Minicourse
University of Maryland
First Lecture
Introduction to Göbner Bases

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Let k be a field (e.g. $k = \mathbf{Q}, \mathbf{R}$, or $\mathbf{C}$).

Let $k[x_1, \ldots, x_n]$ be the set of all polynomials in n variables with coefficients in k. E.g., with $k = \mathbf{Q}$ and $n = 3$

$$f = x_1^3 x_2 + 8x_1x_2x_3 - \frac{17}{2}x_1^2x_3^3 + 37$$

is such a polynomial.

Problem: Given $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$, find the solutions to the system

$$\begin{cases} f_1 = 0 \\ f_2 = 0 \\ \quad\vdots \\ f_s = 0 \end{cases}$$

We will concentrate on the case where the number of solutions is finite, but much of what will be said holds in the more general case of infinitely many solutions.

Example 1. Lagrange Multiplyers.

Find the minimum and maximum of

$$f(x, y, z) = x^3 + 2xyz - z^2$$

subject to the constraint

$$g(x, y, z) = x^2 + y^2 + z^2 = 1$$

So we need to solve the equations given by the vector equation

$$\bigtriangledown f = \lambda \bigtriangledown g$$

together with

$$g(x, y, z) = 1$$

The system of equations is

$$\begin{cases} x^2 + y^2 + z^2 = 1 \\ 3x^2 + 2yz = \lambda(2x) \\ 2xz = \lambda(2y) \\ 2xy - 2z = \lambda(2z) \end{cases}$$

$$\begin{cases} \lambda = \frac{3}{2}x + \frac{3}{2}yz + \frac{167616}{3835}z^6 + \frac{36717}{590}z^4 + \frac{134419}{7670}z^2 \\ x^2 + y^2 + z^2 - 1 = 0 \\ xy - \frac{19584}{3835}z^5 + \frac{1999}{295}z^3 - \frac{6403}{3835}z = 0 \\ xz + yz^2 - \frac{1152}{3835}z^5 - \frac{108}{295}z^3 + \frac{2556}{3835}z = 0 \\ y^3 + yz^2 - y - \frac{9216}{3835}z^5 + \frac{906}{295}z^3 - \frac{2562}{3835}z = 0 \\ y^2z - \frac{6912}{3835}z^5 + \frac{827}{295}z^3 - \frac{3839}{3835}z = 0 \\ yz^3 - yz - \frac{576}{59}z^6 + \frac{1605}{118}z^4 - \frac{453}{118}z^2 = 0 \\ z^7 - \frac{1763}{1152}z^5 + \frac{655}{1152}z^3 - \frac{11}{288}z = 0 \end{cases}$$

The solutions are

$$\begin{cases} z = 0, y = 0, x = \pm 1. \\ z = 0, y = \pm 1, x = 0. \\ z = \pm 1, y = 0, x = 0. \\ z = 2/3, y = 1/3, x = -2/3. \\ z = -2/3, y = -1/3, x = -2/3. \end{cases}$$

Why exact methods instead of, or together with, numerical methods?

- exact methods avoid issues of conditioning and stability;
- numerical methods cannot guarantee that all solutions will be found or prove that no solutions exist;
- many systems which arise in practice contain parameters, and hence certain systems of equations must be solved over non-numerical domains.

Basic assumption: we can compute the roots of a single polynomial equation in one variable (using algebraic methods, such as factoring, and numerical methods, such as Newton's method).

Our strategy for solving a system of polynomial equations: reduce the problem of solving a system of non-linear equations into a problem involving solving a single polynomial in one variable (if possible).

Example 2. Parametrizations.

Consider the following parametrization of a curve in $\mathbf{R}^2$.

$$\begin{cases} x = c + dt^2 + 1 \\ y = ct + dt^3 + 1 \end{cases}$$

where t is the parameter and c, d are constants. What is this curve?.

The 2 equations above can be viewed as two equations in three unknowns (x, y, t) with coefficients in an extension of the rationals, $\mathbf{Q}[c, d]$. Using the algorithm presented later, we can change this system into the following equivalent system:

$$\begin{cases} dt^2 - x + c + 1 = 0 \\ y - xt + t + 1 = 0 \\ x^2 - (c+2)x - dty + dt + c + 1 = 0 \\ dy^2 - x^3 - 2dy - (2c+3)x + \\ (c+3)x^2 + c + d + 1 = 0 \end{cases}$$

From the last equation, one can see that if $d = 0$, then we get a cubic equation in x of degree 3. The solutions are

$$x = 1, y = 1, \text{ and } x = c + 1, y = 1 - c$$

If $d \neq 0$, then the last equation is the elliptic curve

$$d(y-1)^2 - (x-1)^3 + c(x-1)^2 = 0$$

The idea in solving the system

$$f_1 = 0, \ldots, f_s = 0$$

is to replace the given equations by equations which have the same solutions but which are "easier" to solve.

