REGULAR FUNCTIONS OF SEVERAL QUATERNIONIC VARIABLES AND THE CAUCHY–FUETER COMPLEX

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Abstract. We employ a classical idea of Ehrenpreis, together with a new algebraic result, to give a new proof that regular functions of several quaternionic variables cannot have compact singularities. As a byproduct we characterize those inhomogeneous Cauchy–Fueter systems which admit solutions on convex sets. Our method readily extends to the case of monogenic functions on Clifford Algebras. We finally study a free resolution of the Cauchy–Fueter complex of differential operators and we obtain some new duality theorems which hint at a hyperfunction theory of several quaternionic variables.

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1. Introduction

The well known Hartogs' theorem [7] shows that holomorphic functions in $\mathbb{C}^n$, $n > 1$, cannot have compact singularities. In [13], Pertici has shown how to use the Cauchy–Fueter formula (and its extension to the n–dimensional quaternionic space $\mathbb{H}^n$, $n > 1$) to prove an analogous result for regular functions of several quaternionic variables. In the first part of this paper we give an alternative, and essentially algebraic, proof of this same result, namely

Theorem 1.1. *Let K be a compact subset of $\mathbb{H}^n$, $n > 1$, such that $\mathbb{H}^n \setminus K$ is connected, and let f be a (left) regular function on $\mathbb{H}^n \setminus K$. Then f extends uniquely to an entire (left) regular function.*

In [4], Ehrenpreis had shown that removability of singularities results for distributions (and in particular Hartogs' theorem) could be seen in the framework of solutions of overdetermined systems of linear partial differential equations. The proof which Ehrenpreis gave of Hartogs' theorem essentially consisted of showing its equivalence with the vanishing of the first cohomology group with compact support, $H^1_c(\mathbb{C}^n, \mathcal{O})$. For systems $\vec{P}(D)$ more general than the Cauchy–Riemann system, one has to establish the vanishing of $H^1_c(\mathbb{R}^n, \mathcal{D}'^{\vec{P}})$, where $\mathcal{D}'^{\vec{P}}$ denotes the sheaf of distribution solutions to the system $\vec{P}$, i.e. the sheaf of germs of distributions f such that

$$\vec{P}(D)f = (P_1(D)f, \ldots, P_r(D)f) = 0,$$

where $P_1, \ldots, P_r$ are polynomials in $\mathbb{C}^n$ and

$$D = (-i\frac{\partial}{\partial x_1}, \ldots, -i\frac{\partial}{\partial x_n}).$$

In order to prove that such a cohomology vanishes, two basic facts are needed: the so-called Ehrenpreis–Malgrange Lemma, on the growth of the division of entire functions by polynomials, and the commutativity of the operators $P_i(D)$ and $P_j(D)$ $(i \neq j)$ which allows an immediate construction of the syzygies of the operator $\vec{P}(D) : \mathcal{D}' \longrightarrow (\mathcal{D}')^n$. If one tries to extend this approach to the case of regular functions of several quaternionic variables $(q_1, \dots, q_n)$, $n > 1$, then the system $\vec{P}(D)$ is the Cauchy–Fueter system, i.e.

$$\frac{\partial}{\partial \bar{q}_1}, \dots, \frac{\partial}{\partial \bar{q}_n}$$

and one immediately notices that

$$\frac{\partial}{\partial \bar{q}_i}(\frac{\partial}{\partial \bar{q}_j}) \neq \frac{\partial}{\partial \bar{q}_j}(\frac{\partial}{\partial \bar{q}_i}), \quad i \neq j.$$

A way out of this difficulty is to regard a regular function $f : \mathbb{H}^n \longrightarrow \mathbb{H}$ as a vector function $\vec{f} : \mathbb{R}^{4n} \longrightarrow \mathbb{R}^4$, and to replace the Cauchy–Fueter system by its "real" counterpart which is now a $4n \times 4$ matrix of linear constant coefficients differential operators, $[P_{ij}(D)]$. Removability of compact singularities for the matrix situation has also been discussed in [4], [5], and [12], and the most recent work on this subject is probably [11], where a sufficient condition on $[P_{ij}(D)]$ is given for the removability of compact singularities of (among other things) hyperfunction solutions. Unfortunately, the condition given in [11] is too strong and does not apply to the case of the Cauchy–Fueter system. However, the result in Section 3 allows us to show that the Cauchy–Fueter system does indeed satisfy a cokernel condition given in [2], and we therefore obtain, as a consequence, an essentially algebraic proof of the removability of compact singularities for regular functions in more than one variable.

