COMPLETE FLAT AFFINE AND LORENTZIAN MANIFOLDS

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1. Introduction

When Hilbert posed the problem of classifying groups of Euclidean isometries with compact fundamental domain, he launched an effort that still continues today. We are at the point where we have a good understanding of Euclidean manifolds, and more generally, compact, complete flat affine manifolds with amenable holonomy. We propose here a survey of a series of results, which have culminated in the study of flat Lorentz manifolds.

Let $\mathrm{Aff}(\mathbb{R}^n)$ denote the group of affine transformations of $\mathbb{R}^n$. A *flat affine structure* on a manifold M^n is a maximal atlas of coordinate charts

$$\psi_\alpha : U_\alpha \longrightarrow \mathbb{R}^n$$

such that the restriction of a coordinate transformation $\psi_\beta \circ \psi_{\alpha^{-1}}$ to a connected component of $U_\alpha \cap U_\beta$ is an element of $\mathrm{Aff}(\mathbb{R}^n)$. That is, an affine structure is a compatible system of local coordinates modeled on affine space.

Recall that a group G of homeomorphisms of a locally compact Hausdorff space E acts *properly discontinuously* if and only only if, when given the discrete topology, G acts *properly* on E. A group acts *properly* if, given any compact subset K of E,

$$G(K,K) = \{g \in G | gK \cap K \neq \emptyset\}$$

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is compact. When G is discrete, this condition means that $G(K,K)$ is finite, in which case E/G is Hausdorff and the quotient map $E \longrightarrow E/G$ is a branched covering space. If, in addition, G acts *freely* on E (that is, the only element of G which fixes a point of E is the identity), then $E \longrightarrow E/G$ is a covering space. In particular, if E is a smooth manifold upon which G acts properly discontinuously and freely by diffeomorphisms, then E/G has an induced smooth structure for which $E \longrightarrow E/G$ is a local diffeomorphism.

If E is an affine space and $G \subset \mathrm{Aff}(\mathbb{R}^n)$ is a subgroup which acts properly discontinuously, then E/G is a flat affine manifold M. Such an affine manifold is said to be *complete,* since this condition is equivalent to geodesic completeness of the corresponding flat affine connection.

In general a discrete subgroup of $\mathrm{Aff}(\mathbb{R}^n)$ may not act properly discontinuously on $\mathbb{R}^n$. Consider, for example, the action of $g : (x, y) \mapsto (\lambda x, \lambda^{-1} y)$, on $\mathbb{R}^2$ $(0 < \lambda < 1)$. The cyclic group generated by g is discrete in $\mathrm{SL}(2,\mathbb{R})$ yet fails to act freely (since the origin is a fixed point). Neither is this action proper (taking K in the above definition to be the origin). Indeed, even if the origin is removed, the resulting free action is not proper.

In this paper, we only consider complete, flat affine manifolds. The classification of such manifolds can be approached by determining which subgroups of $\mathrm{Aff}(\mathbb{R}^n)$ can occur as the fundamental group of a complete, flat affine manifold. Equivalently, we try to determine which subgroups of $\mathrm{Aff}(\mathbb{R}^n)$ act properly discontinuously and freely on $\mathbb{R}^n$.

2. Classification of compact flat affine manifolds

Hilbert's 18th problem was posed for *crystallographic groups*, that is, discrete groups of Euclidean isometries such that a fundamental domain for their action is compact. The problem was to find a finite classification, up to conjugacy in $\mathrm{Aff}(\mathbb{R}^n)$, of the n-dimensional crystallographic groups for every n. Note that by the Hopf-Rinow theorem, the quotient by a crystallographic group must be complete.

Bieberbach solved the problem entirely in 1911 by showing the following results, described by Milnor [27] :

1. Every discrete group of Euclidean motions of $\mathbb{R}^n$ with compact fundamental domain contains n linearly independent translations;
2. The subgroup of translations of such a group forms a maximal, free, abelian, normal subgroup of finite index;
3. For each n, there are only finitely many possibilities for such groups, up to conjugation in $\mathrm{Aff}(\mathbb{R}^n)$.

Suppose now that Γ is any discrete subgroup of $\mathrm{Aff}(\mathbb{R}^n)$ such that its action is properly discontinuous and $\mathbb{R}^n/\Gamma$ is compact. This is what Fried and Goldman called an *affine crystallographic group.* In [2], L. Auslander announced that an affine crystallographic group had to be *virtually solvable*, that is, it must contain a solvable subgroup of finite index. However, his proof was flawed and this question, which has come to be known as Auslander's conjecture, remains unanswered.

