1. Define \( f(x) \) by
\[
f(x) = \frac{1}{1 + 25x^2}, \quad -1 \leq x \leq 1.
\]

(a) For \( n = 5, 10, 15 \), plot \( p_n(x) \), the polynomial interpolating \( f(x) \) at \( n + 1 \) equally spaced points, along with the graph of \( f(x) \). Use the MATLAB functions POLYFIT and POLYVAL. Observe what is happening to the graphs. Where is the polynomial fit getting better? Where is it getting worse?

(b) Repeat part (a) but now use the interpolation points
\[
x_j = \cos \left( \frac{(2j - 1)\pi}{2n + 2} \right), \quad j = 1, \ldots, n + 1.
\]

What difference do you observe?

2. Ex. 3.10, p.90, Shampine, Allen & Preuss.

3. For \( f(x) = \cosh x \) we are given that
\[
f(0) = 1, \quad f'(0) = 0, \quad f(1) = 1.5431, \quad f'(1) = 1.1752.
\]

Calculate an approximation to \( f(0.5) \) using cubic Hermite interpolation. Compare the result with \( f(0.5) = 1.1276 \).

4. Ex. 3.16, p.98, Shampine, Allen & Preuss.

5. Consider the function \( S(x) \) defined as
\[
S(x) = \begin{cases} 
28 + 25x + 9x^2 + x^3, & -3 \leq x \leq -1, \\
30 + 31x + 15x^2 + 3x^3, & -1 \leq x \leq 0, \\
30 + 31x + 15x^2 - 3x^3, & 0 \leq x \leq 3, \\
-159 + 220x - 48x^2 + 4x^3, & 3 \leq x \leq 4.
\end{cases}
\]

Show that \( S(x) \) is a natural cubic spline function with the knots \{−3, −1, 0, 3, 4\}. (A natural cubic spline is a spline \( S(x) \) which satisfies \( S''(x_1) = S''(x_N) = 0 \) Be sure to state explicitly each of the properties of \( S(x) \) which are necessary for this to be true.

6. Ex. 3.23, p.116, Shampine, Allen & Preuss. Use the MATLAB function SPLINE. Your output should be a graph. To judge how well the spline does compare values at the remaining seven points.

7. Repeat part (a) of the previous problem using \( p_8(x) \), the polynomial of degree \( \leq 8 \) which interpolates \( f \) at the nine indicated points. How does it do compared with the spline?

8. Take the nine points from Ex. 3.23 and find and plot the polynomials of degree 2, 3 and 4 which best fit this data in the sense of least squares. The MATLAB functions POLYVAL and POLYFIT give you exactly what you need.
For the data of problem 8 find the cubic polynomial \( p_3(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3 \) interpolating the data in the sense of least squares by constructing the \( 9 \times 4 \) data matrix \( A \) and finding the vector \((p_0, p_1, p_2, p_3)^T\) in four different ways:

(a) By using the backslash operator.

(b) By forming and solving the normal equations. Note the condition number of the matrix \( A^T A \).

(c) By using the QR decomposition.

(d) By using the Singular-Value Decomposition.

All of this is quite easy in MATLAB. Compare with the values found in problem 8.