An obvious byproduct of our computations is the explicit calculation (via CoCoA) of a complete set of syzygies for the map $(q_1, q_2) : \mathbb{H}^2 \longrightarrow \mathbb{H}$. This result is interesting in view of the non–commutative nature of $\mathbb{H}$, and because it shows that even though $\frac{\partial}{\partial \bar{q}_1}(\frac{\partial}{\partial \bar{q}_2}) \neq \frac{\partial}{\partial \bar{q}_2}(\frac{\partial}{\partial \bar{q}_1})$, one can restore Ehrenpreis argument by constructing operators $P_1(D)$, $P_2(D)$

such that $P_1(D)(\frac{\partial}{\partial \bar{q}_2}) = P_2(D)(\frac{\partial}{\partial \bar{q}_1})$. This same computation (or, rather, a small variation of it) is also used in Section 5 to find necessary and sufficient conditions for the solvability, on convex sets, of the inhomogeneous Cauchy–Fueter system. This result is a non-trivial generalization of the well-known result for the Cauchy–Riemann system.

An important feature of our approach is that it shows how the phenomenon discovered by Pertici is actually a special case of the more general principle of Ehrenpreis (surprisingly enough, Pertici quotes [4] in his paper [13], but apparently fails to see the connection with his own result); even more interesting is the fact that our approach readily carries over to the case of monogenic functions on more general Clifford Algebras, in which case it provides a totally new result. As we have said, the removability of compact singularities and, in particular, the vanishing of $\mathrm{Ext}^1(M, S)$, where M is the module associated to the Cauchy–Fueter system and S is the ring of polynomials in $4n$ variables over $\mathbb{C}$, can be translated into an algebraic statement. In the second part of this paper we use CoCoA to explicitely compute a resolution for M which allows us to show that $\mathrm{Ext}^2(M, S)$ also vanishes, so that a duality theorem follows which completely describes the dual space of the space of germs of plurivariate regular functions on a compact K. This last result suggests a way to define compactly supported hyperfunctions as higher cohomology classes of regular functions.

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2. Algebraic Treatment of Systems of Differential Equations

We begin this section by following [11] to provide a quick overview of those aspects of the algebraic treatment of systems of differential equations which are relevant to our goal. The main ideas of this section are essentially due to Ehrenpreis [5] and Palamodov [12]. We will then construct our specific system in Section 4 and show how it satisfies the conditions for the applicability of the general theory. Finally we show how all of this can be extended to monogenic functions and we provide some related applications.

Consider an s–tuple $\vec{f} = (f_1, \ldots, f_s)$ of functions which are infinitely differentiable on $\mathbb{R}^n \backslash K$, for K a compact convex set such that $\mathbb{R}^n \backslash K$ is connected. Suppose that $\vec{f}$ satisfies a $q \times s$ system of linear constant coefficients partial differential equations

$$\sum_{j=1}^{s} P_{ij}(D) f_j = 0 \qquad i = 1, \ldots, q$$

where $P = [P_{ij}]$ is a $q \times s$ matrix of complex polynomials in $\mathbb{C}^n$, $1 \leq s \leq q$, $q \geq 2$ and $D = (-i\frac{\partial}{\partial x_1}, \ldots, -i\frac{\partial}{\partial x_n})$ so that $P_{ij}(D)$ is simply the differential operator obtained by the polynomial P_{ij} by replacing $z = (z_1, \ldots, z_n)$ by D. The functions f_j can now be extended, allowing a small modification near ∂K, to globally defined $\tilde{f}_j$, with f_j equal to $\tilde{f}_j$ on $\mathbb{R}^n \backslash K_\varepsilon$, for small $\varepsilon > 0$ and $K_\varepsilon = \{x \in \mathbb{R}^n | d(x, K) \leq \varepsilon\}$. If one is willing to invoke the theory of hyperfunctions, one can actually assume $\tilde{f}_j \in \mathcal{B}(\mathbb{R}^n)$, $j = 1, \ldots, q$, and now $\tilde{f}_j = f_j$ on $\mathbb{R}^n \backslash K$, in view of the flabbiness of the sheaf $\mathcal{B}$, and of the injection $\mathcal{E} \hookrightarrow \mathcal{B}$. Define now

$$g_i = \sum_{j=1}^{s} P_{ij}(D) \tilde{f}_j, \qquad i = 1, \ldots, q.$$

Clearly, $g_i \in \mathcal{B}_K(\mathbb{R}^n)$, and if $Q = [Q_{jt}]$ is a matrix which generates the module of relations of the rows of P, then