We define $V(f_1, \ldots, f_s)$ to be the set of all points $(a_1, \ldots, a_n) \in \mathbf{C}^n$ such that $f_i(a_1, \ldots, a_n) = 0$, for $i = 1, \ldots, s$. This set is called a *variety* or *algebraic set*.

So we want to find a collection of "nice" polynomials $g_1, \ldots, g_t$ such that

$$V(f_1, \ldots, f_s) = V(g_1, \ldots, g_t).$$

To find these polynomials we translate this problem into another algebra problem.

We define $\langle f_1, \ldots, f_s \rangle$ to be the set of all possible linear combinations $h_1 f_1 + \cdots + h_s f_s$ with $h_i \in k[x_1, \ldots, x_n]$

This set is called the *ideal* generated by $f_1, \ldots, f_s$.

Note that

$$V(f_1, \ldots, f_s) = V(\langle f_1, \ldots, f_s \rangle)$$

and so if $\langle f_1, \ldots, f_s \rangle = \langle g_1, \ldots, g_t \rangle$, then

$$V(f_1, \ldots, f_s) = V(g_1, \ldots, g_t).$$

So our strategy is to find a "good" generating set for the ideal $\langle f_1, \ldots, f_s \rangle$.

We will use 3 methods for changing $f_1, \ldots, f_s$ into a better set $g_1, \ldots, g_t$.

- If $h = f_1 + h_2 f_2 + \cdots + h_s f_s$, then
$$\langle f_1, \ldots, f_s \rangle = \langle h, f_2, \ldots, f_s \rangle$$
and
$$V(f_1, \ldots, f_s) = V(h, f_2, \cdots, f_s)$$
- If $h = h_1 f_1 + h_2 f_2 + \cdots + h_s f_s$, then
$$\langle f_1, \ldots, f_s \rangle = \langle h, f_1, \ldots, f_s \rangle$$
and
$$V(f_1, \ldots, f_s) = V(h, f_1, \ldots, f_s)$$
- If c is a non-zero element of k we have
$$\langle f_1, \ldots, f_s \rangle = \langle c f_1, \ldots, f_s \rangle$$
and
$$V(f_1, \ldots, f_s) = V(c f_1, \ldots, f_s)$$

We note that the first and third operations are similar to row operations.

Linear Example.

$$\begin{cases} f_1 = x_1 - 5x_2 - 3x_3 + 2 = 0 \\ f_2 = -2x_1 + 11x_2 + 9x_3 - 3 = 0 \\ f_3 = -x_1 + 6x_2 + 8x_3 - 1 = 0 \end{cases}$$

These equations define an ideal $\langle f_1, f_2, f_3 \rangle$ in $\mathbf{Q}[x_1, x_2, x_3]$

Let $h_2 = f_2 + 2f_1 = x_2 + 3x_3 + 1$. Then

$$\langle f_1, f_2, f_3 \rangle = \langle f_1, h_2, f_3 \rangle$$

and

$$V(f_1, f_2, f_3) = V(f_1, h_2, f_3).$$

Let $h_3 = f_3 + f_1 = x_2 + 5x_3 + 1$. Then

$$\langle f_1, h_2, f_3 \rangle = \langle f_1, h_2, h_3 \rangle$$

and

$$V(f_1, h_2, f_3) = V(f_1, h_2, h_3)$$

Let $h_4 = h_3 - h_2 = 2x_3$. Then $\langle f_1, h_2, h_3 \rangle = \langle f_1, h_2, h_4 \rangle$ and $V(f_1, h_2, h_3) = V(f_1, h_2, h_4)$

We have triangularize the system, and the solutions to $f_1 = 0, h_2 = 0, h_4 = 0$ are easy to compute.

Indeed, we use back substitution:

The solution to $h_4 = 0$ gives $x_3 = 0$.

Substituting $x_3 = 0$ in h_2 gives $x_2+1 = 0$ and so $x_2 = -1$.

