrm{Aut}(\pi, K)$$

where $\epsilon = \pm 1$. (In fact, $\epsilon = \det(h([\phi]))$, where h is the homomorphism defined by (1.2.1) below.) Since the centralizer of K in π equals the cyclic group $\langle K \rangle$, the automorphism ϕ determines the coset of g modulo $\langle K \rangle$ uniquely.

We obtain a short exact sequence

$$1 \longrightarrow \langle \iota_K \rangle \hookrightarrow \mathrm{Aut}(\pi, K) \longrightarrow \mathrm{Out}(\pi) \longrightarrow 1$$

The action on the homology $H_1(M; \mathbb{Z}) \cong \mathbb{Z}^2$ defines a homomorphism

$$h : \mathrm{Out}(\pi) \longrightarrow \mathrm{GL}(2, \mathbb{Z}). \tag{1.2.1}$$

By Nielsen [29], h is an isomorphism. (Surjectivity follows by realizing an element of $\mathrm{GL}(2, \mathbb{Z})$ as a *linear* homeomorphism of the torus $\mathbb{R}^2/\mathbb{Z}^2$. See Lyndon-Schupp [21], Prop. 4.5 or Magnus-Karrass-Solitar [23], §3.5, Corollary N4.) We obtain an isomorphism

$$\mu : \mathrm{GL}(2, \mathbb{Z}) \longrightarrow \mathrm{Aut}(\pi, K)/\langle \iota_K \rangle.$$

Restriction of the composition $\mu \circ h$ to $\mathrm{Aut}(\pi, K)$ equals the quotient homomorphism

$$\mathrm{Aut}(\pi, K) \longrightarrow \mathrm{Aut}(\pi, K)/\langle \iota_K \rangle.$$

2. Structure of the character variety

2.1. **Trace functions.** The relevant moduli space is the *character variety,* the categorical quotient of $\mathrm{Hom}(\pi, G)$ by the G-action by inner automorphisms, where $G = \mathrm{SL}(2, \mathbb{C})$. Since π is freely generated by two elements X and Y, the set $\mathrm{Hom}(\pi, G)$ of homomorphisms $\pi \longrightarrow G$ identifies with the set of pairs $(\xi, \eta) \in G \times G$, via the mapping

$$\begin{aligned} \mathrm{Hom}(\pi, G) &\longrightarrow G \times G \\ \rho &\longmapsto (\rho(X), \rho(Y)). \end{aligned}$$

This mapping is equivariant respecting the action of G on $\mathrm{Hom}(\pi, G)$ by

$$g : \rho \longmapsto \iota_g \circ \rho$$

and the action of G on $G \times G$ by

$$g : (\xi, \eta) \longmapsto (g \xi g^{-1}, g \eta g^{-1}).$$

The moduli space $\mathrm{Hom}(\pi, G)/\!\!/G$ consists of equivalence classes of elements of $\mathrm{Hom}(\pi, G) \cong G \times G$ where the equivalence class of a homomorphism ρ is defined as the closure of the G-orbit $G\rho$. Then $\mathrm{Hom}(\pi, G)/\!\!/G$ is the categorical quotient in the sense that its coordinate ring identifies with the ring of G-invariant regular functions on

$\mathrm{Hom}(\pi, G)$. For a single element $g \in G$, the conjugacy class $\mathrm{Inn}(G)(g)$ is determined by the trace $t = \mathrm{tr}(g)$ if $t \neq \pm 2$. That is,

$$\mathrm{Inn}(G)(g) = \mathrm{tr}^{-1}(t). \tag{2.1.1}$$

For $t = \pm 2$, then

$$\mathrm{tr}^{-1}(\pm 2) = \{\pm \mathbb{I}\} \bigcup \mathrm{Inn}(G) \cdot \left(\pm \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right). \tag{2.1.2}$$

By Fricke [8] and Fricke-Klein [9], the traces of the generators X, Y, XY parametrize $\mathrm{Hom}(\pi, G)/\!\!/G$ as the affine space $\mathbb{C}^3$. As X and Y freely generate π, we may identify:

$$\begin{aligned} \mathrm{Hom}(\pi, G) &\longleftrightarrow \quad G \times G \\ \rho \quad &\longleftrightarrow (\rho(X), \rho(Y)) \end{aligned}$$

The character mapping

$$\begin{aligned} \chi : \mathrm{Hom}(\pi, G)/\!\!/G &\longrightarrow \mathbb{C}^3 \\ [\rho] &\longmapsto \begin{bmatrix} x(\rho) \\ y(\rho) \\ z(\rho) \end{bmatrix} = \begin{bmatrix} \mathrm{tr}(\rho(X)) \\ \mathrm{tr}(\rho(Y)) \\ \mathrm{tr}(\rho(XY)) \end{bmatrix} \end{aligned}$$

is an isomorphism. (Compare the discussion in Goldman [12], 4.1, [13], §§4–5 and [14].)

For example, given $(x, y, z) \in \mathbb{C}^3$, the representation ρ defined by

$$\rho(X) = \begin{bmatrix} x & -1 \\ 1 & 0 \end{bmatrix}, \quad \rho(Y) = \begin{bmatrix} 0 & \zeta^{-1} \\ -\zeta & y \end{bmatrix} \tag{2.1.3}$$

satisfies $\chi(\rho) = (x, y, z)$ where $\zeta \in \mathbb{C}$ is chosen so that

$$\zeta + \zeta^{-1} = z.$$

Conversely, if $(x, y, z) \in \mathbb{C}^3$ then $\chi^{-1}(x, y, z)$ consists of a single G-orbit if and only if $\kappa(x, y, z) \neq 2$ where κ is defined below in (2.1.4). (This is also the condition that ρ is an *irreducible representation* of π. Compare Lubotzky-Magid [20], Brumfiel-Hilden [4], Culler-Shalen [6].)

For any word $w(X, Y)$, the function

$$\begin{aligned} \mathrm{Hom}(\pi, G) &\longrightarrow \mathbb{C} \\ [\rho] &\longmapsto \mathrm{tr}(\rho(w(X, Y)) \end{aligned}$$

is G-invariant. Hence there exists a polynomial $f_w(x, y, z) \in \mathbb{C}[x, y, z]$ such that

$$\mathrm{tr}(\rho(w(X, Y)) = f_w(x(\rho), y(\rho), z(\rho)).$$

A particularly important example occurs for $w(X,Y) = [X,Y] = K$, in which case we denote $f_w(x,y,z)$ by $\kappa(x,y,z)$. By an elementary calculation (see, for example [14]),

$$\operatorname{tr}(\rho(K)) = \kappa(x,y,z) := x^2 + y^2 + z^2 - xyz - 2. \tag{2.1.4}$$

The level set $\kappa^{-1}(t)$ consists of equivalence classes of representations $\rho : \pi \to G$ where $\rho(K)$ is constrained to lie in $\operatorname{tr}^{-1}(t)$. (Compare (2.1.1) and (2.1.2) above.)

2.2. **Automorphisms.** Let $G = \mathrm{SL}(2,\mathbb{C})$. The group $\mathrm{Aut}(\pi)$ acts on the character variety $\mathrm{Hom}(\pi, G)/\!\!/G$ by:

$$\phi_*([\rho]) = [\rho \circ \phi^{-1}]$$

and since

$$\rho \circ \iota_\gamma = \iota_{\rho(\gamma)} \circ \rho$$

the subgroup $\mathrm{Inn}(\pi)$ acts trivially. Thus $\mathrm{Out}(\pi)$ acts on

$$\mathrm{Hom}(\pi, G)/\!\!/G \cong \mathbb{C}^3$$

and since an automorphism ϕ of π is determined by

$$(\phi(X), \phi(Y)) = (w_1(X,Y), w_2(X,Y)),$$

the action of ϕ on $\mathbb{C}^3$ is given by the three polynomials $f_{w_1}, f_{w_2}, f_{w_3}$:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} f_{w_1}(x,y,z) \\ f_{w_2}(x,y,z) \\ f_{w_3}(x,y,z) \end{bmatrix}.$$

where $w_3(X,Y) = w_1(X,Y)w_2(X,Y)$ is the product word. Hence $\mathrm{Out}(\pi)$ acts on $\mathbb{C}^3$ by *polynomial automorphisms.* Nielsen's theorem (Proposition 1.1.1) implies that any such automorphism preserves $\kappa : \mathbb{C}^3 \longrightarrow \mathbb{C}$, that is

$$\kappa\left(f_{w_1}(x,y,z), f_{w_2}(x,y,z), f_{w_1 w_2}(x,y,z)\right) = \kappa(x,y,z).$$

2.2.1. *Sign-change automorphisms.* Some automorphisms of the character variety are not induced by automorphisms of π. Namely, the homomorphisms of π into the center $\{\pm\mathbb{I}\} \subset G$ form a group acting on $\mathrm{Hom}(\pi, G)$ by pointwise multiplication. Let $\zeta \in \mathrm{Hom}(\pi, \{\pm\mathbb{I}\})$ and $\rho \in \mathrm{Hom}(\pi, G)$. Then

$$\zeta \cdot \rho : \gamma \longmapsto \zeta(\gamma)\rho(\gamma) \tag{2.2.1}$$

is a homomorphism. This defines an action

$$\mathrm{Hom}(\pi, \{\pm\mathbb{I}\}) \times \mathrm{Hom}(\pi, G) \longrightarrow \mathrm{Hom}(\pi, G).$$

Furthermore, since $\{\pm\mathbb{I}\}$ is central in G and $K \in \pi$ is a commutator

$$(\zeta \cdot \rho)(K) = \rho(K). \tag{2.2.2}$$

Since π is free of rank two,

$$\mathrm{Hom}(\pi, \{\pm\mathbb{I}\}) \cong \mathbb{Z}/2 \times \mathbb{Z}/2.$$

The three nontrivial elements $(0,1), (1,0), (1,1)$ of $\mathbb{Z}/2 \times \mathbb{Z}/2$ act on representations by $\sigma_1, \sigma_2, \sigma_3$ respectively:

$$\sigma_1 \cdot \rho : \begin{cases} X & \longmapsto \rho(X) \\ Y & \longmapsto -\rho(Y) \end{cases}$$

$$\sigma_2 \cdot \rho : \begin{cases} X & \longmapsto -\rho(X) \\ Y & \longmapsto \rho(Y) \end{cases}$$

$$\sigma_3 \cdot \rho : \begin{cases} X & \longmapsto -\rho(X) \\ Y & \longmapsto -\rho(Y) \end{cases}$$

The corresponding action on characters is:

$$(\sigma_1)_* : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} x \\ -y \\ -z \end{bmatrix}$$
$$(\sigma_2)_* : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} -x \\ y \\ -z \end{bmatrix}$$
$$(\sigma_3)_* : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} -x \\ -y \\ z \end{bmatrix}.$$

We call this group the *group of sign-changes* and denote it by Σ.

