COMPLEX HYPERBOLIC GEOMETRY

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This article summarizes a series of 3 90-minute talks presented at the joint TGRC-KOSEF workshop held at Kyungpook National University in Taegu on July 4–8, 1994. Most of the material I discussed may be found in the survey article, "Complex hyperbolic Kleinian groups," (published in "Complex Geometry," Proceedings of the Osaka International Conference, Marcel Dekker, pp. 31–52 (1992)), and my forthcoming book, "Complex Hyperbolic Geometry," (accepted for publication by Oxford Mathematical Monographs, Oxford University Press). I see no need to duplicate material already available in the literature. Furthermore a more informal and intuitive treatment of this materials may be valuable. For that reason I outline my lectures from the workshop, referring to the Osaka survey article and the monograph (henceforth abbreviated as [CHG]) for details.

1. The projective model of complex hyperbolic space

The first talk began with a motivation of complex hyperbolic geometry. Complex hyperbolic manifolds are the simplest examples of Kähler manifolds of negative curvature; such manifolds appear as moduli spaces of geometric structures of various sorts. For example Teichmüller space is a moduli space for hyperbolic-geometry structures on closed surfaces and carries, for example, a negatively curved Kähler structure. In his beautiful preprint, "Shapes of polyhedra," Thurston shows that the moduli space of Riemannian metrics on S^2, which are locally Euclidean except for a prescribed set of isolated singularities with prescribed cone angles, has the natural structure as a (possibly singular) complex hyperbolic manifold. Thus complex hyperbolic manifolds are prototypes of moduli spaces.

A complex hyperbolic manifold is a quotient $\mathbf{H}^{\mathbf{n}}_{\mathbb{C}}/\Gamma$ where Γ is a discrete subgroup of $\mathbf{PU}(\mathbf{n},\mathbf{1})$, the biholomorphic automorphism group of $\mathbf{H}^{\mathbf{n}}_{\mathbb{C}}$. Our investigation was motivated largely to construct discrete

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groups (and hence complex hyperbolic manifolds) by direct geometric means, specifically *Poincare's fundamental polyhedron technique.* Poincare's technique begins with a polyhedron P with a collection Υ of isometries identifying some of the faces. Under certain conditions, Υ generates a discrete group Γ having P as its fundamental domain.

Complex hyperbolic manifolds furnish also the simplest examples of negatively curved Riemannian manifolds not having *constant* curvature. The variation of the sectional curvature leads to intriguing complications in the geometry. In particular totally geodesic real hypersurfaces do not exist. The best substitute for such hypersurfaces seems to be metric bisectors which, although not totally geodesic, in complex hyperbolic space are foliated by totally geodesic submanifolds in two different ways.

Here is the definition of a (metric) bisector. Let (X, d) be a metric space. If $x, y \in X$ then the *equidistant hypersurface* $\mathfrak{E}\{\mathfrak{x}, \mathfrak{y}\}$ consists of all $u \in X$ satisfying

$$\rho(x, u) = \rho(y, u).$$

There are two types of totally geodesic submanifolds in $\mathbf{H}^{\mathbf{n}}_{\mathbb{C}}$, namely complex linear subspaces and totally real submanifolds. A complex linear subspace is the intersection of complex hyperbolic space with a projective subspace in the ambient projective space. An important feature of complex hyperbolic geometry is that such subspaces are totally geodesic: indeed this property essentially characterizes complex hyperbolic space. Such a complex linear subspace is itself a complex hyperbolic space of lower dimension.

Particularly important are the complex lines, sometimes called *complex geodesics.* In its standard model, a complex geodesic is a copy of the disc with the Poincaré model of hyperbolic geometry with constant curvature -1. Two distinct points x, y in $\mathbf{H}^{\mathbf{n}}_{\mathbb{C}} \cup \partial\mathbf{H}^{\mathbf{n}}_{\mathbb{C}}$ lie on a unique complex geodesic $l(x, y)$. The Bergman distance between x, y is just the distance between x and y with respect to the Poincaré metric on $l(x, y)$. Similarly, the geodesic joining x, y is just the geodesic from x to y in the Poincaré metric on $l(x, y)$. The boundary of a complex geodesic is called a *chain,* following É.Cartan.

