INVARIANT FUNCTIONS AND MODULI SPACES

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Let $G = \mathrm{SU}(n)$ and π the fundamental group of a closed oriented surface S. Let $\mathrm{Hom}(\pi, G)$ be the moduli space of conjugacy classes of representations $\pi \longrightarrow G$ and let $X \subset \mathrm{Hom}(\pi, G)$ be the dense open subset corresponding to irreducible representations. Arising from the natural symplectic structure on $\mathrm{Hom}(\pi, G)$ is a smooth finite measure μ on X. Suppose that $\gamma \in \pi$. The function

$$\begin{aligned} X &\longrightarrow \mathbb{C} \\ \phi &\longmapsto \mathrm{tr}(\phi(\gamma)) \end{aligned}$$

is $\mathrm{Inn}(G)$-invariant and defines a function $t_\gamma : X \longrightarrow \mathbb{C}$.

Theorem. *If γ defines a nonzero homology class in $H_1(S; \mathbb{Z}/n)$,*

$$\int_X t_\gamma d\mu = 0.$$

I thank Steve Zelditch for suggesting this problem.

Let Z denote the center of G, the group consisting of all scalar matrices $\zeta\mathbb{I}$ where $\zeta^n = 1$. Then $\mathrm{Hom}(\pi, Z) \cong H^1(M; \mathbb{Z}/n)$ acts on $\mathrm{Hom}(\pi, G)/G$ by pointwise multiplication: if $\phi \in \mathrm{Hom}(\pi, G)$ and $u \in \mathrm{Hom}(\pi, Z)$, then $u \cdot [\phi] : \gamma \longmapsto [\phi(\gamma)u(\gamma)]$. The corresponding $\mathrm{Hom}(\pi, Z)$-action on functions t_γ is:

$$u : t_\gamma \longmapsto t_\gamma \circ u^{-1} = u(\gamma)^{-1} t_\gamma. \tag{1}$$

Recall the definition [1] of the symplectic structure on X. Suppose that $\phi \in \mathrm{Hom}(\pi, G)$ is an irreducible representation. By Weil [6] (compare also Raghunathan [5]), the Zariski tangent space to $\mathrm{Hom}(\pi, G)$ at ϕ identifies with the space $Z^1(\pi, \mathfrak{g}_{\mathrm{Ad}\,\phi})$ of cocycles where $\mathfrak{g}_{\mathrm{Ad}\,\phi}$ is the π-module defined by the composition

$$\pi \xrightarrow{\phi} G \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\mathfrak{g})$$

and g is the Lie algebra of G. Since ϕ is irreducible, G acts locally freely and $\mathrm{Hom}(\pi, G)/G$ has a smooth structure in a neighborhood of $[\phi]$. The tangent space to the orbit $G \cdot \phi$ equals the space of coboundaries $B^1(\pi, \mathfrak{g}_{\mathrm{Ad}\,\phi})$ and the tangent space to X at $[\phi]$ identifies with the cohomology group $H^1(\pi, \mathfrak{g}_{\mathrm{Ad}\,\phi})$.

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Let $u \in \mathrm{Hom}(\pi, Z)$. Since $\mathrm{Ad}(Z)$ is trivial, the action of u induces an identification of π-modules $\mathfrak{g}_{\mathrm{Ad}\,\phi} \longrightarrow \mathfrak{g}_{\mathrm{Ad}(u\cdot\phi)}$ and hence of tangent spaces

$$T_{[\phi]}X = H^1(\pi, \mathfrak{g}_{\mathrm{Ad}\,\phi}) \longrightarrow T_{[u\cdot\phi]}X = H^1(\pi, \mathfrak{g}_{\mathrm{Ad}(u\cdot\phi)}).$$

The symplectic form ω_ϕ at $[\phi]$ is defined by the cup product

$$H^1(\pi, \mathfrak{g}_{\mathrm{Ad}\,\phi}) \times H^1(\pi, \mathfrak{g}_{\mathrm{Ad}\,\phi}) \longrightarrow H^2(\pi; \mathbb{R})$$

using the pairing of π-modules induced by an Ad-invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$ as a coeefficient pairing. Evidently the symplectic form ω, and therefore the corresponding measure μ, are $\mathrm{Hom}(\pi, Z)$-invariant.

Since the homology class of γ in $H_1(\pi, \mathbb{Z}/n)$ is nonzero, $u(\gamma) \neq 1$ for some $u \in \mathrm{Hom}(\pi, Z)$. Since $\mu \circ u^{-1} = \mu$,

$$\int_X t_\gamma d\mu = \int_X (t_\gamma \circ u^{-1})(d\mu \circ u^{-1}) = u(\gamma)^{-1} \int t_\gamma d\mu$$

by (1). Since $u(\gamma)^{-1} \neq 1$, the integral is zero.

References

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