Here is why virtual solvability is desirable. If Γ is virtually solvable, there exists a subgroup $G \subset \mathrm{Aff}(\mathbb{R}^n)$ with finitely many components such that Γ is a lattice inside G; furthermore, G has a complete left-invariant affine structure. Geometrically speaking, Γ is the fundamental group of a manifold that is finitely covered by a solvmanifold, and such manifolds are relatively well understood. For details on this construction (which is based on earlier work by Zassenhaus, Wang, and Auslander) see §1 of Fried–Goldman [16].

Fried and Goldman [16] proved that Auslander's conjecture holds in dimensions two and three. Soifer [32] and Tomanov [34] extended the result up to dimension eight. Goldman and Kamishima [19] proved Auslander's conjecture for arbitrary dimensions in the particular case where $\Gamma \subset \mathrm{SO}(n-1,1)$.

Carrière [7] proved that compact flat Lorentzian manifolds are geodesically complete. Mess [26] proved that compact 3-dimensional Lorentzian manifolds of constant negative curvature are complete (see the proof of Proposition 27, p.53 of [26]). This result was extended to other constant curvature Lorentzian manifolds by Klingler [21]. Independently, Morrill [28] proved compact Lorentzian manifolds of positive curvature are complete. In particular every constant curvature Lorentzian structure on a closed manifold is a quotient X/G, where X is *de Sitter space* (when the curvature is positive) and *anti-de Sitter space* (when the curvature is negative). Combining this fact with work of Calabi-Markus [6] on proper isometric actions on de Sitter space, the theorems of Morrill and Klingler imply that compact de Sitter manifolds do not exist.

For examples of closed Lorentzian manifolds of constant negative curvature, we refer to Kulkarni-Raymond [22], Ghys [17], Goldman [18], Zeghib [35] and Salein [29, 30].

Few examples of incomplete locally homogeneous pseudo-Riemannian on closed manifolds are known. However, Guediri and Lafontaine [20] constructed compact, locally homogeneous, pseudo-Riemannian manifolds which are not geodesically complete. By Marsden [25], a compact *homogeneous* pseudo-Riemannian manifold is geodesically complete.

3. Non-compact manifolds and Milnor's question

After showing in [27] that a torsion-free, virtually polycyclic subgroup is isomorphic to $\pi_1(M)$ for some complete, flat affine manifold M, Milnor asked whether the converse must hold:

Question 3.1 (Milnor [27]). *Let M be a complete, flat affine manifold. Is $\pi_1(M)$ virtually polycyclic?*

This is the best generalization of Auslander's conjecture to the non-compact case, since a discrete, solvable subgroup of a connected Lie group must actually be polycyclic.

Milnor justified his question by pointing out that a connected Lie group admits a free affine action on some $\mathbb{R}^n$ if and only if it is simply connected and solvable. Furthermore, he remarked that by a result of Tits [33], if Γ is a discrete subgroup of a Lie group with finitely many components, then either Γ is virtually polycyclic or Γ contains a free, non-abelian subgroup on two (or more) generators. Tits's construction of an example of the latter case was based on *Schottky groups* (see below).

Milnor suggested a procedure in $\mathbb{R}^{2,1}$, which we describe here. For

$$\mathsf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \mathsf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

in $\mathbb{R}^3$, set $\mathbb{B}(\mathsf{x},\mathsf{v}) = x_1v_1 + x_2v_2 - x_3v_3$. When $\mathbb{R}^3$ is equipped with this indefinite bilinear form, we will call it $\mathbb{R}^{2,1}$. Thus $\mathrm{O}(2,1)$ is the group of linear isometries of $\mathbb{R}^{2,1}$. Furthermore, a vector $\mathsf{v} \in \mathbb{R}^{2,1}$ will be called *spacelike* (resp. *null*, *timelike*) if and only if $\mathbb{B}(\mathsf{v},\mathsf{v}) > 0$ (resp. $\mathbb{B}(\mathsf{v},\mathsf{v}) = 0$, $\mathbb{B}(\mathsf{v},\mathsf{v}) < 0$).