2. Heisenberg geometry

The second talk discussed Heisenberg geometry and ended with a showing of my videotape, "Complex hyperbolic Kleinian groups," which was first presented at an invited hour address of the American Mathematical Society in October 1990. This videotape shows picture prepared with the computer program, "HEISENBERG," (written in collaboration with Mark Phillips and Robert Miner), which depicts objects on the boundary of complex hyperbolic 2-space. Complex hyperbolic 2-space is a ball of real dimension 4 with boundary a 3-sphere. Since it seems hard to visualize 4-dimensional objects, we chose to see how they intersect the boundary and use a natural kind of "stereographic projection" to view the objects in complex hyperbolic geometry. This is a very practical reason to consider the geometry of the boundary.

The natural structure on the boundary is that of the one-point compactification of the Heisenberg group, just as the boundary of real hyperbolic n-space is the one-point compactification of Euclidean $n-1$-space with conformal geometry. However the automorphisms of complex hyperbolic space do not act conformally in the usual sense. They act "confomally" with respect to the natural CR-structure on $\partial\mathbf{H}^{\mathbf{n}}_{\mathbb{C}}$ induced from the complex structure of complex projective space. (In the literature such transformations are sometimes called "pseudoconformal.") Explicitly, the boundary $\partial\mathbf{H}^{\mathbf{n}}_{\mathbb{C}}$ is S^{2n-1} embedded as a real hyperquadric in $\mathbb{P}^{\ltimes}_{\mathbb{C}}$. For each $x \in S^{2n-1}$, the tangent space $T_x(S^{2n-1})$ contains a *maximal complex subspace* $E_x = T_x(S^{2n-1}) \cap JT_x(S^{2n-1})$. These subspaces define a complex $n-1$-dimensional vector bundle E embedded as a field of real hyperplanes in TS^{2n-1}. This structure is what we call the CR-structure on the boundary of complex hyperbolic space. Corresponding to special choices (such as a point in complex hyperbolic space, a point on the boundary, or a totally geodesic subspace) there are special choices of 1-forms η on S^{2n-1} such that E is the kernel of $\eta : TS^{2n-1} \longrightarrow \mathbb{R}$. (Such 1-forms are called *calibrations of the CR-structure* in [CHG].) The restriction of $d\eta$ to E is a symplectic form, which together with the complex structure on E defines a Hermitian structure on E. (In the literature this extra structure — the calibration η of E as well as the Hermitian structure h on E — is called a *pseudo-Hermitian structure.*) Although this structure is not invariant under all automorphisms, it is "conformally invariant" in the sense that if g is an automorphism there exist functions f and f' on S^{2n-1} such that $g^*\eta = f \cdot \eta$ and $g^*h = f' \cdot h$.

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3. The geometry of bisectors

The third talk dealt with the geometry of bisectors. Although no totally geodesic real hypersurfaces exist, bisectors try their best to

FIGURE 1. Slices, meridian and complex spine of a vertical spinal sphere

be totally geodesic. In particular bisectors are minimal hypersurfaces of cohomogeneity one. They admit two different foliations by totally geodesic submanifolds. For example if $\epsilon \neq 0$, then the bisector E equidistant from the points $(\pm i\epsilon, 0)$ equals the set of all $z \in B^2$ such that $z_1 \in \mathbb{R}$.

Clearly $\mathfrak{E} = \mathfrak{E}\{\mathfrak{x}, \mathfrak{y}\}$ meets $l(x, y)$ in the orthogonal bisector σ of the geodesic segment $\overline{xy}$ in $l(x, y)$, called the *spine* of $\mathfrak{E}$. Let Π : $\mathbb{B}^{\ltimes} \longrightarrow \lessdot(\curvearrowleft, \curvearrowright)$ denote orthogonal projection onto the complex line $l = l(x, y)$. It is proved in Mostow's paper ("A remarkable class of polyhedra in complex hyperbolic space," Pac.J.Math. 86 (1980), 171-276) and also by Giraud ("Sur certaines fonctions automorphes de deux variables," Annales de l'École Normale (3^e série), **38** (1921), 43–164) that

$$\Pi(\mathfrak{E}) = \sigma, \qquad \mathfrak{E}\{\mathfrak{x}, \mathfrak{y}\} = \Pi^{-1}(\sigma),$$

i.e., $\mathfrak{E}$ is a fibration over σ with fibers complex hyperplanes orthogonal to l. If $\partial\sigma = \{v_1, v_2\} \subset \partial\mathbb{B}^{\ltimes}$, then these fibers (which Mostow calls the *slices* of $\mathfrak{E}$) are complex hyperplanes H characterized by the property that the corresponding inversion ι_H interchanges v_1 and v_2.