Evidently Σ preserves $\kappa(x, y, z)$. (Alternatively apply (2.2.2).)

2.2.2. *Permutations.* Since $\kappa(x, y, z)$ is symmetric in x, y, z the full symmetric group $\mathfrak{S}_3$ also acts on $\mathbb{C}^3$ preserving κ. Unlike Σ, elements of $\mathfrak{S}_3$ are induced by automorphisms of π. The group of all *linear* automorphisms of $(\mathbb{C}^3, \kappa)$ is generated by Σ and $\mathfrak{S}_3$ and forms a semidirect product $\Sigma \rtimes \mathfrak{S}_3$.

$\mathfrak{S}_3$ is actually a quotient of Γ. The *projective line* $\mathbb{P}^1(\mathbb{Z}/2)$ *over* $\mathbb{Z}/2$ has three elements, and every permutation of this set is realized by a projective transformation. Since $\mathbb{Z}/2$ has only one nonzero element, the projective automorphism group of $\mathbb{P}^1(\mathbb{Z}/2)$ equals $\mathrm{GL}(2, \mathbb{Z}/2)$ and $\mathrm{GL}(2, \mathbb{Z}/2) \cong \mathfrak{S}_3$. The action of $\mathrm{GL}(2, \mathbb{Z})$ on $(\mathbb{Z}/2)^2$ defines a homomorphism

$$\mathrm{GL}(2, \mathbb{Z}) \longrightarrow \mathrm{GL}(2, \mathbb{Z}/2) \cong \mathfrak{S}_3$$

whose kernel $\mathrm{GL}(2,\mathbb{Z})_{(2)}$ is generated by the involutions

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \tag{2.2.3}$$

and $-\mathbb{I} \in \mathrm{GL}(2,\mathbb{Z})_{(2)}$. The sequence

$$\mathrm{PGL}(2,\mathbb{Z})_{(2)} := \mathrm{GL}(2,\mathbb{Z})_{(2)}/\{\pm\mathbb{I}\} \longrightarrow \mathrm{PGL}(2,\mathbb{Z}) \longrightarrow \mathfrak{S}_3$$

is exact. The kernel $\Gamma_{(2)}$ of the composition

$$\Gamma \longrightarrow \mathrm{PGL}(2,\mathbb{Z}) \longrightarrow \mathfrak{S}_3,$$

equals the semidirect product

$$\mathrm{PGL}(2,\mathbb{Z})_{(2)} \ltimes (\mathbb{Z}/2 \oplus \mathbb{Z}/2)$$

and Γ is an extension

$$\Gamma_{(2)} \longrightarrow \Gamma \longrightarrow \mathfrak{S}_3.$$

(See Goldman-Neumann [15] for more extensive discussion of Γ and its action on the the set of *complex points* of the character variety.)

2.2.3. *Other automorphisms.* For later use, here are several specific elements of Γ.

The *elliptic involution* is the automorphism

$$\begin{aligned} X &\longmapsto X^{-1} \\ Y &\longmapsto Y^{-1} \end{aligned}$$

corresponding to $-\mathbb{I} \in \mathrm{GL}(2,\mathbb{Z})$ and acts trivially on characters.

The *Dehn twist* about X is the automorphism $\tau_X \in \mathrm{Aut}(\pi)$

$$\begin{aligned} X &\longmapsto X \\ Y &\longmapsto YX \end{aligned}$$

inducing the automorphism of characters

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} x \\ xy - z \\ y \end{bmatrix}$$

and corresponds to

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathrm{GL}(2,\mathbb{Z}).$$

The *Dehn twist* about Y is the automorphism $\tau_Y \in \operatorname{Aut}(\pi)$

$$\begin{aligned} X &\longmapsto XY \\ Y &\longmapsto Y \end{aligned}$$

inducing the automorphism of characters

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} xy - z \\ y \\ x \end{bmatrix}.$$

and corresponds to

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \in \mathrm{GL}(2, \mathbb{Z}).$$

The *quadratic reflection* Q_z is the automorphism

$$\begin{aligned} X &\longmapsto X \\ Y &\longmapsto Y^{-1} \end{aligned}$$

inducing the automorphism of characters

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} x \\ y \\ xy - z \end{bmatrix}.$$

and corresponds to

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathrm{GL}(2, \mathbb{Z}).$$

2.3. **Reducible characters.** Any representation ρ having character in $\kappa^{-1}(2)$ is reducible, as can be checked by direct calculation of commutators in $\mathrm{SL}(2, \mathbb{C})$. Namely, let $\xi, \eta \in \mathrm{SL}(2, \mathbb{C})$. We may write

$$\eta = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where $ad - bc = 1$. By applying an inner automorphism, we may assume that ξ is in Jordan canonical form. If ξ is diagonal,

$$\xi = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$$

then

$$\mathrm{tr}[\xi, \eta] = 2 + bc(\lambda - \lambda^{-1})^2$$

implies that if $\operatorname{tr}[\xi,\eta] = 2$, then either $\xi = \pm\mathbb{I}$ or $bc = 0$ (so ξ is upper-triangular or lower-triangular). Otherwise

$$\xi = \pm \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix},$$

(where $s \neq 0$), in which case

$$\operatorname{tr}[\xi,\eta] = 2 + s^2c^2 \geq 2 \tag{2.3.1}$$

so $\operatorname{tr}[\xi,\eta] = 2$ implies that $c = 0$ and η is upper-triangular. Thus if $\operatorname{tr}(\rho(K)) = 2$, then ρ is conjugate to an upper-triangular representation.

We may replace ρ by its *semisimplification,* that is the upper-triangular matrices by the corresponding diagonal matrices (their semisimple parts) to obtain a representation by diagonal matrices having the same character:

$$\begin{aligned} \rho(X) &= \begin{bmatrix} \xi & * \\ 0 & \xi^{-1} \end{bmatrix} \\ \rho(Y) &= \begin{bmatrix} \eta & * \\ 0 & \eta^{-1} \end{bmatrix} \\ \rho(XY) &= \begin{bmatrix} \zeta & * \\ 0 & \zeta^{-1} \end{bmatrix}. \end{aligned} \tag{2.3.2}$$

where $\xi\eta = \zeta$. Thus

$$\begin{aligned} x &= \xi + \xi^{-1}, \\ y &= \eta + \eta^{-1}, \\ z &= \zeta + \zeta^{-1}. \end{aligned} \tag{2.3.3}$$

(2.3.3) corresponds to the following factorization (2.3.4) of

$$\kappa(x,y,z) - 2 = x^2 + y^2 + z^2 - xyz - 4.$$

Under the embedding

$$\mathbb{C}[x,y,z] \longrightarrow \mathbb{C}[\xi,\xi^{-1},\eta,\eta^{-1},\zeta,\zeta^{-1}]$$

defined by (2.3.3), the polynomial $\kappa(x,y,z) - 2$ factors:

$$\begin{aligned} &\kappa(\xi+\xi^{-1},\eta+\eta^{-1},\zeta+\zeta^{-1}) - 2 = \\ &\qquad \zeta^{-2}(1-\xi\eta\zeta)(1-\xi^{-1}\eta\zeta)(1-\xi\eta^{-1}\zeta)(1-\xi\eta\zeta^{-1}). \end{aligned} \tag{2.3.4}$$

Given (x,y,z) with $\kappa(x,y,z) = 2$, the triple (ξ,η,ζ) is only defined up to an action of $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. Namely the automorphisms

$$\begin{cases} X & \longmapsto X^{-1} \\ Y & \longmapsto Y \end{cases}$$

$$\begin{cases} X & \longmapsto X \\ Y & \longmapsto Y^{-1} \end{cases}$$

$$\begin{cases} X & \longmapsto X^{-1} \\ Y & \longmapsto Y^{-1} \end{cases}$$

act on (ξ, η, ζ) by:

$$\begin{cases} \xi & \longmapsto \xi^{-1} \\ \eta & \longmapsto \eta \\ \zeta & \longmapsto \zeta^{-1} \end{cases}$$

$$\begin{cases} \xi & \longmapsto \xi \\ \eta & \longmapsto \eta^{-1} \\ \zeta & \longmapsto \zeta^{-1} \end{cases}$$

$$\begin{cases} \xi & \longmapsto \xi^{-1} \\ \eta & \longmapsto \eta^{-1} \\ \zeta & \longmapsto \zeta \end{cases}$$

respectively.

2.4. **The Poisson structure.** As in Goldman [13],§5.3, the automorphisms preserving κ are unimodular and therefore preserve the exterior bivector field as well

$$\begin{aligned} \Xi &= \frac{1}{2\pi^2} d\kappa \cdot \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} \\ &= \frac{1}{2\pi^2}\Bigg((2x - yz)\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} \\ &\qquad + (2y - zx)\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} \\ &\qquad\quad + (2z - xy)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \Bigg) \end{aligned}$$

which restricts to (the dual of) an area form on each level set $\kappa^{-1}(t)$ which is invariant under $\mathrm{Out}(\pi)$. We shall always consider this invariant measure on $\kappa^{-1}(t) \cap \mathbb{R}^3$. In the case when $\kappa^{-1}(t)$ has a rational parametrization by an affine plane, that is, when $t = 2$, the above bivector field is the image of the constant bivector field under the parametrization:

$$\begin{aligned} \mathbb{C} \times \mathbb{C} &\longrightarrow \kappa^{-1}(2) \\ (\xi, \eta) &\longmapsto \begin{bmatrix} e^{\xi} + e^{-\xi} \\ e^{\eta} + e^{-\eta} \\ e^{\xi+\eta} + e^{-(\xi+\eta)} \end{bmatrix} \\ \frac{\partial}{\partial \xi} \wedge \frac{\partial}{\partial \eta} &\longmapsto 2\pi^2 \Xi \end{aligned}$$

The bivector field Ξ defines a *Poisson structure* on the moduli space $\mathbb{C}^3$ for which κ defines a *Casimir function.*

(The coefficient $2\pi^2$ occurs to normalize the area of the compact component C_K of reducible $\mathrm{SU}(2)$-characters to 1.) Here we study the action of $\mathrm{Out}(\pi)$ on the set $\kappa^{-1}(t) \cap \mathbb{R}^3$ of $\mathbb{R}$-points of $\kappa^{-1}(t)$ where $t \in \mathbb{R}$, with respect to this invariant measure.