Substituting $x_2 = -1$ and $x_3 = 0$ in $f_1 = 0$ gives $x_1 + 7 = 0$ and so $x_1 = -7$.

One Variable Example. In this case, a system

$$f_1 = 0, f_2 = 0, \ldots, f_s = 0$$

can be reduced to a single polynomial equation $g = 0$, where $g = \text{gcd}(f_1, \ldots, f_s)$, i.e. $V(f_1, \ldots, f_s) = V(g)$.

We illustrate this with a system of 2 polynomials $f_1 = 0, f_2 = 0$.

Divide f_1 by f_2 to get $f_1 = q_1 f_2 + r_1$. We have $r_1 = f_1 - q_1 f_2$ and so $\langle f_1, f_2 \rangle = \langle f_2, r_1 \rangle$ and $V(f_1, f_2) = V(f_2, r_1)$.

Now divide f_2 by r_1 to get $f_2 = q_2 r_1 + r_2$. We have $r_2 = f_2 - q_2 r_1$ and so $\langle f_2, r_1 \rangle = \langle r_1, r_2 \rangle$ and $V(f_2, r_1) = V(r_1, r_2)$.

Continue until $r_{\ell+1} = 0$, and we get

$$\langle f_1, f_2 \rangle = \langle f_2, r_1 \rangle = \langle r_1, r_2 \rangle = \cdots \langle r_\ell, 0 \rangle = \langle r_\ell \rangle$$

and

$$V(f_1, f_2) = V(f_2, r_1) = \cdots = V(r_\ell, 0) = V(r_\ell).$$

One recognizes that $r_\ell = \text{gcd}(f_1, f_2)$.

How do we divide?

Consider, for example, $f_1 = 4x^4 - 7x^3 + 3x + 1$ and $f_2 = 2x^3 + 3x + 1$.

The first division is

$$r_1 = f_1 - 2xf_2 = -7x^3 - 6x^2 + x + 1$$

Notation: $f = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, $a_n \neq 0$.

$\text{lt}(f) = a_n x^n =$ the leading term of f
$\text{lp}(f) = x^n =$ the leading power product of f
$\text{lc}(f) = a_n =$ the leading coefficient of f

So the division above can be viewed as follows: since $\text{lp}(f_2) | \text{lp}(f_1)$

$$r_1 = f_1 - \frac{\text{lt}(f_1)}{\text{lt}(f_2)} f_2.$$

To generalize to n variables we need an order on the power products.

We use lexicographical order with

$$x_1 > x_2 > \cdots > x_n$$

(alphabitize with x_1 first, x_2 second etc.)

$$x_1^{\nu_1} x_2^{\nu_2} \cdots x_n^{\nu_n} > x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n} \Longleftrightarrow$$

the first j such that $\nu_j \neq \mu_j$,

is such that $\nu_j > \mu_j$.

Examples. $x_1^3 x_3 > x_1^2 x_2^5 x_3^4$ and $x_1^3 x_2 > x_1^3 x_3^4$.

Given $f = cX +$ lower terms , where $c \in k - \{0\}$ and X is a power product, we define

$$\begin{aligned} &\mathrm{lt}(f) = cX = \text{the leading term of } f \\ &\mathrm{lp}(f) = X = \text{the leading power product of } f \\ &\mathrm{lc}(f) = c = \text{the leading coefficient of } f. \end{aligned}$$

Example. $f = 7x_1^3x_2 + 5x_1^3x_3 + 8x_1^2x_2^3x_4^5$.

$$\begin{aligned} &\mathrm{lt}(f) = 7x_1^3x_2 \\ &\mathrm{lp}(f) = x_1^3x_2 \\ &\mathrm{lc}(f) = 7. \end{aligned}$$

Now we can define what we meant by a "nice" system of equations.

First note that if lp$(g) = x_n^{\nu_n}$, then $g \in k[x_n]$. If lp$(g) = x_{n-1}^{\nu_{n-1}}$, then $g \in k[x_{n-1}, x_n]$. Etc.

So if $V(f_1, \ldots, f_s) = V(g_1, \ldots, g_t)$, then $g_1, \ldots, g_t$ will be "nice" if lp$(g_j) = x_j^{\nu_j}$ for $j = 1, \ldots, n$. (Some g_j may be left over).

THEOREM. This can always be done in the case where $V(f_1, \ldots, f_s)$ is finite.