Now, suggests Milnor, start with a Schottky group in $\mathrm{SO}(2,1)$ and add translation components so that the resulting affine group might possibly act freely and properly discontinuously on $\mathbb{R}^{2,1}$. In the wake of this remark, Fried and Goldman [16] showed that, indeed, the only way that a discrete $\Gamma \subset \mathrm{Aff}(\mathbb{R}^{2,1})$ could act properly discontinuously without being virtually solvable was if

- $\mathbb{R}^{2,1}/\Gamma$ is non-compact; and
- $\mathbb{L} : \Gamma \longrightarrow \mathrm{SO}(2,1)$ is an embedding onto a discrete subgroup of $\mathrm{SO}(2,1)$.

Schottky's classical construction [31] (also see Ford [15], p.59) involves g pairs of circles

$$C_i, C_i' \subset \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$$

for $i = 1, \ldots, g$ and transformations $\gamma_i \in \mathrm{PSL}(2,\mathbb{C})$ such that γ_i maps the inside of C_i to the outside of C_i'. Schottky proved that $\{\gamma_i \mid i = 1, \ldots, g\}$ freely generates a discrete subgroup Γ which acts properly discontinuously on an open subset $\Omega \subset \mathbb{C}P^1$, where the complement of the insides of all C_i, C_i' determines a fundamental domain. In particular the quotient Ω/Γ is a closed orientable surface of genus g. The *limit set* Λ of Γ is the complement $\mathbb{C}P^1 - \Omega$, and is homeomorphic to a Cantor set.

More precisely, consider the action of Γ on hyperbolic 3-space $\mathbb{H}^3$. For any subset $S \subset \overline{\mathbb{H}^3}$, let $h(S)$ denote its convex hull in $\overline{\mathbb{H}^3}$. Thus each circle C_i, C_i' bounds a totally geodesic hyperplane $h(C_i), h(C_i') \subset \mathbb{H}^3$ respectively. Let $H_i \subset \mathbb{H}^3$ be the convex hull of the inside of C_i and $H_i' \subset \mathbb{H}^3$ be the convex hull of the outside of C_i'. Then γ_i maps the half-space H_i to the half-space H_i'. Thus $\Delta_i = H_i' - H_i$ is a fundamental domain for the cyclic group $\langle\gamma_i\rangle$ acting on the complement of the fixed points of γ_i. Call the region bounded by two disjoint hyperplanes a *slab* . Each Δ_i is a slab. They can be *combined,* in that their intersection

$$\Delta = \Delta_1 \cap \cdots \cap \Delta_g$$

is a fundamental domain for the action of Γ acting on

$$\overline{\mathbb{H}^3} - \Lambda = \mathbb{H}^3 \cup \Omega.$$

The quotient $(\mathbb{H}^3 \cup \Omega)/\Gamma$ is homeomorphic to a solid handlebody bounded by Ω/Γ.

To combine fundamental domains for cyclic groups, the essential point is that the slabs Δ_i have disjoint complements in $\overline{\mathbb{H}^3}$. Then, as Schottky proved, a nontrivial reduced word

$$\gamma = \gamma_{i_1}^{\epsilon_{i_1}} \dots \gamma_{i_k}^{\epsilon_{i_k}}$$

(where each $\epsilon_{i_k} = \pm 1$) takes Δ to $\overline{\mathbb{H}^3} - H'_{i_k}$ if $\epsilon_{i_k} = +1$ or to H_{i_k} if $\epsilon_{i_k} = -1$. In particular γ maps Δ off itself. One immediately deduces that Γ is discrete and Δ is a fundamental domain for the action of Γ on $\Gamma(\Delta)$.

In particular since $\mathbb{H}^3 \approx \mathbb{R}^3$, there do exist properly discontinuous actions of free groups on $\mathbb{R}^3$ by homeomorphisms. Thus Milnor's question is very specific to its *affine* context.

Margulis ended the suspense in 1983 by showing that Milnor's suggestion does indeed work:

Theorem 3.2 (Margulis [23, 24])**.** *There exist free non-abelian subgroups of* $\mathrm{Isom}(\mathbb{R}^{2,1})$ *acting properly discontinuously on* $\mathbb{R}^{2,1}$.