The slices of $\mathfrak{E}$ are the maximal holomorphic submanifolds, i.e., the integral submanifolds of the CR-structure on $\mathfrak{E}$ induced by the complex

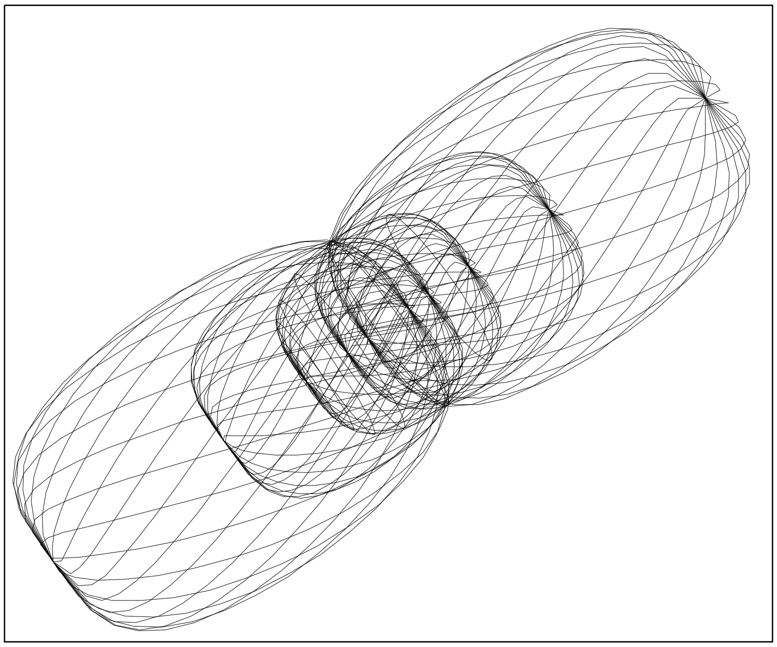

FIGURE 2. Spinal spheres with a common slice

structure of $\mathbb{B}^{\ltimes}$. It follows that the spine of $\mathfrak{E}$ determines $\mathfrak{E}$ and is determined by $\mathfrak{E}$; thus there is a natural one-to-one correspondence between equidistant hypersurfaces (real codimension one) and geodesics (real dimension one). This "Poincaré duality" is manifest by the symplectic geometry of the Kähler form on $\mathbb{B}^{\ltimes}$: the function

$$
\begin{aligned}
z \mapsto \sinh(\rho(\Pi(z), \sigma)) \cosh^2(\frac{1}{2}\rho(z, \Pi(z))) \\
= \sinh(\rho(z, \mathfrak{E})) \cosh^2(\frac{1}{2}\rho(\mathfrak{z}, \mathfrak{l}))
\end{aligned}
$$

is a Hamiltonian potential function for the hyperbolic one-parameter subgroup stabilizing σ.

There is another characterization of equidistant hypersurfaces in terms of the other class of totally geodesic submanifolds: $\mathfrak{E}$ is the union of all totally real geodesic submanifolds containing σ. We call these submanifolds the *meridians* of $\mathfrak{E}$. Each meridian intersects a slice in a geodesic γ; the corresponding *conjugation* preserves the slice and acts by reflection in γ.

We choose coordinates on $\mathfrak{E}$ as follows. Let $t : \mathfrak{E} \longrightarrow \mathbb{R}$ be the distance along the spine, having the slices as level sets. Let $u : \mathfrak{E} \longrightarrow \mathbb{R}$ be the distance to the spine and let θ be the corresponding angular variable on each slice. The meridians of $\mathfrak{E}$ are the level sets of θ. An orthonormal frame (at least away from σ) is given by:

$$e_1 = \operatorname{sech}(u/2)\frac{\partial}{\partial t}, \qquad e_2 = \frac{\partial}{\partial u}, \qquad e_3 = \operatorname{csch}(u)\frac{\partial}{\partial \theta}$$

and with respect to this frame the second fundamental form is given by:

$$\begin{bmatrix} 0 & 0 & f(u) \\ 0 & 0 & 0 \\ f(u) & 0 & 0 \end{bmatrix}$$

where

$$f(u) = \sqrt{\frac{\tanh(\frac{u}{2})}{2\tanh(u)} - \frac{1}{4}}$$

and the sectional curvatures of $\mathfrak{E}$ are bounded between -1 and $-1/4$ (as in the ambient space itself). Furthermore $\mathfrak{E}$ is minimal although not totally geodesic. The equidistant hypersurfaces nonetheless enjoy the weak convexity property that a geodesic whose endpoints lie on $\partial\mathfrak{E}$ is either completely disjoint from $\mathfrak{E}$ or contained in either a slice or a meridian of $\mathfrak{E}$.