2.5. **The orthogonal representation.** To describe the structure of the character variety more completely, we use a 3-dimensional orthogonal representation associated to a character $(x, y, z) \in \mathbb{C}^3$. (Compare §4.2 of Goldman [12] and Brumfiel-Hilden [4].) This will be used to identify real characters as characters of representations into the real forms $\mathrm{SL}(2, \mathbb{R})$ and $\mathrm{SU}(2)$ of $\mathrm{SL}(2, \mathbb{C})$.

Consider the complex vector space $\mathbb{C}^3$ with the standard basis $\{e_1, e_2, e_3\}$ and bilinear form defined by the symmetric matrix

$$\mathbb{B} = \begin{bmatrix} 2 & z & y \\ z & 2 & x \\ y & x & 2 \end{bmatrix}.$$

Since

$$\det(\mathbb{B}) = -2\big(\kappa(x, y, z) - 2\big)$$

(where $\kappa(x, y, z)$ is defined in (2.1.4)), the symmetric bilinear form is nondegenerate if and only if $\kappa(x, y, z) \neq 2$.

Assume $\kappa(x, y, z) \neq 2$ so that $\mathbb{B}$ is nondegenerate. Let $\mathrm{SO}(\mathbb{C}^3, \mathbb{B}) \cong \mathrm{SO}(3, \mathbb{C})$ denote the group of unimodular linear transformations of $\mathbb{C}^3$ orthogonal with respect to $\mathbb{B}$. The local isomorphism

$$\Phi : \mathrm{SL}(2, \mathbb{C}) \longrightarrow \mathrm{SO}(\mathbb{C}^3, \mathbb{B}).$$

is a surjective double covering, equivalent to the adjoint representation (or the representation on the second symmetric power of $\mathbb{C}^2$), and is unique up to composition with automorphisms of $\mathrm{SL}(2, \mathbb{C})$ and $\mathrm{SO}(3, \mathbb{C})$.

For any vector $v \in \mathbb{C}^3$ such that $\mathbb{B}(v,v) \neq 0$, the reflection

$$R_v : x \longmapsto x - 2\,\frac{\mathbb{B}(v,x)}{\mathbb{B}(v,v)} v \tag{2.5.1}$$

is a $\mathbb{B}$-orthogonal involution.

Let $\mathbb{Z}/2 \star \mathbb{Z}/2 \star \mathbb{Z}/2$ be the free product of cyclic groups $\langle R_i \rangle$, where $i = 1, 2, 3$, and $R_i^2 = \mathbb{I}$. The homomorphism

$$\begin{aligned} \pi &\longrightarrow \mathbb{Z}/2 \star \mathbb{Z}/2 \star \mathbb{Z}/2 \\ X &\longmapsto \quad R_2R_3 \\ Y &\longmapsto \quad R_3R_1 \end{aligned}$$

embeds π as an index two subgroup of $\mathbb{Z}/2 \star \mathbb{Z}/2 \star \mathbb{Z}/2$.

Lemma 2.5.1. *Let $\rho \in \mathrm{Hom}(\pi, G)$ satisfy $\kappa(x,y,z) \neq 2$. The restriction to π of the representation $\hat{\rho} : \mathbb{Z}/2\star\mathbb{Z}/2\star\mathbb{Z}/2 \longrightarrow \mathrm{SO}(\mathbb{C}^3, \mathbb{B})$ defined by $R_i \longmapsto R_{e_i}$ is conjugate to the composition $\Phi \circ \rho : \pi \longrightarrow \mathrm{SO}(\mathbb{C}^3, \mathbb{B})$.*

See §4.2 of Goldman [12], or Brumfiel-Hilden [4] for a proof.

2.6. **The set of $\mathbb{R}$-points.** A real character is the character of a representation into a *real form* of G. (Proposition III.1.1 of Morgan-Shalen [26]). The above orthogonal representation gives an explicit form of this result.

Suppose $(x, y, z) \in \mathbb{R}^3$ and $\kappa(x, y, z) \neq 2$. Then the restriction of $\mathbb{B}$ to $\mathbb{R}^3$ is a nondegenerate symmetric $\mathbb{R}$-valued bilinear form (also denoted $\mathbb{B}$), which is either indefinite or (positive) definite. Let $\mathrm{SO}(\mathbb{R}^3, \mathbb{B})$ denote the group of unimodular linear transformations of $\mathbb{R}^3$ orthogonal with respect to $\mathbb{B}$.

There are two conjugacy classes of real forms of G, compact and noncompact. Every compact real form of G is conjugate to $\mathrm{SU}(2)$ and every noncompact real form of G is conjugate to $\mathrm{SL}(2,\mathbb{R})$. Specifically, if $\mathbb{B}$ is positive definite, then $\mathrm{SO}(\mathbb{R}^3, \mathbb{B})$ is conjugate to $\mathrm{SO}(3)$ and $\Phi^{-1}(\mathrm{SO}(\mathbb{R}^3, \mathbb{B}))$ is conjugate to $\mathrm{SU}(2)$. If $\mathbb{B}$ is indefinite, then $\mathrm{SO}(\mathbb{R}^3, \mathbb{B})$ is conjugate to either $\mathrm{SO}(2, 1)$ or $\mathrm{SO}(1, 2)$ (depending on whether $\kappa(x, y, z) > 2$ or $\kappa(x, y, z) < 2$ respectively) and $\Phi^{-1}(\mathrm{SO}(\mathbb{R}^3, \mathbb{B}))$ is conjugate to $\mathrm{SL}(2,\mathbb{R})$. Thus every real character $(x, y, z) \in \mathbb{R}^3$ is the character of a representation into either $\mathrm{SU}(2)$ or $\mathrm{SL}(2,\mathbb{R})$.

When $x, y, z \in \mathbb{R}$, the restriction of $\mathbb{B}$ to $\mathbb{R}^3$ is definite if and only if $-2 \leq x, y, z \leq 2$ and $\kappa(x, y, z) < 2$.

The coordinates of characters of representations $\pi_1(M) \longrightarrow \mathrm{SU}(2)$ satisfy:

$$-2 \leq x, y, z \leq 2, \qquad x^2 + y^2 + z^2 - xyz - 2 \leq 2$$

and, for fixed $t \in [-2,2]$, comprise a component of $\kappa^{-1}(t) \cap \mathbb{R}^3$. The other four connected components of $\kappa^{-1}(t) \cap \mathbb{R}^3$ consist of characters of $\mathrm{SL}(2,\mathbb{R})$-representations. These four components are freely permuted by Σ. For $t < -2$, all four components of the relative character variety $\kappa^{-1}(t) \cap \mathbb{R}^3$ consist of characters of $\mathrm{SL}(2,\mathbb{R})$-representations and are freely permuted by Σ. For $t > 2$, the relative character variety $\kappa^{-1}(t) \cap \mathbb{R}^3$ is connected and consists of characters of $\mathrm{SL}(2,\mathbb{R})$-representations.

The two critical values ± 2 of $\mathrm{tr} : G \longrightarrow \mathbb{C}$ deserve special attention. When $t = -2$, a representation $\rho \in \mathrm{Hom}(\pi, G)$ with $\mathrm{tr}\,\rho(K) = -2$ is a regular point of the mapping

$$\begin{aligned} E_K : \mathrm{Hom}(\pi, G) &\longrightarrow G \\ \rho \qquad &\longmapsto \rho(K). \end{aligned}$$

Such a representation is a regular point of the composition

$$\mathrm{tr} \circ E_K : \mathrm{Hom}(\pi, G) \longrightarrow \mathbb{C}$$

unless $\rho(K) = -I$. In the latter case (when $[\rho(X), \rho(Y)] = -I$), the representation is conjugate to the *quaternion representation in* $\mathrm{SU}(2)$

$$\rho(X) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \rho(Y) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{2.6.1}$$

In both cases, G acts locally freely on the subset $(\mathrm{tr} \circ E_K)^{-1}(-2)$ of $\mathrm{Hom}(\pi, G)$ with quotient $\kappa^{-1}(-2)$.

Now consider the case $t = 2$. As in §2.3, the $\mathbb{R}$-points of $\kappa^{-1}(2)$ correspond to reducible representations, and in fact are characters of representations with values in the Cartan subgroups (maximal tori) of the real forms $\mathrm{SU}(2)$ and $\mathrm{SL}(2,\mathbb{R})$. Every Cartan subgroup of $\mathrm{SU}(2)$ is conjugate to $\mathrm{U}(1)$, and every Cartan subgroup of $\mathrm{SL}(2,\mathbb{R})$ is conjugate to either $\mathrm{SO}(2)$ or $\mathrm{SO}(1,1)$. Characters of reducible $\mathrm{SU}(2)$-representations form the compact set

$$C_K = \kappa^{-1}(2) \cap [-2,2]^3$$

which identifies with the quotient of the 2-torus $\mathrm{U}(1) \times \mathrm{U}(1)$ by $\{\pm \mathbb{I}\}$ under the extension (2.3.3).

Characters of reducible $\mathrm{SL}(2,\mathbb{R})$-representations comprise four components (related by $C_i = \sigma_i C_0$ for $i = 1, 2, 3$):

$$\begin{aligned}
C_0 &= \kappa^{-1}(2) \cap \Big([2,\infty) \times [2,\infty) \times [2,\infty) \Big), \\
C_1 &= \kappa^{-1}(2) \cap \Big([2,\infty) \times (-\infty,-2] \times (-\infty,-2] \Big), \\
C_2 &= \kappa^{-1}(2) \cap \Big((-\infty,-2] \times [2,\infty) \times (-\infty,-2] \Big), \\
C_3 &= \kappa^{-1}(2) \cap \Big((-\infty,-2] \times (-\infty,-2] \times [2,\infty) \Big),
\end{aligned}$$

each of which identifies with the quotient of $\mathbb{R}_+ \times \mathbb{R}_+$ by $\{\pm\mathbb{I}\}$ under (2.3.3). These four components are freely permuted by Σ. There is a compact component of equivalence classes of $\mathrm{SL}(2,\mathbb{R})$-representations which are irreducible but not *absolutely irreducible* — that is, although $\mathbb{R}^2$ is an irreducible π-module, its complexification is reducible. In that case the representation is conjugate to a representation in $\mathrm{SO}(2)$. This component agrees with the component C_K consisting of characters of reducible $\mathrm{SU}(2)$-representations.

3. Hyperbolic structures on tori ($t < 2$)

The topology and dynamics change dramatically as t changes from $t < -2$ to $t > 2$. The level sets $t < -2$ correspond to Fricke spaces of one-holed tori with geodesic boundary. The level set for $t = -2$ consists of four copies the Teichmüller space of the punctured torus, together with $\{(0,0,0)\}$. (The origin is the character of the quaternion representation in $\mathrm{SU}(2)$.) The origin is fixed under the action, while Γ acts properly on its complement in $\kappa^{-1}(-2)$. (However the action on the set of *complex points* of $\kappa^{-1}(-2)$ is extremely nontrivial and mysterious; see Bowditch [3].)