We outline the proof later in this survey, and refer the reader to the original papers as well as to Drumm–Goldman [12]. Margulis then asked what sufficient conditions on a discrete subgroup of $\mathrm{Aff}(\mathbb{R}^3)$ would insure that it acts properly discontinuously on $\mathbb{R}^3$. By Fried-Goldman [16], the linear representation $\mathbb{L} : \Gamma \longrightarrow \mathrm{Aff}(\mathbb{R}^3)$ must be conjugate to an embedding of Γ onto a discrete subgroup of $\mathrm{O}(2,1)$. Therefore we shall restrict our attention to affine transformations whose linear part lies in $\mathrm{O}(2,1)$.

One might try to imitate Schottky's construction above in affine geometry. Namely, begin with a Schottky subgroup of $\mathrm{O}(2,1)$ and add translational components. Specifically, let $\{\gamma_i | i = 1, \dots, k\}$ be a collection of affine transformations of $\mathbb{R}^{2,1}$ whose linear parts $\mathbb{L}(\gamma_i)$ generate a Schottky group acting on $\mathbb{H}^2$. We view $\mathbb{H}^2$ as the set of timelike lines (lines that are parallel to a timelike vector) in $\mathbb{R}^{2,1}$. Then each $\mathbb{L}(\gamma_i)$ is *hyperbolic* in the sense that it has three positive real eigenvalues $\lambda < 1 < \lambda^{-1}$. Furthermore the eigenvectors corresponding to eigenvalues $\lambda^{\pm 1}$ are null and the other eigenvector is spacelike. Geometrically, $\mathbb{L}(\gamma_i)$ acts properly discontinuously on $\mathbb{H}^2$ by translation along a geodesic (joining the two null eigenlines). Let S_i be the plane spanned by the $\lambda^{\pm 1}$-eigenspaces of $\mathbb{L}(\gamma_i)$. The invariant geodesic in $\mathbb{H}^2$ is the subset of $\mathbb{H}^2$ consisting of lines in S_i.

The affine transformation γ_i also acts properly discontinuously on $\mathbb{R}^{2,1}$, at least for certain choices of translational part. Choose a plane P_i which is parallel to S_i and a translational part not lying in S_i. Then P_i and $\gamma_i(P_i)$ are parallel planes bounding a "slab" in $\mathbb{R}^{2,1}$. This slab is a fundamental domain for the cyclic group $\langle\gamma_i\rangle$ and the quotient $\mathbb{R}^{2,1}/\langle\gamma_i\rangle$ is an open solid torus.

One might hope to "combine" these cyclic groups $\langle\gamma_i\rangle$ and their slab fundamental domains to obtain a fundamental domain for $\langle\gamma_1, \dots, \gamma_g\rangle$. However the complements of *any* two such slabs are *never* disjoint. Thus, a direct generalization of Schottky's construction using slabs fails.

In the following two sections, we describe two approaches in the investigation of complete Lorentz space-times. Margulis's approach [23, 24] depends on the definition of a signed Lorentzian length of geodesics. The more geometric approach, developed by Drumm in [8], is a generalization of Schottky's constructions where flat hyperplanes are replaced by *crooked planes* which bound a new type of slab that could be combined.

4. Lorentzian length of geodesics

Margulis associates to a hyperbolic element $g \in \mathrm{SO}^0(2,1)$ three vectors x_g^-, x_g^o, and x_g^+ as follows. Let

$$\lambda(g) < 1 < \lambda(g)^{-1}$$

be the eigenvalues of g. Then

- x_g^+ is an eigenvector of g with eigenvalue $\lambda(g)^{-1} > 1$ and x_g^- is an eigenvector of g with eigenvalue $\lambda(g) < 1$; both these eigenvectors are necessarily null vectors;
- x_g^o is an eigenvector of g with eigenvalue 1; it is necessarily a spacelike vector;
- $\mathsf{x}_g^\pm$ are chosen to lie on the upper nappe of the null cone and have Euclidean length 1;
- x_g^o is chosen so that $\mathbb{B}(\mathsf{x}_g^o, \mathsf{x}_g^o) = 1$ and its direction is chosen so that $\{\mathsf{x}_g^-, \mathsf{x}_g^+, \mathsf{x}_g^o\}$ is a right-handed basis for $\mathbb{R}^{2,1}$.

An element γ in the affine group whose linear part is g can be written as $\gamma(x) = g(x) + \mathsf{v}$, where $g = \mathbb{L}(\gamma)$ is the linear part of γ and v is its *translational part.* We say that γ is *hyperbolic* if g is hyperbolic; furthermore, we set $\mathsf{x}_\gamma^o = \mathsf{x}_g^o$ and $\mathsf{x}_\gamma^\pm = \mathsf{x}_g^\pm$.