Once the polynomials g_j have been computed, we have in effect triangularize the system.

We solve $g_n = 0$ to obtain all the possible x_n values, then, for each of these values, we solve $g_{n-1} = 0$, substituting for x_n, to obtain all the possible values of x_{n-1}. Etc.

The solutions thus found are then plugged into $g_{n+1} = 0, \ldots, g_t = 0$.

We now define division for multivariate polynomials.

Given $f, g \in k[x_1, \ldots, x_n]$ with $\mathsf{lt}(g) | \mathsf{lt}(f)$ we write

$$f \xrightarrow{g} h$$

provided that $h = f - \frac{\mathsf{lt}(f)}{\mathsf{lt}(g)} g$.

Note:

- We used the leading term of g to cancel the leading term of f.
- $V(f, g, \ldots) = V(h, g, \ldots)$
- $\langle f, g, \ldots \rangle = \langle h, g, \ldots \rangle$.
- this mimics the linear and single variable cases.

If $F = \{f_1, \ldots, f_s\}$, we write

$$f \xrightarrow{F}_{+} h$$

provided that

$$f \xrightarrow{f_{i_1}} h_1 \xrightarrow{f_{i_2}} h_2 \cdots$$

$$\cdots \xrightarrow{f_{i_{\ell-1}}} h_{\ell-1} \xrightarrow{f_{i_\ell}} h$$

(We say f reduces to h.)

Again

$$V(f, f_1, \ldots, f_s) = V(h, f_1, \ldots, f_s)$$

and

$$\langle f, f_1, \ldots, f_s \rangle = \langle h, f_1, \ldots, f_s \rangle$$

Example. Consider the following system

$$\begin{cases} f_1 = x^2 + y^2 + z^2 - 1 = 0 \\ f_2 = 4x^2 + xy + y^2 + z^2 - 1 = 0 \\ f_3 = x + y - z - 1 = 0 \end{cases}$$

We will use lex with $x > y > z$.

Then

$$f_2 \xrightarrow{f_1} xy - 3y^2 - 3z^2 + 3$$

$$\xrightarrow{f_3} -4y^2 + yz + y - 3z^2 + 3.$$

NOTE: It is not clear now what this gains us (except that we now have a polynomial in y and z alone). But it is what we did in both the linear and one variable cases to get a nicer set of equations to solve.

Properties of our nice sets in the special cases.

ONE VARIABLE CASE: In this case we had

$$V(f_1, \ldots, f_s) = V(g)$$

where $g = \text{gcd}(f_1, \ldots, f_s)$. Indeed,

$$\langle f_1, \ldots, f_s \rangle = \langle g \rangle.$$

Moreover, with $G = \{g\}$ then $f \in \langle g \rangle$ if and only if $g|f$ if and only if $f \xrightarrow{G}_+ 0$. This latter statement is FALSE if $G = \{f_1, \ldots, f_s\}$. For example,

$$\langle x^2 + x, x^2 - x \rangle = \langle x \rangle$$

but $x \in \langle x^2 + x, x^2 - x \rangle$ and x does not reduce to zero using $\{x^2 + x, x^2 - x\}$.

LINEAR CASE: We got for linear $f_1, \ldots, f_s$ that

$$\langle f_1, \ldots, f_s \rangle = \langle g_1, \ldots, g_t \rangle$$

where the set $G = \{g_1, \ldots, g_t\}$ is triangularized. Moreover given $f \in \langle f_1, \ldots, f_s \rangle$ you can check that $f \xrightarrow{G}_+ 0$, but that that is false if we replace $\{g_1, \ldots, g_t\}$ with $\{f_1, \ldots, f_s\}$.

MAIN DEFINITION:
A FINITE set $G = \{g_1, \ldots, g_t\}$ is a Gröbner basis for $I = \langle f_1, \ldots, f_s \rangle$ if and only if for all $f \in I$ we have

$$f \xrightarrow{G}_+ 0.$$

For a subset $S \subseteq k[x_1, \ldots, x_n]$ let

$$\mathrm{Lt}(S) = \langle \mathrm{lt}(s) | s \in S \rangle.$$

USEFUL EQUIVALENT DEFINITIONS: The following are equivalent

1. G is a Gröbner basis for I.

2. For all $f \in I$ there is an i such that $\mathrm{lp}(g_i) | \mathrm{lp}(f)$.

3. $\mathrm{Lt}(G) = \mathrm{Lt}(I)$.

