1. (10 points) Let \( f(x) = \frac{1}{x} \). Let \( x_0, x_1, \ldots, x_k \) be distinct non-zero numbers. Show that
\[
f[x_0, x_1, \ldots, x_k] = (-1)^k \prod_{i=0}^{k} \frac{1}{x_i}
\]
(Hint: Use induction.)

2. (10 points) Construct a polynomial \( p(x) \) of degree at most 2 for which
\[
p(x_0) = y_0, \quad p'(x_1) = z_0, \quad p(x_2) = y_2,
\]
where \( x_0, x_1, x_2, y_0, z_0, y_2 \) are given numbers with \( x_0 \neq x_2 \). Under what conditions will \( p(x) \) exist and be unique?

3. (12 points) Let \( f \in C[a,b] \). Let \( S_n \) be the result of applying the \( n \) panel Simpson’s rule to \( f \) (of course \( n \) must be even). Show that
\[
\lim_{n \to \infty} S_n = \int_a^b f(x) \, dx.
\]
(Hint: Show \( S_n \) is a Riemann Sum.)

4. (13 points) Let
\[
\int_a^b f(x)w(x) \, dx \approx \sum_{i=1}^{n} w_i f(x_i)
\]
be an \( n \)-point Gaussian integration rule.
(a) What are the \( x_i \)?
(b) For which polynomials is the formula exact?
(c) Prove \( w_i > 0, \ i = 1, \ldots, n \).

5. (15 points) Let \( A \in R^{m \times n} \).
(a) What is the Singular Value Decomposition for \( A \)?
(b) Define
\[
\|A\|_2 = \max_{\|x\|_2=1} \frac{\|Ax\|_2}{\|x\|_2} = \max_{\|x\|_2=1} \|Ax\|_2
\]
Show that \( \|A\|_2 = \sigma_1 \), the largest singular value of \( A \).

6. (15 points) Let \( A \) be an invertible matrix. Let \( C_0 \) be given. Define \( R_0 = I - AC_0 \) and assume \( \|R_0\| < 1 \) for some matrix norm. Define the iteration method
\[
C_{m+1} = C_m(I + R_m), \quad R_{m+1} = I - AC_{m+1}, \quad m \geq 0
\]
(a) Show that \( C_m \) converges to \( A^{-1} \) by first relating the error \( A^{-1} - C_m \) to the residual \( R_m \). Then examine the behavior of the residual \( R_m \) by showing \( R_{m+1} = R_m^2, \ m \geq 0 \).
(b) Relate $C_m$ to the expansion

$$A^{-1} = C_0(I - R_0)^{-1} = C_0 \sum_{j=0}^{\infty} R_0^j.$$ 

7. (25 points) Let $A \in \mathbb{R}^{n \times n}$ be nonsingular and $b \in \mathbb{R}^n$. Consider the following iteration for the solution of $Ax = b$.

$$x_{k+1} = x_k + \alpha (b - Ax_k).$$

(a) Show that if all the eigenvalues of $A$ have positive real part then there will be some real $\alpha$ such that the iterates converge for any starting vector $x_0$. Discuss how to choose $\alpha$ optimally in case $A$ is symmetric and determine the rate of convergence.

(b) Show that if some eigenvalues of $A$ have negative real part and some have positive real part, then there is no real $\alpha$ for which the iterations converge.

(c) Let $\rho = \|I - \alpha A\| < 1$ for a matrix norm associated to a vector norm. Show that the error can be expressed in terms of the difference between consecutive iterates, namely

$$\|x - x_{k+1}\| \leq \frac{\rho}{1 - \rho} \|x_k - x_{k+1}\|$$

(The proof of this is short but a little tricky.)

(d) Let $A$ be the tridiagonal matrix

$$A = \begin{pmatrix}
3 & 1 & 0 & 0 & \cdots & 0 \\
-1 & 3 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 3 & 1 \\
0 & \cdots & \cdots & \cdots & -1 & 3
\end{pmatrix}$$

Find a value of $\alpha$ that guarantees convergence.