The level sets for $-2 < t < 2$ correspond to Teichmüller spaces of singular hyperbolic structures with one singularity, as well as a (compact) component consisting of characters of unitary representations. Except for the component of unitary representations, there are four components, freely permuted by Σ. Except for the component of unitary representations, the Γ-action is proper.

3.1. **Complete hyperbolic structures ($t \leq -2$).**

The theory of deformations of geometric structures implies that Γ acts properly on certain components of $\kappa^{-1}(t)$. When $t < 2$, the moduli space $\kappa^{-1}(t)$ contains 4 contractible noncompact components, freely permuted by Σ.

(When $-2 \leq t < 2$, an additional compact component corresponds to $\mathrm{SU}(2)$-representations.) These contractible components correspond to $\mathrm{SL}(2,\mathbb{R})$-representations, and each one identifies with the Teichmüller space of M, with certain boundary conditions. Specifically, if $t < -2$, then these components correspond to hyperbolic structures on $\mathrm{int}(M)$ with geodesic boundary of length $2\cosh^{-1}(-t/2)$. For $t = -2$, these components correspond to complete hyperbolic structures on $\mathrm{int}(M)$ and identify with the usual Teichmüller space $\mathfrak{T}_M$. For $-2 < t < 2$, these components correspond to singular hyperbolic structures on a torus whose singularity is an isolated point with cone angle

$$\theta = 2\cos^{-1}(-t/2).$$

3.2. **Complete structures and proper actions.** We begin with the case $t < -2$. Then each component of $\kappa^{-1}(t) \cap \mathbb{R}^3$ parametrizes complete hyperbolic structures on $\mathrm{int}(M)$ with a closed geodesic parallel to ∂M having length $2\cosh^{-1}(-t/2)$. Thus the union

$$\bigcup_{t \leq -2} \kappa^{-1}(t)$$

consists of four copies of the Fricke space of M. This space contains equivalence classes of marked complete hyperbolic structures on $\mathrm{int}(M)$, not necessarily of finite area. The properness of the action of the mapping class group $\pi_0(\mathrm{Homeo}(M))$ on the Fricke space of M implies properness of the Γ-action of $\kappa^{-1}(t)$. (For proof, see §2.2 of Abikoff [1], Bers-Gardiner [2] 6.5.6 (p.156) of Buser [5], 6.3 of Imayoshi-Tanigawa [19], 2.4.1 of Harvey [16], or §2.7 of Nag [28].)

When $t = -2$, then $\kappa^{-1}(t)$ has five connected components. It is the union of the single (singular) point $(0,0,0)$ and four copies of the Teichmüller space $\mathfrak{T}_M$ of M. Teichmüller space $\mathfrak{T}_M$ consists of equivalence classes of marked complete finite-area hyperbolic structures on M. The group Γ acts properly on $\kappa^{-1}(t)$ and fixes $(0,0,0)$.

3.3. **Singular hyperbolic structures on tori ($-2 < t < 2$).** We next consider the case $2 > t > -2$. Here $\kappa^{-1}(t)$ has five connected components. One component C_t is compact and consists of unitary characters, while the other four components correspond to singular hyperbolic structures on a torus with a cone point. C_t is diffeomorphic to S^2, and the symplectic structure is a smooth area form. Γ acts ergodically on C_t. See §5 of Goldman [13] for a detailed discussion.

Consider next the four noncompact components of $\kappa^{-1}(t)$. This case is similar to the previous case, except that the ends of M are replaced by cone points on a torus. Using the properness of the mapping class group action on Teichmüller space, the action on these components of the relative character variety remains proper. However, none of the corresponding representations in $\mathrm{Hom}(\pi, G)$ are discrete embeddings. Generically these representations are isomorphisms onto dense subgroups of $\mathrm{SL}(2,\mathbb{R})$.

We first show (Theorem 3.4.1) that every representation in these components is a lift (to $\mathrm{SL}(2,\mathbb{R})$) of the holonomy representation of a singular hyperbolic structure. §3.5 proves that In §3.5), we deduce that Γ acts properly on the level sets $\kappa^{-1}(t)$ where $-2 < t < 2$.

3.4. **Construction of hyperbolic structures on T^2 with one cone point.** Let $\theta > 0$ and let $\mathfrak{C}_\theta$ denote the space of hyperbolic structures on T^2 with a conical singularity of cone angle θ. By results of McOwen [25] and Troyanov [33], $\mathfrak{C}_\theta$ identifies with the Teichmüller space $\mathfrak{T}(\theta)$, the deformation space of conformal structures on T^2 singular at one point with cone angle θ.

Theorem 3.4.1. *Suppose $\xi, \eta \in \mathrm{SL}(2,\mathbb{R})$ and $\gamma := [\xi, \eta]$ is elliptic of rotation angle θ. Let M denote a one-holed torus whose fundamental group has the presentation* (1.1.2). *Then there exists a singular hyperbolic structure on $\bar{M}$ with a singularity of cone angle θ having holonomy representation ρ defined by*

$$\rho(X) = \xi, \;\; \rho(Y) = \eta, \;\; \rho(K) = \gamma.$$

The proof will be based on the following.

Lemma 3.4.2. *Let $p \in \mathbf{H}^2$ be a point fixed by γ and consider the points*

$$\begin{aligned} p_4 &= p, \\ p_3 &= \eta^{-1}p, \\ p_2 &= \xi^{-1}\eta^{-1}p, \\ p_1 &= \eta\xi^{-1}\eta^{-1}p. \end{aligned}$$

The four points p_1, p_2, p_3, p_4 are the vertices of an embedded quadrilateral $\mathbf{Q}$. In other words the four segments

$$\begin{aligned} l_1 &= \overline{p_1p_2} \\ l_2 &= \overline{p_2p_3} \\ l_3 &= \overline{p_3p_4} \\ l_4 &= \overline{p_4p_1} \end{aligned}$$

are disjoint and bound a quadrilateral.

Proof of Theorem 3.4.1 assuming Lemma 3.4.2.

ξ maps p_1 (respectively p_2) to p_4 (respectively p_3). Therefore ξ maps the directed edge l_1 to l_3 with the opposite orientation. Similarly η maps p_2 (respectively p_3) to p_1 (respectively p_4) so η maps l_2 to l_4 with the opposite orientation.

From the embedding of $\mathbf{Q}$ and the identifications of its opposite sides by ξ and η, we construct a developing map for a (singular) hyperbolic structure on M as follows. Let $\mathcal{Q}$ denote an *abstract quadrilateral,* that is, a cell complex with a single 2-cell, four 1-cells (the *sides*), and four 0-cells (the *vertices*). Denote the oriented edges $\mathcal{L}_i$ numbered in cyclic order and the vertices p_i $(i = 1, 2, 3, 4)$ where

$$\partial\mathcal{L}_1 = \{p_1, p_2\}, \quad \partial\mathcal{L}_2 = \{p_2, p_3\}$$
$$\partial\mathcal{L}_3 = \{p_3, p_4\}, \quad \partial\mathcal{L}_4 = \{p_4, p_1\}.$$

Let $\bar{\mathcal{L}}_i$ denote $\mathcal{L}_i$ with the opposite orientation. Write

$$\ddddot{\mathcal{Q}} = \mathcal{Q} - \{p_1, p_2, p_3, p_4\}.$$

Choose orientation-preserving homeomorphisms

$$\tilde{\xi} : \mathcal{L}_1 \longrightarrow \bar{\mathcal{L}}_3, \qquad \tilde{\eta} : \mathcal{L}_2 \longrightarrow \bar{\mathcal{L}}_4.$$

Let $\Pi = \langle X, Y\rangle$ denote the free group generated by X, Y. A model for the universal covering $\tilde{M}$ of M is the quotient space of the Cartesian product

$$\ddddot{\mathcal{Q}} \times \Pi$$

by the equivalence relation generated by the identifications

$$(x, \eta) \longleftrightarrow (\tilde{\xi}(x), X\eta)$$

for $x \in \mathcal{L}_1$, and

$$(x, \eta) \longleftrightarrow (\tilde{\eta}(x), Y\eta)$$

for $x \in \mathcal{L}_2$. The action of Π on $\mathcal{Q} \times \Pi$ defined by the trivial action on $\mathcal{Q}$ and right-multiplication on Π descends to a free proper action of Π on $\tilde{M}$. This action corresponds to the action of deck transformations. Let $\rho : \Pi \longrightarrow G$ be the homomorphism defined by:

$$\rho(X) = \xi$$
$$\rho(Y) = \eta$$

The embedding $\ddddot{\mathcal{Q}} \longrightarrow \mathbf{Q}$ together with the identifications of the sides of $\mathbf{Q}$ extends uniquely to a ρ-equivariant local homeomorphism $\tilde{M} \longrightarrow \mathbf{H}^2$. The holonomy around the puncture

$$\rho([X, Y]) = [\xi, \eta] = \gamma$$

is elliptic of rotation angle θ fixing p. The resulting hyperbolic structure extends to a singular hyperbolic structure on $\bar{M}$ with a singularity of cone angle θ. Thus Theorem 3.4.1 reduces to Lemma 3.4.2. □