Sketch of the proof:

(1. $\Longrightarrow$ 2.) If $f \in I$ then $f \xrightarrow{G}_+ 0$ means that $\mathsf{lt}(f)$ is canceled by a g_i and so $\mathsf{lp}(g_i)|\mathsf{lp}(f)$.

(2. $\Longrightarrow$ 3.) Trivial.

(3. $\Longrightarrow$ 1.) If $f \in I$ then

$$\mathsf{lt}(f) = \sum_{i=1}^{t} h_i \mathsf{lt}(g_i)$$

and from this it is easy to see that we can cancel the leading term of f and so $f \xrightarrow{G}_+ h$ with $h \in I$ also. Repeating this argument we would see that $f \xrightarrow{G}_+ 0$.

Corollary: Every ideal has a Gröbner basis.

NEED: Hilbert Basis Theorem. Given any set $\mathcal{S}$ of ideals in $k[x_1, \ldots, x_n]$ there is an $I \in \mathcal{S}$ such that no ideal of $\mathcal{S}$ is bigger than I.

Given this, we let $\mathcal{S}$ be the set of all ideals of the form Lt(F) for finite subsets F of I.

Choose a largest such ideal $Lt(G)$. Then if $f \in I$ and lt(f) $\notin$ Lt(G) we would have Lt($G \cup \{f\}$) is strictly bigger that Lt(G) which cannot be. So G is a Gröbner basis of I.

MAIN THEOREM: Let $G = \{g_1, \ldots, g_t\}$ be a Gröbner basis for $I = \{f_1, \ldots, f_s\}$ where we assume that g_i is monic for all i. Then $V(f_1, \ldots, f_s)$ is finite if and only if for each $i = 1, \ldots, n$ there is a j such that

$$\mathrm{lp}(g_j) = x_i^{\nu_i}.$$

So we have an easy to use criterion to determine when a system of polynomial equations has a finite number of solutions.

To prove this theorem, we need another result of Hilbert.

HILBERT NULLSTELLENSATZ:
If $I = \langle f_1, \ldots, f_s \rangle$ is an ideal of $k[x_1, \ldots, x_n]$ and g vanishes on $V(f_1, \ldots, f_s)$ then there is an integer ν such that $g^\nu \in I$.

We first consider the case where we assume that $V(f_1,\ldots,f_s)$ is finite. We will prove the result only in the case that $k = \mathbf{C}$.

We fix an i. Let $a_1,\ldots,a_m$ denote the ith coordinates of all the points in $V(f_1,\ldots,f_s)$. Then

$$g(x_i) = (x_i - a_1)(x_i - a_2)\cdots(x_i - a_m)$$

vanishes on $V(f_1,\ldots,f_s)$ and so by the Hilbert Nullstellensatz $g^\nu \in I$, for some integer ν.

Thus there is a j such that $\text{lt}(g_j)|\text{lt}(g^\nu)$. But $\text{lt}(g^\nu) = x_i^{m\nu}$ and so $\text{lp}(g_j)$ must be a power of x_i as desired.

For the other direction, we prove that we have actually triangularized the basis. By reordering the g_i's we may assume that $\mathrm{lt}(g_i) = x_i^{\nu_i}$. Then $\mathrm{lt}(g_n) = x_n^{\nu_n}$ implies that g_n is a polynomial in x_n alone (any power product involving an x_i for $i < n$ is bigger than $x_n^{\nu_n}$ by the definition of the lexicographical order.)

Similarly $\mathrm{lt}(g_{n-1}) = x_{n-1}^{\nu_{n-1}}$ implies that g_{n-1} is a polynomial in x_n and x_{n-1} only. Continuing in this way, we see that g_i involves only the variables $x_i, \ldots, x_n$.

Now to see that there are only finitely many solutions to the equations, we note that there are only finitely many solutions to $g_n(x_n) = 0$. For each of these we only get finitely many solutions x_{n-1} to $g_{n-1}(x_{n-1}, x_n) = 0$ (note that since $\mathrm{lt}(g_{n-1}) = x_{n-1}^{\nu_{n-1}}$ it is non-zero after plugging in a root of $g_n(x_n)$.

Continuing in this way we see that there are only finitely many solutions to the system of equations.

