

THE MAPPING CLASS GROUP ACTS REDUCIBLY ON $SU(n)$ -CHARACTER VARIETIES

WILLIAM M. GOLDMAN

ABSTRACT. When G is a connected compact Lie group, and π is a closed surface group, then $\text{Hom}(\pi, G)/G$ contains an open dense $\text{Out}(\pi)$ -invariant subset which is a smooth symplectic manifold. This symplectic structure is $\text{Out}(\pi)$ -invariant and therefore defines an invariant measure μ , which has finite volume. The corresponding unitary representation of $\text{Out}(\pi)$ on $L^2(\text{Hom}(\pi, G)/G, \mu)$ contains no finite-dimensional subrepresentations besides the constants. This note gives a short proof that when $G = SU(n)$, the representation $L^2(\text{Hom}(\pi, G)/G, \mu)$ contains many other invariant subspaces.

Let $G = SU(n)$ and π be the fundamental group of a closed oriented surface Σ . Let $\text{Hom}(\pi, G)$ be the space of representations $\pi \rightarrow G$. The group $\text{Aut}(\pi) \times \text{Aut}(G)$ acts on $\text{Hom}(\pi, G)$. Let $\text{Hom}(\pi, G)/G$ be the quotient of $\text{Hom}(\pi, G)$ by $\{1\} \times \text{Inn}(G)$. Then $\text{Out}(\pi) := \text{Aut}(\pi)/\text{Inn}(\pi)$ acts on $\text{Hom}(\pi, G)/G$. The $\text{Out}(\pi)$ -action preserves a symplectic structure on $\text{Hom}(\pi, G)/G$ which determines a finite invariant smooth measure μ (on an invariant dense open subset which is a smooth manifold). When G is compact, the total measure is finite (Jeffrey-Weitsman [6, 7], Huebschmann [5]). There results a unitary representation of $\text{Out}(\pi)$ on the Hilbert space

$$\mathcal{H} := L^2(\text{Hom}(\pi, G)/G, \mu).$$

Let $C \subset \mathcal{H}$ denote the subspace corresponding to the constant functions.

The following theorem is proved in Goldman [3] for $n = 2$ and Pickrell-Xia [8, 9] in general:

Theorem. *The only finite dimensional $\text{Out}(\pi)$ -invariant subspace of \mathcal{H} is C .*

According to [3], the only finite-dimensional invariant subspace of \mathcal{H} consists of constants. Let \mathcal{H}_0 denote the orthocomplement of the constants in \mathcal{H} , that is, the set of $f \in \mathcal{H}$ such that $\int f d\mu = 0$.

The goal of this note is an elementary proof of the following:

Date: September 12, 2005.

Partially supported by NSF grants DMS0405605 and DMS-0103889.

Theorem. *The representation of $\text{Out}(\pi)$ on \mathcal{H}_0 is reducible.*

In general for a compact connected Lie group G , the components of $\text{Hom}(\pi, G)$ and $\text{Hom}(\pi, G)/G$ are indexed by $\pi_1(G')$ where G' is the commutator subgroup. In that case the ergodicity/mixing results of [3, 8, 9] imply that the only invariant finite dimensional subspaces are subspaces of the space of locally constant functions, a vector space of dimension $|\text{vert}\pi_1(G')|$. The method of proving reducibility requires a nontrivial element of the center of G . For simplicity, we only discuss the case $G = \text{SU}(n)$ in this paper.

I am grateful to Steve Zelditch, Charlie Frohman, and Jeff Hakim for helpful conversations, and to David Fisher for carefully reading the manuscript and pointing out several corrections.

