1. The Runge function is
\[ r(x) = \frac{1}{1 + 25x^2}, \quad -1 \leq x \leq 1. \]

(a) For \( n = 5, 10, 15 \), plot \( p_n(x) \), the polynomial interpolating \( r(x) \) at \( n + 1 \) equally spaced points, along with the graph of \( r(x) \). Use the MATLAB functions POLYFIT and POLYVAL. Observe what is happening to the graphs. Where is the polynomial fit getting better? Where is it getting worse?

(b) Repeat part (a) but now use the interpolation points
\[ x_j = \cos \left( \frac{(2j - 1)\pi}{2n + 2} \right), \quad j = 1, \ldots, n + 1. \]

What difference do you observe?

2. Given the data \((-1, 2), (0, 2), (1, 2), (2, 5)\), calculate \( P_3(x) \), the cubic polynomial interpolating this data (by hand) in three ways:

(a) Solve the Vandermonde system.

(b) Use Lagrange Polynomials.

(c) Find the Newton form, using the divided difference table.

Show that you get the same polynomial in each case.

3. For \( f(x) = \sinh x \) we are given that
\[ f(0) = 0, \quad f'(0) = 1, \quad f(1) = 1.1752, \quad f'(1) = 1.5431. \]

Calculate an approximation to \( f(0.5) \) using cubic Hermite interpolation. Compare the result with \( f(0.5) = .5211 \).

4. Consider the function \( S(x) \) defined as
\[
S(x) = \begin{cases} 
28 + 25x + 9x^2 + x^3, & -3 \leq x \leq -1, \\
26 + 19x + 3x^2 - x^3, & -1 \leq x \leq 0, \\
26 + 19x + 3x^2 - 2x^3, & 0 \leq x \leq 3, \\
-163 + 208x - 60x^2 + 5x^3, & 3 \leq x \leq 4.
\end{cases}
\]

Show that \( S(x) \) is a natural cubic spline function with the knots \( \{-3, -1, 0, 3, 4\} \). (A natural cubic spline is a spline \( S''(x_1) = S''(x_N) = 0 \) Be sure to state explicitly each of the properties of \( S(x) \) which are necessary for this to be true.

5. The vapor pressure \( P \) of water (in bars) as a function of temperature \( T (\degree C) \) is

<table>
<thead>
<tr>
<th>( T )</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(T) )</td>
<td>.006107</td>
<td>.012277</td>
<td>.023378</td>
<td>.042433</td>
</tr>
<tr>
<td>( T )</td>
<td>40</td>
<td>50</td>
<td>60</td>
<td>70</td>
</tr>
<tr>
<td>( P(T) )</td>
<td>.073774</td>
<td>.12338</td>
<td>.19924</td>
<td>.31166</td>
</tr>
<tr>
<td>( T )</td>
<td>80</td>
<td>90</td>
<td>100</td>
<td>110</td>
</tr>
<tr>
<td>( P(T) )</td>
<td>.47364</td>
<td>.70112</td>
<td>1.01325</td>
<td>1.22341</td>
</tr>
</tbody>
</table>
Interpolate these data with the cubic spline $S(x)$ using the MATLAB function SPLINE and plot the results. It is also known that $P(5) = 0.08721$, $P(45) = 0.095848$ and $P(95) = 0.84528$. How well does $S(x)$ do at these points?


11. For the CENSUSGUI data on p.144 find the cubic polynomial $p_3(s) = \beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4$, where $s$ is the translated and scaled time variable, which interpolates the data in the sense of least squares by constructing the $11 \times 4$ data matrix $A$ and finding the vector $(\beta_1, \beta_2, \beta_3, \beta_4)^T$ in four different ways:

   (a) By using the backslash operator.

   (b) By forming and solving the normal equations. Note the condition number of the matrix $A^T A$.

   (c) By using the $QR$ decomposition.

   (d) By using the Singular-Value Decomposition.

All of this is quite easy in MATLAB. Compare with the values given by POLYFIT.