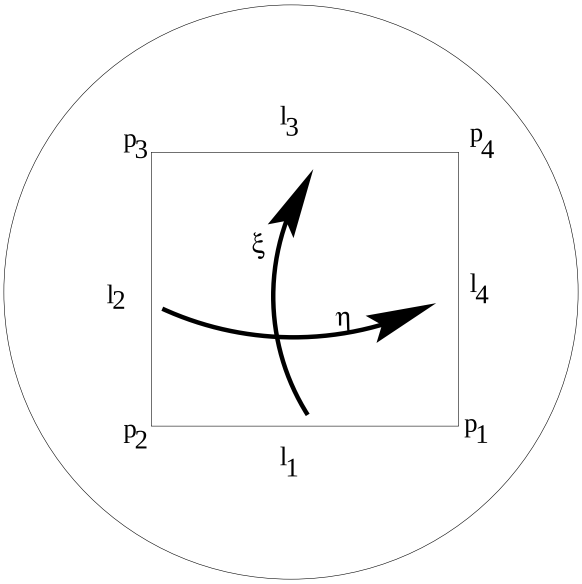

FIGURE 9. A convex embedded quadrilateral

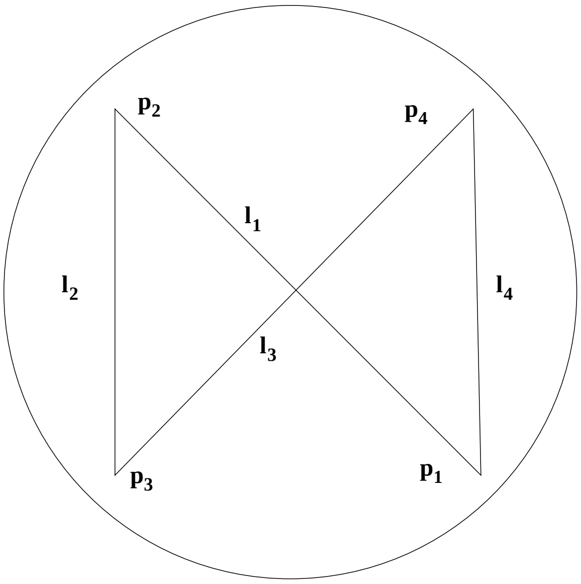

FIGURE 10. When $\overline{p_3p_4}$ meets $\overline{p_1p_2}$

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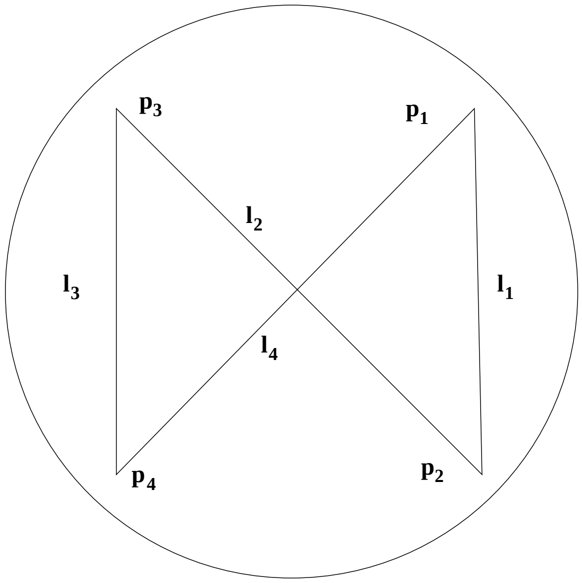

FIGURE 11. When $\overline{p_2p_3}$ meets $\overline{p_4p_1}$

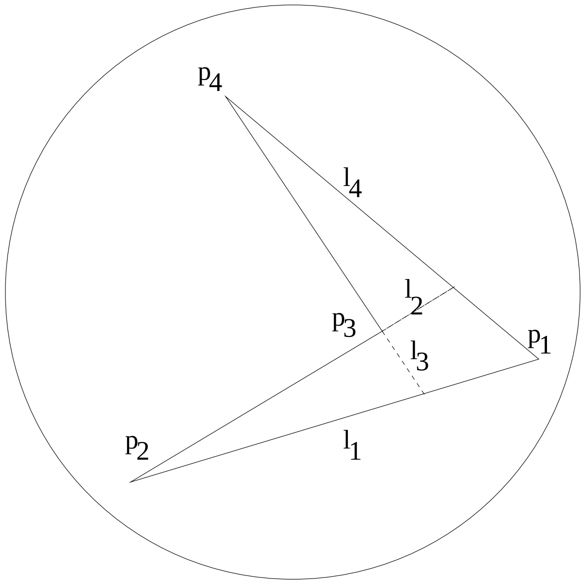

FIGURE 12. A nonconvex embedded quadrilateral

The proof of Lemma 3.4.2 is based on the following two lemmas.

Lemma 3.4.3. *The points p_1, p_2, p_3, p_4 are not collinear.*

Lemma 3.4.4. *If the sequence of points p_1, p_2, p_3, p_4 are not the vertices of an embedded quadrilateral, then either l_1 and l_3 intersect, or l_2 and l_4 intersect.*

Proof of Lemma 3.4.3: Suppose that p_1, p_2, p_3, p_4 are collinear. If more than one line contains p_1, p_2, p_3, p_4, then all the points coincide and ρ fixes this point, contradicting $\gamma \neq \mathbb{I}$. Thus a unique line l contains p_1, p_2, p_3, p_4. We claim that l is ρ-invariant.

If $p_3 = p_4$, then η fixes p and since $\gamma = \xi\eta\xi^{-1}\eta^{-1}$ fixes p, so must $\xi\eta\xi^{-1}$ fix p. Since $\gamma = [\xi, \eta]$ is nontrivial, both ξ and η are nontrivial. Thus ξ maps the unique fixed point of η to the unique fixed point of $\xi\eta\xi^{-1}$, and both ξ and η fix p, contradicting $\gamma \neq \mathbb{I}$. Thus $p_3 \neq p_4$ and l is the unique line containing p_3 and p_4.

$\xi(p_2) = p_3$ and $\xi(p_1) = p_4$ imply that $p_1 \neq p_2$. Thus l is the unique line containing p_1 and p_2, and is ξ-invariant. Similarly l is η-invariant and thus $\rho(\pi)$-invariant, as claimed.

The commutator subgroup of the stabilizer of a line l consists of hyperbolic elements, contradicting ellipticity of γ. $\square$

Proof of Lemma 3.4.4: If $x, y \in \mathbf{H}^2$ are distinct points, let $\overleftrightarrow{xy}$ denote the line passing through x and y. If $z \notin \overleftrightarrow{xy}$, let $H_{x,y}(z)$ denote the component (a half-space) of $\mathbf{H}^2 - \overleftrightarrow{xy}$ containing z, and $\hat{H}_{x,y}(z)$ denote the complementary component so that:

$$\mathbf{H}^2 = H_{x,y}(z) \amalg \overleftrightarrow{xy} \amalg \hat{H}_{x,y}(z)$$

Suppose first that no three of the points p_1, p_2, p_3, p_4 are collinear. There are four cases, depending on where p_4 lies in relation to the decompositions determined by the lines $\overleftrightarrow{p_1p_2}$ and $\overleftrightarrow{p_2p_3}$:

(1) $p_4 \in H_{p_1p_2}(p_3) \cap H_{p_2p_3}(p_1)$;
(2) $p_4 \in \hat{H}_{p_1p_2}(p_3) \cap H_{p_2p_3}(p_1)$;
(3) $p_4 \in H_{p_1p_2}(p_3) \cap \hat{H}_{p_2p_3}(p_1)$;
(4) $p_4 \in \hat{H}_{p_1p_2}(p_3) \cap \hat{H}_{p_2p_3}(p_1)$.

In the first case p_1, p_2, p_3, p_4 are the vertices of a convex quadrilateral. (Compare Figure 9.) In the second case, $\overline{p_3p_4}$ meets $\overline{p_1p_2}$. (Compare Figure 10.) In the third case, $\overline{p_2p_3}$ meets $\overline{p_4p_1}$. (Compare Figure 11.) In the fourth case p_1, p_2, p_3, p_4 are the vertices of an embedded (non-convex) quadrilateral. (Compare Figure 12.)

Suppose next that three of the vertices are collinear. By Lemma 3.4.3, not all four vertices are collinear. By possibly conjugating ρ by ξ, η etc., we may assume that p_1, p_2, p_3 are collinear. If p_2 lies between p_1 and p_3, then p_1, p_2, p_3, p_4 are the vertices of an embedded quadrilateral (with a straight angle at p_2). Otherwise $\overline{p_2p_3}$ meets $\overline{p_4p_1}$ at p_1. This completes the proof of Lemma 3.4.4. $\square$

Returning to the proof of Lemma 3.4.2, we show that l_1 and l_3 cannot intersect; an identical proof implies l_2 and l_4 cannot intersect.

Claim. *Suppose that l_1 and l_3 intersect. Then $l_1 \cap l_3$ is a point q and the triangles $\triangle(qp_1p_4)$ and $\triangle(qp_3p_2)$ are congruent.*

Proof of Claim. If $l_1 \cap l_3$ is not a point, then it is a segment, contradicting Lemma 3.4.3. Since $\xi(l_1) = l_3$ and $\eta(l_2) = l_4$, the lengths of opposite sides are equal:

$$\begin{aligned} d(p_1, p_2) = \ell(l_1) = \ell(l_3) = d(p_4, p_3), \\ d(p_2, p_3) = \ell(l_2) = \ell(l_4) = d(p_1, p_4) \end{aligned}$$

Since the lengths of the corresponding sides are equal, the triangle $\triangle(p_2p_1p_4)$ is congruent to $\triangle(p_4p_3p_2)$. In particular

$$\angle(p_2p_1p_4) = \angle(p_4p_3p_2).$$

Similarly, $\angle(p_3p_4p_1) = \angle(p_1p_2p_3)$. Now

$$\angle(qp_1p_4) = \angle(p_2p_1p_4) = \angle(p_4p_3p_2) = \angle(qp_3p_2)$$

and similarly, $\angle(qp_4p_1) = \angle(qp_2p_3)$. Since $l_4 = \overline{p_1p_4}$ is congruent to $l_2 = \overline{p_3p_2}$, triangles $\triangle(qp_1p_4)$ and $\triangle(qp_3p_2)$ are congruent as claimed. $\square$

Now let l be the line through q bisecting the angle $\angle(p_1qp_3)$ such that reflection R in l interchanges $\triangle(qp_1p_4)$ and $\triangle(qp_3p_2)$. Since

$$R : \begin{cases} p_1 & \longmapsto p_3 \\ p_2 & \longmapsto p_4 \\ p_3 & \longmapsto p_1 \\ p_4 & \longmapsto p_2, \end{cases}$$

$$\xi : \begin{cases} p_1 & \longmapsto p_4 \\ p_2 & \longmapsto p_3 \end{cases}$$

$$\eta : \begin{cases} p_2 & \longmapsto p_1 \\ p_3 & \longmapsto p_4, \end{cases}$$

$\xi \circ R$ interchanges p_3 and p_4, and $R \circ \eta$ interchanges p_2 and p_3. Since an orientation-reversing isometry of $\mathbf{H}^2$ which interchanges two points must be reflection in a line, $\xi \circ R$ and $R \circ \eta$ have order two. Thus R conjugates ξ to ξ^{-1} and η to η^{-1}.

One of two possibilities must occur:

- At least one of ξ and η is elliptic or parabolic;
- Both ξ and η are hyperbolic, and their invariant axes are each orthogonal to l.