Example. We go back to the earlier system.

$$\begin{cases} f_1 = x^2 + y^2 + z^2 - 1 = 0 \\ f_2 = 4x^2 + xy + y^2 + z^2 - 1 = 0 \\ f_3 = x + y - z - 1 = 0 \end{cases}$$

One can show that the following polynomials is a Gröbner basis for $\langle f_1, f_2, f_3 \rangle$ (we use $x > y > z$).

$$\begin{cases} g_1 = x + y - z - 1 \\ g_2 = 3y^2 + 2z^2 - z - 3 \\ g_3 = 3yz + 3y - z^2 - 4z - 3 \\ g_4 = 7z^3 + 10z^2 + 3z \end{cases}$$

The last polynomial has solutions $0, -1, -\frac{3}{7}$. For $z = 0$ we get $y = 1$ and so $x = 0$. For $z = -1$ we get $y = 0$ and so $x = 0$. Finally, for $z = -\frac{3}{7}$ we get $y = \frac{6}{7}$ and so $x = -\frac{2}{7}$.

The examples given at the beggining also illsutrate the results.

BIG QUESTION:

Can we find Gröbner bases? That is, is there an algorithm for computing them?

YES! It is due to Bruno Buchberger in Austria and was discovered by him in his PhD thesis in 1965 working under W. Gröbner .

IDEA:
Given $F = \{f_1, \ldots, f_s\}$ and $I = \langle f_1, \ldots, f_s \rangle$ we want $\mathsf{Lt}(F) = \mathsf{Lt}(I)$.

So seek $f \in I$ with $\mathsf{lt}(f) \notin \mathsf{Lt}(F)$. When we find one, if it exists, then add it into F (set $f_{s+1} = f$) and check again if $\mathsf{Lt}(F \cup \{f_{s+1}\}) = \mathsf{Lt}(I)$.

This would have to end by the Hilbert Basis theorem.

BUT: How do we find such f's?

To be in I we need that $f = h_1 f_1 + \cdots + h_s f_s$. We clearly need one such that the $\mathsf{lt}(f_j)$'s cancel out.

The easiest way to do this is to cancel them out in pairs.

Example: Let$f_1 = x_1^2x_2x_3 - x_1x_3^2 + x_2$ and$f_2 = x_1x_2^2 + x_2x_3$ then we note that

$$x_2f_1 - x_1x_3f_2 = -x_1x_2x_3^2 + x_2^2 - x_1x_2x_3^2$$

and you see that the leading terms of the two summands $x_2f_1, x_1x_3f_2$ have canceled out.

In general for f_i, f_j with $i \neq j$ let

$$X = \mathrm{lcm}(\mathrm{lp}(f_i), \mathrm{lp}(f_j))$$

and set

$$S(f_i, f_j) = \frac{X}{\mathrm{lt}(f_i)}f_i - \frac{X}{\mathrm{lt}(f_j)}f_j.$$

This polynomial is called the S-polynomial of f_i and f_j.

Note that

$$\mathrm{lp}(\frac{X}{\mathrm{lt}(f_i)}f_i) = \mathrm{lp}(\frac{X}{\mathrm{lt}(f_j)}f_j) = X.$$

and so these terms cancel out in $S(f_i, f_j)$.

BRILLIANT DISCOVERY OF BUCHBERGER: This is enough!

Specifically, we have Buchberger's Theorem.

Given $G = \{g_1, \ldots, g_t\}$, then G is a Gröbner basis of $I = \langle f_1, \ldots, f_s \rangle$ if and only if for all $i \neq j$ we have

$$S(g_i, g_j) \xrightarrow{G}_+ 0.$$

So this gives us:

BUCHBERGER'S ALGORITHM:

1. Given $I = \{f_1, \ldots, f_s\}$
2. Choose i and j with $i \neq j$
3. Let $S(f_i, f_j) \xrightarrow{F}_+ r$, where r cannot be reduced any more by F.
4. If $r \neq 0$ then set $f_{s+1} = r$ and add it to F. (note that $\langle f_1, \ldots, f_s\rangle = \langle f_1, \ldots, f_s, f_{s+1}\rangle$)
5. Repeat until all $S(f_i, f_j) \xrightarrow{G}_+ 0$.

It ends: Each time we iterate the loop we see that Lt($f_1, \ldots, f_s$) is a proper subset of Lt($f_1, \ldots, f_s, f_{s+1}$) (or we could reduce r) and this set of ideals must have a largest one by the Hilbert Basis Theorem.