Any hyperbolic γ leaves invariant a unique line C_γ parallel to the vector x_γ^o. It is easy to see that γ's action on C_γ is translation by $\alpha(\gamma)\mathsf{x}_\gamma^o$, where

$$\alpha(\gamma) = \mathbb{B}(\mathsf{v}, \mathsf{x}_\gamma^o).$$

Alternatively, $\alpha(\gamma)$ can be defined as

$$\alpha(\gamma) = \mathbb{B}(\gamma(x) - x, \mathsf{x}_\gamma^o),$$

where x is an arbitrary point in $\mathbb{R}^{2,1}$. It is easily shown that the value of $\alpha(\gamma)$ does not depend on the choice of x. As a matter of fact, the first definition of $\alpha(\gamma)$ is the same as the second, but with x taken to be the origin. If $\alpha(\gamma) \neq 0$, then

$$C_\gamma/\langle\gamma\rangle \hookrightarrow \mathbb{R}^{2,1}/\langle\gamma\rangle$$

is a closed geodesic and $\alpha(\gamma)$ is its *signed Lorentzian length.*

A full understanding of how α varies over a free group is elusive. However, we can state some elementary, yet important, properties that α has for any particular hyperbolic γ.

- γ has a fixed point if and only if $\alpha(\gamma) = 0$.
- $\alpha(\gamma) = \alpha(\gamma^{-1})$. This follows from the fact that $\mathsf{x}^o_{\gamma^{-1}} = -\mathsf{x}^o_\gamma$ since the expanding and contracting eigenvectors of γ^{-1} are reversed from those of γ.

Margulis proved that the sign of $\alpha(\gamma)$, also called the *sign of* γ, is of central importance. In particular, he showed the following.

Lemma 4.1 (Margulis [23],[24])**.** *If γ_1 and γ_2 have opposite signs then $\langle\gamma_1, \gamma_2\rangle$ does not act properly on $\mathbb{R}^{2,1}$.*

We will not prove this lemma here, but instead we refer the reader to the original papers noted above and expositions in [12, 10, 11].

Having established the necessary notation, we can now give an overview of the proof of Theorem 3.2 leaving the details to be found in the original and in a related theorem in [12]. In the proof, Margulis starts with a group Γ generated by γ_1 and γ_2 with some restrictive but satisfiable assumptions on the linear and translational parts. With these assumptions he finds a lower bound for $\alpha(\gamma)$, as γ varies over Γ, which grows linearly with the length of γ viewed as a word in the generators γ_1 and γ_2. He also shows that for any point $x \in \mathbb{R}^{2,1}$, the Euclidean distance between x and $\gamma(x)$ is greater than a quantity depending on $\alpha(\gamma)$ and the Euclidean distance between the vectors $\mathsf{x}^\pm_\gamma$. To get around the fact that the Euclidean distance between $\mathsf{x}^\pm_\gamma$ can approach 0, Margulis actually performs his calculations with a "suitable" coset of Γ inside the connected group $\mathrm{SO}^0(2,1) \ltimes \mathbb{R}^{2,1}$.

Using many of the same techniques pioneered by Margulis, the conditions on the linear and translational parts of the generators were refined by Drumm and Goldman in [12], where they proved

Theorem 4.2. *Let Γ_0 be a Schottky subgroup of $\mathrm{SO}^0(2,1)$. Then there exists Γ such that $\mathbb{L}(\Gamma) = \Gamma_0$ and Γ acts properly discontinuously on $\mathbb{R}^{2,1}$.*

In addition, Abels, Margulis and Soifer [1] have been able to extend the definition of α to higher dimensions in the case where the linear part of the Γ lies in $\mathrm{SO}(n+1, n)$. Here, the results depend on the parity of n.

Theorem 4.3. *Suppose that Γ is a noncyclic free subgroup of $\mathrm{Aff}(\mathbb{R}^{2n+1})$ such that the Zariski closure of $\mathbb{L}(\Gamma)$ is conjugate to $\mathrm{SO}(n+1, n)$.*

- *For odd n, there exist such Γ which act properly discontinuously on $\mathbb{R}^{2n+1}$.*

- *For even n, there do not exist such Γ which act properly discontinuously on $\mathbb{R}^{2n+1}$.*

The statement of the theorem for even n is a corollary of the extension of Lemma 4.1 to higher dimensions, i.e. in $\mathbb{R}^{n+1,n}$, and from the easy fact that $\alpha(\gamma^{-1}) = (-1)^{n+1}\alpha(\gamma)$.