Neither possibility occurs, due to the following:

Lemma 3.4.5. *Let $\xi, \eta \in \mathrm{SL}(2,\mathbb{R})$. The following conditions are equivalent:*

- $\mathrm{tr}[\xi,\eta] < 2$*;*
- *ξ, η are hyperbolic elements and their invariant axes cross.*

Proof. Assuming $\mathrm{tr}[\xi,\eta] < 2$, we first show that ξ and η must be hyperbolic. We first show that ξ is not elliptic. If ξ is elliptic (or $\pm\mathbb{I}$), we may assume that $\xi \in \mathrm{SO}(2)$, that is, we represent ξ, η by matrices

$$\xi = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \quad \eta = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where $ad - bc = 1$, whence

$$\mathrm{tr}[\xi,\eta] = 2 + \sin^2(\theta)(a^2 + b^2 + c^2 + d^2 - 2) \geq 2.$$

Similarly if ξ is parabolic, (2.3.1) implies that $\mathrm{tr}[\xi,\eta] \geq 2$.

Thus ξ is hyperbolic. Since $[\eta,\xi] = [\xi,\eta]^{-1}$, an identical argument shows that η is hyperbolic. Denote their invariant axes by l_ξ, l_η respectively.

It remains to show that $\mathrm{tr}[\xi,\eta] < 2$ if and only if $l_\xi \cap l_\eta \neq \emptyset$.

By conjugation, we may assume that the fixed points of ξ are ± 1 and that the fixed points of η are r, ∞. Thus $l_\xi \cap l_\eta \neq \emptyset$ if and only if $-1 < r < 1$. Represent ξ, η by matrices

$$\xi = \begin{bmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{bmatrix}, \quad \eta = \begin{bmatrix} e^{\phi} & -2r\sinh(\phi) \\ 0 & e^{-\phi} \end{bmatrix},$$

where $\theta, \phi \neq 0$ and

$$\mathrm{tr}[\xi,\eta] = 2 + 4(r^2 - 1)\sinh^2(\theta)\sinh^2(\phi).$$

Then $\mathrm{tr}[\xi,\eta] < 2$ if and only if $-1 < r < 1$ and $\mathrm{tr}[\xi,\eta] = 2$ if and only if $r = \pm 1$, as desired. □

This completes the proof of Lemma 3.4.2 (and also Theorem 3.4.1). □

3.5. **Properness.** That Γ acts properly on $\kappa^{-1}(t) \cap \mathbb{R}^3$ now follows easily. The construction in Theorem 3.4.1 gives a map

$$\kappa^{-1}(t) \cap [2, \infty)^3 \longrightarrow \mathfrak{T}(\theta) \tag{3.5.1}$$

which is evidently $\mathrm{Out}(\pi)$-equivariant. Every hyperbolic structure with conical singularities has an underlying *singular conformal structure,* where the singularities are again conical singularities, that is, they are defined by local coordinate charts to a model space, which in this case is a cone. However, there is an important difference. Conical singularities in conformal structures are *removable,* while conical singularities in Riemannian metrics are not.

Here is why conformal conical singularities are removable: Let $D^2 \subset \mathbb{C}$ be the unit disk and let

$$S_\theta := \{z \in D^2 \mid 0 \leq \arg(z) \leq \theta\}$$

be a sector of angle θ. Then

$$\begin{aligned} \Pi_\theta : S_\theta - \{0\} &\longrightarrow D^2 - \{0\} \\ z &\longmapsto z^{2\pi/\theta} \end{aligned}$$

is conformal. The model coordinate patch for a cone point of angle θ is the *cone* C_θ of angle θ, defined as identification space C_θ of S_θ by the equivalence relation defined by

$$z \longleftrightarrow e^{\pm i\theta} z$$

for $z \in \partial S_\theta$. That is, a cone point p has a coordinate patch neighborhood U and a coordinate chart $\psi : U \longrightarrow C_\theta$ in the atlas defining the singular geometric structure. The power map Π_θ defines a conformal isomorphism between the C_θ punctured at the cone point and a punctured disk (a "cone" of angle 2π). Replacing the coordinate chart $\psi : U \longrightarrow \mathbb{C}$ at a cone point p of angle θ by the composition $\Pi_\theta \circ \psi$ gives a coordinate atlas for a conformal structure which is nonsingular at p and isomorphic to the original structure on the complement of p.

The resulting map $\mathfrak{T}(\theta) \longrightarrow \mathfrak{T}_M$ is evidently $\mathrm{Out}(\pi)$-equivariant. Since $\mathrm{Out}(\pi)$ acts properly on $\mathfrak{T}_M$, $\mathrm{Out}(\pi)$ acts properly on $\mathfrak{T}(\theta)$, and hence on $\kappa^{-1}(t) \cap \mathbb{R}^3$ as well.

With more work one can show that (3.5.1) is an isomorphism. For any hyperbolic structure on T^2 with a cone point p of angle $0 < \theta < 2\pi$, the Arzelá-Ascoli theorem (as in Buser [5] §1.5) applies to represent ξ, η by geodesic loops based at p. Furthermore these geodesics intersect only at p. By developing this singular geometric structure one obtains a polygon $\mathbf{Q}$ as above.

4. Reducible characters ($t = 2$)

The level set $\kappa^{-1}(2)$ consists of characters of reducible representations. Over $\mathbb{C}$ such a representation is upper-triangular (2.3.2) with character defined by (2.3.3). By (2.3.4), every $(x, y, z) \in \kappa^{-1}(2)$ lies in the image of the map

$$\begin{aligned} \Phi : \mathbb{C}^* \times \mathbb{C}^* &\longrightarrow \kappa^{-1}(2) \\ (\xi, \eta) &\longmapsto \begin{bmatrix} \xi + \xi^{-1} \\ \eta + \eta^{-1} \\ \xi\eta + \xi^{-1}\eta^{-1} \end{bmatrix}. \end{aligned} \tag{4.0.2}$$

The set of $\mathbb{R}$-points $\kappa^{-1}(2) \cap \mathbb{R}^3$ is a singular algebraic hypersurface in $\mathbb{R}^3$, with singular set

$$S_0 = \left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} \right\}.$$

Characters in S_0 correspond to unipotent representations twisted by central characters (as in the sense of (2.2.1)). A *central character* is a homomorphism taking values in the center $\{\pm\mathbb{I}\}$ of G and a *unipotent representation* is a representation in $\pm U$, where U is a unipotent subgroup of G. A reductive representation with character in S_0 is itself a central character. The most general representation with character in S_0 is one taking values in $\pm U$, where U is a unipotent subgroup of G. The character $(2, 2, 2)$ is the character of any unipotent representation, for example the trivial representation. The other three points are images of $(2, 2, 2)$ by the three nontrivial elements of Σ.

The smooth stratum of $\kappa^{-1}(2)$ is the complement

$$\kappa^{-1}(2) \cap \mathbb{R}^3 - S_0.$$

Denote its five components by C_K and C_i, where $i = 0, 1, 2, 3$. Here C_0 denotes the component $\kappa^{-1}(2) \cap [2, \infty)^3$ and $C_i = (\sigma_i)_* C_0$ where $(\sigma_i)_* \in \Sigma$ is the sign-change automorphism fixing the i-th coordinate (§2.2.1).

The component C_K corresponds to reducible $\mathrm{SU}(2)$-representations which are *non-central,* that is, their image does not lie in the center $\{\pm\mathbb{I}\}$ of $\mathrm{SU}(2)$. The closure of C_K is the union of C_K with S_0. The map

$$\begin{aligned} \mathrm{U}(1) \times \mathrm{U}(1) &\longrightarrow \bar{C}_K \\ (\xi, \eta) &\longmapsto (\xi + \xi^{-1}, \eta + \eta^{-1}, \xi\eta + \xi^{-1}\eta^{-1}) \end{aligned}$$

is a double branched covering space, with deck transformation

$$(\xi, \eta) \longmapsto (\xi^{-1}, \eta^{-1})$$

and four branch points

$$(\xi, \eta) = (\pm 1, \pm 1)$$

which map to S_0.

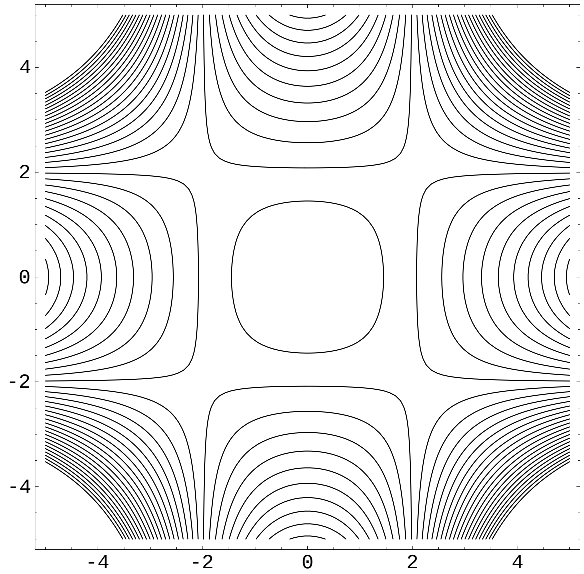

FIGURE 13. Level sets of $(x^2 - 4)(y^2 - 4)$

Similarly, each of the other components identifies with the quotient $(\mathbb{R}_+)^2/\{\pm\mathbb{I}\}$ with the action of $\mathrm{GL}(2, \mathbb{Z})$. For example, $\bar{C}_0$ is the image of the double branched covering

$$\begin{aligned} \mathbb{R}_+ \times \mathbb{R}_+ &\longrightarrow \bar{C}_0 \\ (\xi, \eta) &\longmapsto (\xi + \xi^{-1}, \eta + \eta^{-1}, \xi\eta + \xi^{-1}\eta^{-1}), \end{aligned}$$

with deck transformation

$$(\xi, \eta) \longmapsto (\xi^{-1}, \eta^{-1}).$$

For $i = 1, 2, 3$, the component C_i is the image of C_0 under the sign-change $(\sigma_i)_*$. Composing with the $\mathrm{GL}(2, \mathbb{Z})$-equivariant diffeomorphisms

$$\begin{aligned} \mathbb{R}^2 &\longrightarrow \mathbb{R}_+ \times \mathbb{R}_+ \\ (\tilde{\xi}, \tilde{\eta}) &\longmapsto (\exp(\tilde{\xi}), \exp(\tilde{\eta})) \end{aligned}$$

and

$$\begin{aligned}(\mathbb{R}/\mathbb{Z})^2 &\longrightarrow \mathrm{U}(1)\times\mathrm{U}(1)\\ (\tilde{\xi},\tilde{\eta}) &\longmapsto (\exp(2\pi i\tilde{\xi}),\exp(2\pi i\tilde{\eta}))\end{aligned}$$

respectively, yields $\mathrm{GL}(2,\mathbb{Z})$-equivariant double branched coverings

$$\mathbb{R}^2 \longrightarrow C_i, \text{ and } (\mathbb{R}/\mathbb{Z})^2 \longrightarrow C_K$$

respectively. Furthermore these mappings pull back the invariant area form on $\kappa^{-1}(2)\cap\mathbb{R}^3$ to Lebesgue measure on $\mathbb{R}^2$.