In general, these hypersurfaces can enjoy rather bizarre intersections. The Riemannian angle of intersection is generally not constant (unless the spines lie in a common complex geodesic) and the intersection need not even be a manifold. For example if the spines σ_i are ultraparallel geodesics in a totally real geodesic 2-plane P, then the intersection of the corresponding equidistant hypersurfaces $\mathfrak{E}_i$ is the union of the common meridian P and the common slice C (the complex geodesic orthogonal to the σ_i) along the real geodesic orthogonal to the spines. This unusual intersection was used in the 1932 paper of É.Cartan ("Sur le groupe de la géométrie hypersphérique," Comm. Math. Helv. **4** (1932), 158–171) to characterize the relation that three points on the boundary of complex hyperbolic 2-space lie on the boundary of a complex line in terms of the relation that points lie on the boundary of a totally real subspace.

This intersection leads to examples of disconnected bisector intersections. Begin with bisectors B_1, B_2 such that $B_1 \cap B_2$ equals the union $R \cup C$ where R is a totally real geodesic submanifold and C is a complex geodesic. This intersection is not transverse (in local coordinates, the intersection has saddle contact) and is thus unstable under perturbation. For some generic perturbations B_1' of B_1, the intersection $B_1' \cap B_2$ will be disconnected.

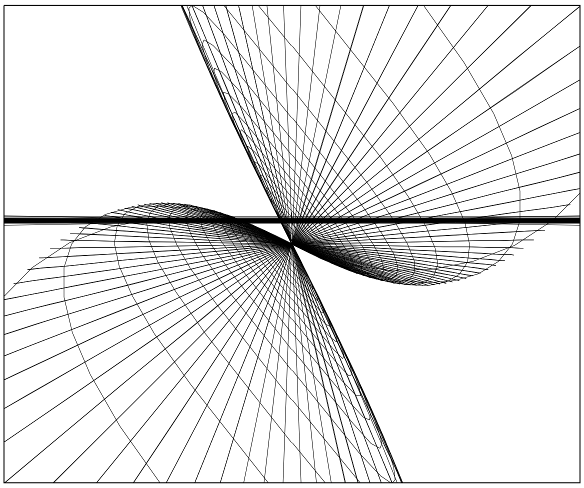

FIGURE 3. A disconnected intersection of two spinal spheres

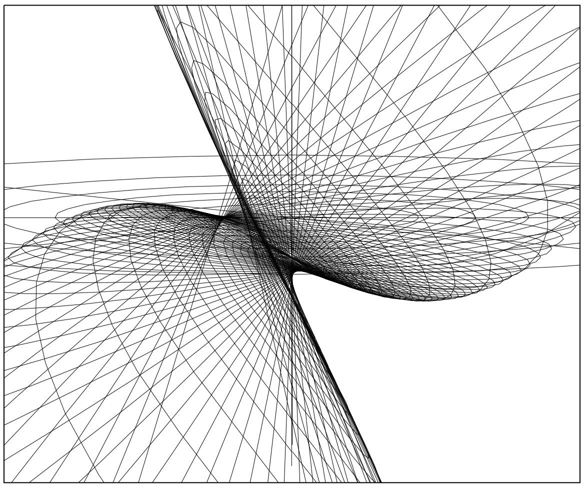

FIGURE 4. Disconnected intersection of spinal spheres: another view

One may take

Figure 5 depicts the level sets of a function defining

Another approach to this problem involves the orthogonal projection. Let $\Pi_i : \mathbf{H}^{\mathbf{n}}_{\mathbb{C}} \longrightarrow \Sigma_{\mathbf{i}}$ be orthogonal projection and let $\sigma_i \subset \Sigma_i$ be the spine. By the Mostow-Giraud decomposition, $B_i = \Pi_i^{-1}(\sigma_i)$ and the restriction

$$\Pi_1 : B_1 \cap B_2 \longrightarrow \sigma_1 \cap \Pi_1(B_2)$$

is a "π_0-isomorphism," that is , it maps the set of components of $B_1 \cap B_2$ bijectively to the set of components of $\sigma_1 \cap \Pi_1(B_2)$. In particular $\Pi_1(B_2)$