5. Crooked planes

The generalization of Schottky's construction was given by Drumm [8] where fundamental polyhedra for the action of many free groups Γ acting properly discontinuously on $\mathbb{R}^{2,1}$ were created. This approach gives a slightly stronger positive result as to which groups act properly discontinuously on $\mathbb{R}^n$.

Theorem 5.1 (Drumm [9])**.** *G is a free discrete subgroup of $\mathrm{SO}(2,1)$ if and only if there exists a Γ such that $\mathbb{L}(\Gamma) = G$ and Γ acts properly discontinuously on $\mathbb{R}^n$.*

The proof constructs fundamental polyhedra which are bounded by disjoint *crooked planes*, the analogue to hyperplanes in Schottky's construction. A crooked plane $C(p, \mathsf{v})$ is determined by a point $p \in \mathbb{R}^{2,1}$ called the *vertex* and a spacelike vector v called the *spine*.

A crooked plane $C(p, \mathsf{v})$ consists of a *stem* and two *wings* and separates $\mathbb{R}^{2,1}$ into two half-spaces which we call *crooked half-spaces*. The stem is the set of points q such that

- $\mathbb{B}(q - p, q - p) \leq 0$, that is $q - p$ is a timelike or null vector, and
- $\mathbb{B}(\mathsf{v}, q - p) = 0$, that is $q - p$ is Lorentzian perpendicular to the spine.

The wings are closed halfplanes $W^{\pm}$ such that

- $\ell \subset W^{+} \cap W^{-}$, where ℓ is the line through p parallel to the spine v;
- the respective boundaries of $W^{\pm}$ are the lines $\ell^{\pm}$ which are both null, i.e. parallel to null vectors, and Lorentzian perpendicular to the spine v.

(Compare Figure 1.)

Given p and v, there are two choices of pairs $W^{\pm}$ which satisfy these conditions. Such a choice of wing pair is related to the sign of $\alpha(\gamma)$ for an appropriate $\gamma \in \Gamma$. One choice of wing pair gives rise to *positively oriented* crooked planes and the other choice gives rise to *negatively oriented* crooked planes. Crooked planes of different orientation always intersect (see [13, 14]) and positively oriented crooked planes will be associated with groups Γ such that $\alpha(\gamma) > 0$ for all $\gamma \in \Gamma$.

Theorem 5.2 (Drumm [10])**.** *If there exist $2n$ disjoint crooked planes $C_i(p_i, \mathsf{v}_i)$ for $0 < |i| \leq n$ such that $\gamma_i(C_{-i}(p_{-i}, \mathsf{v}_{-i})) = C_i(p_i, \mathsf{v}_i)$ (and γ_i maps the interior of the region bounded by the $C_{\pm i}$'s to a disjoint region) then $\langle \gamma_1, \ldots, \gamma_n \rangle$ is a free group on n generators which acts properly discontinuously and freely on $\mathbb{R}^{2,1}$.*

It is still not well understood which $\Gamma \subset \mathrm{Aff}(\mathbb{R}^3)$ act properly when $\mathbb{L}(\Gamma)$ is a discrete subgroup of $\mathrm{SO}(2,1)$. To this end, Drumm and Goldman have established exact conditions under which two crooked planes intersect in [14].

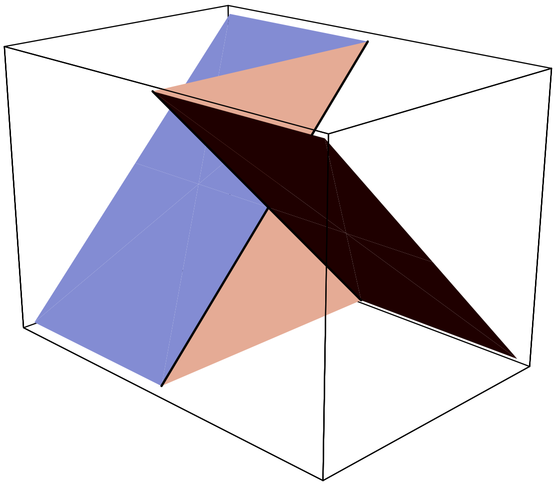

FIGURE 1. A crooked plane

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