1. THE CENTER OF $\text{SU}(n)$

Let $\mathfrak{Z} \cong \mathbb{Z}/n$ denote the center of G , the group consisting of all scalar matrices $\zeta \mathbb{I}$ where $\zeta^n = 1$. Then $\text{Hom}(\pi, \mathfrak{Z}) \cong H^1(M; \mathbb{Z}/n)$ acts on $\text{Hom}(\pi, G)/G$ by pointwise multiplication: If $\rho \in \text{Hom}(\pi, G)$ and $u \in \text{Hom}(\pi, \mathfrak{Z})$, then define the action $u \cdot \rho$ of u on ρ by:

$$(1) \quad \gamma \xrightarrow{u \cdot \rho} \rho(\gamma)u(\gamma).$$

Recall the definition [2] of the symplectic structure on $\text{Hom}(\pi, G)/G$. Suppose that $\rho \in \text{Hom}(\pi, G)$ is an irreducible representation. By Weil [11] (compare also Raghunathan [10]), the Zariski tangent space to $\text{Hom}(\pi, G)$ at ρ identifies with the space $Z^1(\pi, \mathfrak{g}_{\text{Ad}\rho})$ of cocycles where $\mathfrak{g}_{\text{Ad}\rho}$ is the π -module defined by the composition

$$\pi \xrightarrow{\rho} G \xrightarrow{\text{Ad}} \text{Aut}(\mathfrak{g})$$

and \mathfrak{g} is the Lie algebra of G . Since ρ is irreducible, G acts locally freely and $\text{Hom}(\pi, G)/G$ has a smooth structure in a neighborhood of $[\rho]$. The tangent space to the orbit $G \cdot \rho$ equals the space of coboundaries $B^1(\pi, \mathfrak{g}_{\text{Ad}\rho})$. The tangent space to $\text{Hom}(\pi, G)/G$ at $[\rho]$ identifies with the cohomology group $H^1(\pi, \mathfrak{g}_{\text{Ad}\rho})$.

Let $u \in \text{Hom}(\pi, \mathfrak{Z})$. Since $\text{Ad}(\mathfrak{Z})$ is trivial, the action of u induces an identification of π -modules $\mathfrak{g}_{\text{Ad}\rho} \longrightarrow \mathfrak{g}_{\text{Ad}(u \cdot \rho)}$ and hence of tangent spaces

$$\begin{aligned} T_{[\rho]} \text{Hom}(\pi, G)/G &= H^1(\pi, \mathfrak{g}_{\text{Ad}\rho}) \\ &\longrightarrow T_{[u \cdot \rho]} \text{Hom}(\pi, G)/G = H^1(\pi, \mathfrak{g}_{\text{Ad}(u \cdot \rho)}). \end{aligned}$$

The symplectic form ω_ρ at $[\rho]$ is defined by the cup product

$$H^1(\pi, \mathfrak{g}_{\text{Ad}\rho}) \times H^1(\pi, \mathfrak{g}_{\text{Ad}\rho}) \longrightarrow H^2(\pi; \mathbb{R})$$

using the pairing of π -modules induced by an \mathbf{Ad} -invariant nondegenerate symmetric bilinear form on \mathfrak{g} as a coefficient pairing. Evidently the symplectic form ω , and therefore the corresponding measure μ , are $\mathbf{Hom}(\pi, \mathfrak{Z})$ -invariant.

2. TRACE FUNCTIONS

Suppose that $\gamma \in \pi$. The function

$$\begin{aligned} \mathbf{Hom}(\pi, G) &\longrightarrow \mathbb{C} \\ \rho &\longmapsto \mathbf{tr}(\rho(\gamma)) \end{aligned}$$

is $\mathbf{Inn}(G)$ -invariant and defines a function

$$\mathbf{Hom}(\pi, G)/G \xrightarrow{t_\gamma} \mathbb{C}.$$

We extend this definition to products of more than one element of π : Let $\gamma = (\gamma_1, \dots, \gamma_s) \in \pi^s$. Define the *trace function* of γ as the product:

$$t_\gamma([\rho]) := t_{\gamma_1}([\rho]) \dots t_{\gamma_s}([\rho])$$

Using the definition (1) of the action of $\mathbf{Hom}(\pi, \mathfrak{Z})$ on $\mathbf{Hom}(\pi, G)/G$, the action of $u \in \mathbf{Hom}(\pi, \mathfrak{Z})$ on the trace function t_γ is given by $u \cdot t_\gamma$, defined by:

$$(2) \quad \rho \xrightarrow{u \cdot t_\gamma} u(\gamma)^{-1} t_\gamma(\rho)$$

since

$$\begin{aligned} (u \cdot t_\gamma)(\rho) &:= t_\gamma(u^{-1} \cdot \rho) \\ &= \mathbf{tr}((u^{-1} \cdot \rho)(\gamma)) \\ &= \mathbf{tr}(u(\gamma)^{-1} \rho(\gamma)) \\ &= u(\gamma)^{-1} \mathbf{tr}(\rho(\gamma)) \\ &= u(\gamma)^{-1} t_\gamma(\rho) \end{aligned}$$

Since $\mathfrak{Z} \cong \mathbb{Z}/n$, the evaluation $u(\gamma)$ depends only on the *total homology class*

$$[\gamma] := [\gamma_1] + \dots + [\gamma_s] \in H_1(\Sigma; \mathbb{Z}/n),$$

defined as the sum of the \mathbb{Z}/n -homology classes of the γ_i . The evaluation of $u \in \mathbf{Hom}(\pi, \mathfrak{Z})$ on γ is just the natural pairing

$$\begin{aligned} \mathbf{Hom}(\pi, \mathfrak{Z}) \times H_1(\Sigma; \mathbb{Z}/n) &\longrightarrow \mathbb{Z}/n \cong Z \\ (u, [\gamma]) &\longmapsto u \cdot [\gamma] \end{aligned}$$

where $[\gamma]$ is the total homology class of γ .

Lemma 1. *If $[\gamma] \neq 0 \in H_1(\Sigma; \mathbb{Z}/n)$, then*

$$\int t_\gamma d\mu = 0.$$

Proof. Since the homology class of γ in $H_1(\Sigma, \mathbb{Z}/n)$ is nonzero, $u(\gamma) \neq 1$ for some $u \in \mathbf{Hom}(\pi, \mathfrak{S})$. Since $(u^{-1})_*\mu = \mu$,

$$\int t_\gamma d\mu = \int (t_\gamma \circ u^{-1}) d(u^{-1})_*\mu = u(\gamma)^{-1} \int t_\gamma d\mu$$

by (2). Since $u(\gamma)^{-1} \neq 1$, the integral is zero as claimed. \square

An element $\gamma \in \pi$ is *nonseparating simple* if it contains a nonseparating simple loop which is homotopically nontrivial and not homotopic to $\partial\Sigma$. Since such an element has nontrivial homology class $[\gamma] \in H_1(\Sigma, \mathbb{Z}/n)$, Lemma 1 implies:

Corollary 2. *Suppose γ is nonseparating simple. Then $t_\gamma \in \mathcal{H}_0$.*

For $n = 2$, Frohman and Kania-Bartoszyńska [1] have calculated $\int t_\gamma d\mu$ for separating simple loops γ in terms of Bernoulli numbers and $6j$ -symbols.

3. INVARIANT SUBSPACES

Let $\mathcal{W} \subset \mathcal{H}_0$ denote the closure of the span of all t_α , where α is nonseparating simple loops. (Alternatively, let α run over all elements of π with nonzero homology class modulo n .) Clearly \mathcal{W} is $\mathbf{Out}(\pi)$ -invariant.

Proposition 3. *\mathcal{W} is a proper subspace of \mathcal{H}_0 .*

Proof. Choose $\gamma \in \pi^s$ such that:

- t_γ is not constant;
- γ has trivial homology class in $H_1(\Sigma, \mathbb{Z}/n)$.

Such elements are easy to find: for example if γ consists of one nontrivial separating simple loop, or if γ consists of one n -th power, or if $\gamma = (\gamma_1, \dots, \gamma_1)$ where s is a positive multiple of n and γ_1 is nontrivial.