Since $\mathrm{SL}(2,\mathbb{Z})$ is a lattice in $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{R})$ acts transitively on $\mathbb{R}^2$ with noncompact isotropy group, Moore's ergodicity theorem (Moore [27]; see also Feres [7], Zimmer [34] or Margulis [24]) implies that $\mathrm{SL}(2,\mathbb{Z})$ acts ergodically on $\mathbb{R}^2$. Thus $\mathrm{GL}(2,\mathbb{Z})$ acts ergodically on $\mathbb{R}^2/\{\pm\mathbb{I}\}$ and $(\mathbb{R}/\mathbb{Z})^2/\{\pm\mathbb{I}\}$, and hence on each of the components

$$C_0, C_1, C_2, C_3, C_K \subset \kappa^{-1}(2)\cap\mathbb{R}^3.$$

Since Σ permutes C_0, C_1, C_2, C_3, the Γ-action on their union

$$\begin{aligned}C_0\cup C_1\cup C_2\cup C_3 &= \kappa^{-1}(2)\cap\mathbb{R}^3 - C_K\cup S_0\\ &\cong (\mathbb{R}^*\times\mathbb{R}^*)/\{\pm\mathbb{I}\}\end{aligned}$$

is ergodic. (In the case of C_K, any hyperbolic element in $\mathrm{GL}(2,\mathbb{Z})$ acts ergodically on $\mathrm{U}(1)\times\mathrm{U}(1)$ and hence on $\bar{C}_0$, a much stronger result.)

5. Three-holed spheres and ergodicity $(t > 2)$

Next we consider the level sets where $t > 2$. There is an important difference between the cases when $t > 18$ and $2 < t \leq 18$. When $t \leq 18$, the Γ-action is ergodic, but when $t > 18$, wandering domains appear, arising from the Fricke spaces of a three-holed sphere P ("pair-of-pants"). The three-holed sphere is the only other orientable surface homotopy-equivalent to a one-holed torus, and homotopy equivalences to hyperbolic manifolds homeomorphic to P define points in these level sets when $t > 18$, which we call *discrete P-characters* However, the Γ-action on the complement of the discrete P-characters is ergodic.

5.1. **The Fricke space of a three-holed sphere.** When $t > 18$, the octant

$$\Omega_0 = (-\infty,-2)\times(-\infty,-2)\times(-\infty,-2)$$

intersects $\kappa^{-1}(t)$ in a wandering domain and the images of $\Omega_0\cap\kappa^{-1}(t)$ are freely permuted by Γ. Let

$$\Omega = \Gamma\cdot\Omega_0 o.$$

Characters in Ω correspond to discrete embeddings $\rho : \pi \longrightarrow \mathrm{SL}(2,\mathbb{R})$ where the complete hyperbolic surface $\mathbf{H}^2/\rho(\pi)$ is diffeomorphic to a three-holed sphere. We call such a discrete embedding a *discrete P-embedding,* and its character a *discrete P-character.*

The fundamental group $\pi_1(P)$ is free of rank two. A pair of boundary components ∂_1, ∂_2, an orientation on P, and a choice of arcs α_1, α_2 from the basepoint to ∂_1, ∂_2 determines a pair of free generators of $\pi_1(P)$:

$$X := (\alpha_1)^{-1} \star \partial_1 \star \alpha_1$$
$$Y := (\alpha_2)^{-1} \star \partial_2 \star \alpha_2.$$

A third generator $Z := (XY)^{-1}$ corresponds to the third boundary component, obtaining a presentation of $\pi_1(P)$ as

$$\langle X, Y, Z \mid XYZ = 1\rangle.$$

Elements of Ω_0 are discrete P-characters such that the generators $X, Y, Z := (XY)^{-1}$ of π correspond to the boundary components of the quotient hyperbolic surface $\mathbf{H}^2/\rho(\pi)$.

Lemma 5.1.1. *A representation $\rho \in \mathrm{Hom}(\pi, \mathrm{SL}(2,\mathbb{C}))$ has character $[\rho] = (x, y, z) \in \Gamma \cdot \Omega_0$ if and only if ρ is a discrete P-embedding such that X, Y, Z correspond to the components of $\partial\mathbf{H}^2/\rho(\pi)$.*

Proof. The condition that $[\rho] \in \Omega_0$ is equivalent to $x, y, z < -2$, which implies that the generators $\rho(X)$, $\rho(Y)$ and $\rho(XY)$ are hyperbolic and their invariant axes $l_{\rho(X)}, l_{\rho(Y)}, l_{\rho(XY)}$ are pairwise ultraparallel. Denoting the common perpendicular to two ultraparallel lines l, l' by $\perp (l, l')$, the six lines

$$l_{\rho(X)},\ \perp\big(l_{\rho(X)}, l_{\rho(Y)}\big),\ l_{\rho(Y)},\ \perp\big(l_{\rho(Y)}, l_{\rho(XY)}\big),\ l_{\rho(XY)},\ \perp\big(l_{\rho(XY)}, l_{\rho(X)}\big)$$

bound a right-angled hexagon H. The union of H with its reflected image in $\perp \big(l_{\rho(X)}, l_{\rho(Y)}\big)$ is a fundamental domain for $\rho(\pi)$ acting on $\mathbf{H}^2$. (Goldman [11], Gilman-Maskit [10], I-7, p. 15). (Figure 14 depicts the identifications corresponding to the generators $\rho(X), \rho(Y)$.) The quotient is necessarily homeomorphic to a three-holed sphere P and the holonomy around components of ∂P are the three generators $\rho(X), \rho(Y), \rho(XY)$. $\square$

Recall (§2.2.2) that Γ decomposes as the semidirect product

$$\Gamma \cong \Gamma_{(2)} \ltimes \mathfrak{S}_3.$$

The mapping class group of a three-holed sphere P is isomorphic to $\mathbb{Z}/2 \times \mathfrak{S}_3$. The $\mathfrak{S}_3$-factor corresponds to the group of permutations of $\pi_0(\partial P)$. The $\mathbb{Z}/2$-factor is generated by the elliptic involution which

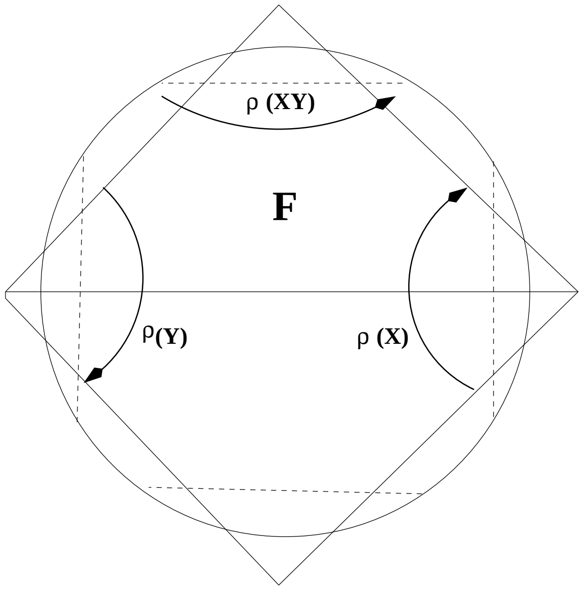

FIGURE 14. Fundamental hexagons for a hyperbolic three-holed sphere

acts by an orientation-reversing homeomorphism of P, whose fixed-point set is the union of three disjoint arcs joining the boundary components. (See §2.2.3 for the corresponding automorphism of π.) Accordingly, $\mathfrak{S}_3$ preserves the Teichmüller space of P. The following result shows that these are the only automorphisms of the character variety which preserve the discrete P-characters.

Proposition 5.1.2. *Ω equals the disjoint union $\coprod_{\gamma\in\Gamma_{(2)}} \gamma\Omega_0$.*

Proof. We show that if for some $\gamma \in \Gamma_{(2)}$, the intersection $\Omega_0 \cap \gamma\Omega_0$ is nonempty, then $\gamma = 1$. Suppose that $[\rho] \in \Omega_0 \cap \gamma\Omega_0$. By Lemma 5.1.1, both ρ and $\rho \circ \gamma$ are discrete P-embeddings such that $X, Y, (XY)^{-1}$ and $\gamma(X), \gamma(Y), \gamma((XY)^{-1})$ correspond to $\partial\mathbf{H}^2/\rho(\pi)$.

The automorphism γ of the character space $\mathbb{C}^3$ corresponds to an automorphism $\tilde{\gamma}$ of π such that

$$[\rho \circ \tilde{\gamma}] = \sigma_* \circ \gamma([\rho])$$

for a sign-change automorphism $\sigma_* \in \Sigma$. Thus $\rho \circ \tilde{\gamma}$ is also a discrete P-embedding, with quotient bounded by curves corresponding to $\tilde{\gamma}(X), \tilde{\gamma}(Y)$ and $\tilde{\gamma}(XY)$. Then $\tilde{\gamma}(X)$ (respectively $\tilde{\gamma}(Y)$,$\tilde{\gamma}(XY)$) is conjugate to $X^{\pm1}$ (respectively $Y^{\pm1}$, $(XY)^{\pm1}$). Such an automorphism is induced by a diffeomorphism of the three-holed sphere P. Since the elliptic involution generates the mapping class group of P, $\tilde{\gamma}$ must be an inner automorphism possibly composed with the elliptic involution of π, and hence must act trivially on characters. Thus $\gamma = 1$ as desired. $\square$

For $t \leq 18$, the domain Ω does not meet $\kappa^{-1}(t)$. (The closure $\bar{\Omega}$ intersects $\kappa^{-1}(18)$ in the Γ-orbit of the character $(-2,-2,-2)$ of the holonomy representation of a complete finite-area hyperbolic structure on P.) For $t > 18$, observe that $\kappa^{-1}(t) - \Omega$ contains the open subset

$$\Big((-2,2) \times \mathbb{R} \times \mathbb{R}\Big) \cap \kappa^{-1}(t).$$

Thus $\kappa^{-1}(t) - \Omega$ has positive measure in $\kappa^{-1}(t)$.

Proposition 5.1.3. *For any $t > 2$, the action of Γ on $\kappa^{-1}(t) - \Omega$ is ergodic.*

The proof uses an iterative procedure (Theorem 5.2.1) due to Kern-Isberner and Rosenberger [18], although their proof contains a gap near the end. See also Gilman and Maskit [10]. Theorem 5.2.1 is proved at the end of the paper.