Normalize t_γ so that it lies in \mathcal{H}_0 :

$$\hat{t}_\gamma := t_\gamma - \int t_\gamma d\mu.$$

We claim that \hat{t}_γ is a nonzero vector in \mathcal{H}_0 orthogonal to \mathcal{W} . It is nontrivial since t_γ is nonconstant. Furthermore if $\alpha \in \pi$ has nonzero homology class $[\alpha] \in H_1(\Sigma, \mathbb{Z}/n)$ (for example if it nonseparating simple), then t_α is orthogonal to \hat{t}_γ .

Since $\rho(\alpha)$ is unitary, $\overline{t_\alpha} = t_{\alpha^{-1}}$. Thus

$$\begin{aligned} \langle \hat{t}_\gamma, t_\alpha \rangle &= \int \hat{t}_\gamma \overline{t_\alpha} d\mu = \int \hat{t}_\gamma t_{\alpha^{-1}} d\mu \\ &= \int t_\gamma t_{\alpha^{-1}} d\mu \text{ (by Corollary 2)} \\ &= \int t_\eta d\mu \end{aligned}$$

where $\eta = (\gamma_1, \dots, \gamma_s, \alpha^{-1})$. Since

$$[\eta] = [\gamma] - [\alpha] \neq 0,$$

Lemma 1 implies that $\langle \hat{t}_\gamma, t_\alpha \rangle = 0$, as desired. \square

When π is a free group and $n = 2$, then ergodicity and weak-mixing of the action of $\mathbf{Out}(\pi)$ on $\mathbf{Hom}(\pi, G)/G$ is proved in Goldman [4]. Exactly the same argument as above (with \mathcal{W} replaced by the closure of the span of t_γ where γ is an element of a free generating set of π) shows that the unitary representation of $\mathbf{Out}(\pi)$ on $L^2(\mathbf{Hom}(\pi, G)/G, \mu)$ contains invariant (necessarily infinite-dimensional) subspaces other than the constants.

REFERENCES

1. Frohman, C. and Kania-Bartoszyńska, J., *Shadow world evaluation of the Yang-Mills measure*, Algebraic and Geometric Topology, **4** (2001), 311–332.
2. Goldman, W., *The symplectic nature of fundamental groups of surfaces*, Adv. Math. **54** (1984), 200–225.
3. ———, *Ergodic theory on moduli spaces*, Ann. Math. **146** (1997), 475–507.
4. ———, *An ergodic action of the outer automorphism group of a free group*, math.DG/0506401 (submitted)
5. Huebschmann, J., *Symplectic and Poisson structures of certain moduli spaces*, Duke Math J. **80** (1995) 737–756.
6. Jeffrey, L. and Weitsman, J., *Bohr-Sommerfeld orbits and the Verlinde dimension formula*, Commun. Math. Phys. **150** (1992) 593–630
7. ———, *Toric structures on the moduli space of flat connections on a Riemann surface: Volumes and the moment map*, Adv. Math. **109**, 151–168 (1994).
8. Pickrell, D. and Xia, E., *Ergodicity of Mapping Class Group Actions on Representation Varieties, I. Closed Surfaces*, Comment. Math. Helv. **77** (2001), 339–362.
9. ———, *Ergodicity of Mapping Class Group Actions on Representation Varieties, II. Surfaces with Boundary*, Transformation Groups **8** (2003), no. 4, 397–402.
10. Raghunathan, M., “Discrete Subgroups of Lie Groups,” Ergebnisse der Math. **58**, Springer-Verlag Berlin-Heidelberg-New York (1972).
11. Weil, A., *Remarks on the cohomology of groups*, Ann. Math. **80** (1964), 149–157.

MATHEMATICS DEPARTMENT, UNIVERSITY OF MARYLAND, COLLEGE PARK,
MD 20742 USA

E-mail address: `wmg@math.umd.edu`