5.2. **The equivalence relation defined by Γ.** Write $u \sim v$ if there exists $\gamma \in \Gamma$ such that $\gamma u = v$. Since κ is Γ-invariant, $u \sim v$ implies that $\kappa(u) = \kappa(v)$.

Theorem 5.2.1. *Suppose that $u \in \mathbb{R}^3$ satisfies $\kappa(u) > 2$. Then there exists $(\tilde{x}, \tilde{y}, \tilde{z}) = \tilde{u} \sim u$ such that either*

- *$\tilde{u} \in (-\infty, -2]^3$, in which case $\tilde{u}$ is the character of a Fuchsian representation whose quotient is a hyperbolic structure on a three-holed sphere P, with boundary mapping to either cusps or closed geodesics;*
- *$\tilde{x} \in [-2, 2]$, in which case $\tilde{u}$ is the character of a representation mapping X to a non-hyperbolic element.*

Recall that a measurable equivalence relation is *ergodic* if the only invariant measurable sets which are unions of equivalence classes are either null or conull. Equivalently an equivalence relation is ergodic if and only if every function constant on equivalence classes is constant almost everywhere. A group action defines an equivalence relation. However, equivalence relations are more flexible since every subset of a space with an equivalence relation inherits an equivalence relation (whether it is invariant or not). Suppose that S is a measurable subset of a measure space X. If every point in X is equivalent to a point of a measurable subset of S, then ergodicity of the equivalence relation on X is equivalent to ergodicity of S with respect to the measure class induced from X.

Proof of Proposition 5.1.3 assuming Theorem 5.2.1. For $i = 1, 2$, let $\mathfrak{E}_t^{(i)}$ denote the subset of $\kappa^{-1}(t) - \Omega$ where at least i of the coordinates lie in $[-2, 2]$. Suppose that $t > 2$ and $u \in \big(\kappa^{-1}(t) - \Omega\big)$. Then Theorem 5.2.1 implies that $\Gamma u \cap \mathfrak{E}_t^{(1)} \neq \emptyset$. Thus ergodicity of the Γ-action on $\kappa^{-1}(t) - \Omega$ is equivalent to the ergodicity of the induced equivalence relation on $\mathfrak{E}_t^{(1)}$.

We now reduce to the equivalence relation on $\mathfrak{E}_t^{(2)}$. By applying a permutation we may assume that $-2 < x < 2$.

The level set
$$E(x_0) := \kappa^{-1}(t) \cap x^{-1}(x_0)$$
is defined by
$$\frac{2 - x_0}{4}(y + z)^2 + \frac{2 + x_0}{4}(y - z)^2 = t - 2 + x_0^2$$
and is an ellipse since $-2 < x_0 < 2$. Furthermore the symplectic measure on $\kappa^{-1}(t)$ disintegrates under the map $x : \kappa^{-1}(t) \longrightarrow \mathbb{R}$ to τ_X-invariant Lebesgue measure on $E(x_0)$. In particular the Dehn twist τ_X (see §2.2.3) acts by the linear map
$$\begin{bmatrix} y \\ z \end{bmatrix} \longrightarrow \begin{bmatrix} x & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix},$$
which is linearly conjugate to a rotation of the circle through angle $\cos^{-1}(x/2)$. Thus for almost every $x_0 \in (-2, 2)$ the Dehn twist τ_X acts on $E(x_0)$ by a rotation of infinite order, and the action is ergodic. Furthermore by applying powers of τ_X, we may assume that $-2 < y < 2$ as well. Thus ergodicity of the Γ-action on $\kappa^{-1}(t) - \Omega$ is equivalent to the ergodicity of the induced equivalence relation on $\mathfrak{E}_t^{(2)}$.

Since Γ acts by polynomial transformations over $\mathbb{Z}$, those points of $\kappa^{-1}(t)$ which are equivalent to a point with $\cos^{-1}(x/2)$ is rational comprise a set of measure zero. We henceforth restrict to the complement of this set.

The quadratic reflection (see §2.2.3)
$$Q_z : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} x \\ y \\ xy - z \end{bmatrix}$$
is the deck transformation for the double covering of $\kappa^{-1}(t)$ given by projection $\Pi_{(x,y)}$ to the (x, y)-plane. The image
$$\Pi_{(x,y)} : \kappa^{-1}(t) \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

is the region

$$\mathcal{R}_t := \big\{(x,y) \in \mathbb{R}^2 \mid (x^2-4)(y^2-4) + t - 2 \geq 0\big\}$$

and $\Pi_{(x,y)} : \kappa^{-1}(t) \longrightarrow \mathcal{R}_t$ is the quotient map for the action of Q_z. Thus ergodicity on $\mathfrak{E}_t^{(2)}$ reduces to ergodicity of the induced equivalence relation on

$$\Pi_{(x,y)}(\mathfrak{E}_t^{(2)}) = [-2,2] \times [-2,2].$$

Ergodicity now follows as in §5.2 of [13]. Suppose that $f : \kappa^{-1}(t) - \Omega \longrightarrow \mathbb{R}$ is a Γ-invariant measurable function. The ergodic decomposition for the equivalence relation induced by the cyclic group $\langle \tau_X \rangle$ is the coordinate function

$$x : [-2,2] \times [-2,2] \longrightarrow [-2,2],$$

and by ergodicity of $\langle \tau_X \rangle$ on the level sets of x, there is a measurable function $g : [-2,2] \longrightarrow \mathbb{R}$ such that f factors as $f = g \circ x$ almost everywhere. Applying the cyclic group $\langle \tau_Y \rangle$ to $[-2,2]$, the function g is constant almost everywhere. Hence f is constant almost everywhere, proving ergodicity. $\square$

5.3. **The trace-reduction algorithm.** The proof of Theorem 5.2.1 is based on the following:

Lemma 5.3.1. *Let $2 < x \leq y \leq z$. Suppose that $\kappa(x,y,z) > 2$. Let $z' = xy - z$. Then $z - z' > 2\sqrt{\kappa(x,y,z) - 2}$.*

The following expression for $\kappa(x,y,z)$ will be useful:

$$\begin{aligned} \kappa(x,y,z) - 2 &= \frac{1}{4}\left\{(2z - xy)^2 - (x^2-4)(y^2-4)\right\} \\ &= \frac{1}{4}\left\{(z-z')^2 - (x^2-4)(y^2-4)\right\}. \end{aligned} \tag{5.3.1}$$

For fixed $x, y > 2$, write $\kappa_{x,y} : \mathbb{R} \to \mathbb{R}$ for the quadratic function

$$\kappa_{x,y} : z \longmapsto \kappa(x,y,z) - 2 = z^2 - xyz + (x^2 + y^2 - 4).$$

Then

$$\kappa_{x,y}^{-1}\Big((-\infty, 0])\Big) = [\zeta_-(x,y), \zeta_+(x,y)]$$

where

$$\zeta_\pm(x,y) = \frac{1}{2}\left(xy \pm \sqrt{(x^2-4)(y^2-4)}\right).$$

Furthermore reflection

$$z \longmapsto z' = xy - z$$

interchanges the two intervals

$$J_- = \big(-\infty,\, \zeta_-(x,y)\big)$$
$$J_+ = \big(\zeta_+(x,y),\, \infty\big)$$

comprising $\kappa_{x,y}^{-1}\big((0,\infty)\big)$.

Lemma 5.3.2. *Suppose that* $2 < x \le y$. *Then* $\zeta_-(x,y) < y < \zeta_+(x,y)$

Proof. The conclusion is equivalent to $\kappa(x,y,y) < 2$, which is what we prove. First observe that $y > 2$ implies $y - 1/2 > 3/2$ so that

$$\begin{aligned} y - y^2 &= -\left(y - \frac{1}{2}\right)^2 + \frac{1}{4} \\ &< -\left(\frac{3}{2}\right)^2 + \frac{1}{4} = -2 \end{aligned}$$

and $x < y$ implies that $x + 2 - y^2 < y + 2 - y^2 < 0$. Therefore

$$\begin{aligned} \kappa(x,y,y) - 2 &= x^2 + 2y^2 - xy^2 - 4 \\ &= (x^2 - 4) + (2y^2 - xy^2) \\ &= (x-2)(x + 2 - y^2) < 0. \end{aligned}$$

□

Conclusion of Proof of Lemma 5.3.1. By Lemma 5.3.2, the quadratic function $\kappa_{x,y}$ is negative at y. By hypothesis $\kappa_{x,y}$ is positive at $z \ge y$. Therefore

$$z > \zeta_+(x,y) > y.$$

Thus $z \in J_+$. Reflection $z \longmapsto z'$ interchanges the intervals J_+ and J_-, so $z' \in J_-$, that is $z' < \zeta_-(x,y) < y$ (Lemma 5.3.2). By (5.3.1),

$$\begin{aligned} (z - z')^2 &= 4\Big(\kappa(x,y,z) - 2\Big) + (x^2 - 4)(y^2 - 4) \\ &> 4(\kappa(x,y,z) - 2) \end{aligned}$$

whence (because $z \ge y > z'$)

$$z - z' > 2\sqrt{\kappa(x,y,z) - 2},$$

completing the proof of Lemma 5.3.1. □

Conclusion of Proof of Theorem 5.2.1. Fix

$$u = (x, y, z) \in \mathbb{R}^3$$

with $\kappa(u) > 2$. We seek $\tilde{u} \sim u$ such that one of the following possibilities occurs:

- one of the coordinates $\tilde{x}, \tilde{y}, \tilde{z}$ lies in the interval $[-2, 2]$;
- $\tilde{x}, \tilde{y}, \tilde{z} < -2$.

It therefore suffices to find $\tilde{u} \sim u$ which lies in $(-\infty, 2]^3$. Suppose that u does not satisfy this. The linear automorphisms in Γ are arbitrary permutations of the coordinates and sign-change automorphisms which allow changing the signs of two coordinates. By applying linear automorphisms, we can assume that $2 < x \leq y \leq z$.

Let $\mu := 2\sqrt{\kappa(u) - 2} > 0$. By Lemma 5.3.1 the quadratic reflection $Q_z \in \Gamma$ given by

$$u = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \tilde{u} = \begin{bmatrix} x \\ y \\ z' \end{bmatrix}$$

reduces z by more than μ. If $z' \leq 2$ then

$$\tilde{u} \sim (-x, -y, z') \in (-\infty, 2]^3,$$

completing the proof. Otherwise, all three coordinates of $\tilde{u}$ are greater than 2 so we repeat the process. Since each repetition decreases $x+y+z$ by more than μ, the procedure ends after at most $(x + y + z - 6)/\mu$ steps. $\square$

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