$$\sum_{t=1}^{q} Q_{jt}(D) g_t = 0, \qquad j = 1, \ldots, s.$$

It is a result of Andreotti–Nacinovich [2] (but essentially due to Ehrenpreis–Malgrange-Palamodov, as pointed out in [5], [12], and see also [11]) that if coker(P) is torsion free then there exist entire functions $k_1, \dots, k_s$ such that

$$\hat{g}_i = \sum_{j=1}^{s} P_{ij} k_j, \qquad i = 1, \dots, q.$$

But since the P_{ij} are polynomials, and since the growth of $\hat{g}_i$ is given by

$$|\hat{g}_i(z)| \leq A_\varepsilon \exp(H_K(z) + \varepsilon |z|),$$

since $\mathcal{B}_K(\mathbb{R}^n) \cong [\mathcal{O}(K)]'$, where H_K is the supporting function on K, one can use Ehrenpreis–Malgrange division lemma (in its version in [5]) to conclude the existence of elements $h_j^* \in \mathcal{B}_K(\mathbb{R}^n)$, $j = 1, \dots, s$, such that $\hat{h}_j^* \in [\mathcal{B}_K(\mathbb{R}^n)]^\wedge$ and

$$\hat{g}_i = \sum_{j=1}^{s} P_{ij} \hat{h}_j^*, \qquad i = 1, \dots, q.$$

If we now define

$$f_j^* := \tilde{f}_j - h_j^*, \qquad j = 1, \dots, s$$

we have that f_j^* are well defined hyperfunctions on $\mathbb{R}^n$, such that

$$\sum_{j=1}^{s} P_{ij}(D) f_j^* = 0$$

everywhere. Moreover, $f_j^* - \tilde{f}_j = f_j$ on $\mathbb{R}^n \setminus K$. What we have proved is the following general result:

Theorem 2.1. *Let $P = [P_{ij}]$ be a $q \times s$ matrix of polynomials in $\mathbb{C}^n$, $1 \leq s \leq q$, $q \geq 2$ such that coker(P) is torsion free. Let $\vec{f} = (f_1, \dots, f_s)$ be a vector of hyperfunctions on $\mathbb{R}^n$ such that*

$$\sum_{j=1}^{s} P_{ij}(D) f_j = 0, \qquad i = 1, \dots, q$$

on $\mathbb{R}^n \setminus K$, for K a compact convex subset in $\mathbb{R}^n$ such that $\mathbb{R}^n \setminus K$ is connected. Then there exists a vector $\vec{f}^* = (f_1^*, \dots, f_s^*)$ of hyperfunctions on $\mathbb{R}^n$ such that

$$f_j^* = f_j \quad \text{on} \quad \mathbb{R}^n \setminus K, \quad j = 1, \dots, s$$

and

$$\sum_{j=1}^{s} P_{ij}(D) f_j^* = 0, \quad i = 1, \ldots, q$$

on all of $\mathbb{R}^n$.

This result was essentially proved in [11] (Theorems 4.1 and 4.3) with the important exception that a rather strong condition was imposed on the minors of the matrix P. This extra condition was suggested to the authors by the need of considering matrices of entire functions (so that infinite order differential operators could be treated) but while relevant in that case, it would prevent us from applying our result to regular functions of quaternionic variables. In our next sections we will show how to use Theorem 2.1 to prove the removability of singularity theorem for regular functions (Theorem 1.1).

3. Torsion Free Cokernels

Let S be a polynomial ring in n variables over $\mathbb{C}$. In this section we will analyze conditions under which a map

$$S^s \longrightarrow S^q$$

given by a matrix A has a torsion free cokernel.

We define the matrix B as the matrix whose rows form a generating set for the syzygy module of the rows of A. It is readily seen that the torsion submodule of $\mathrm{coker}(A)$ is $\ker(B)/\langle A\rangle$, where $\langle A\rangle$ is the submodule generated by the columns of A. Using techniques from Gröbner basis theory (see [1]), $\ker(B)/\langle A\rangle$ can be explicitely computed in our case, where S is a polynomial ring over a field. But this requires, in addition to computing the syzygies of the rows of A, computing the syzygies of the columns of B and determining wheter this module is equal to the module generated by the columns of A (this is equivalent to checking the short exact sequence given by the matrices A and B is exact).

In this section we give a condition which is more readily applicable to the situation in this paper. It only considers the case when A is of maximal rank, but it will be clear that this case is all we need.

Since there is nothing to be gained by being more specific, we will prove the following Lemma for any Unique Factorization Domain (UFD), and so the result will immediately specialize to a polynomial ring over the complex numbers.

So let A be a $q \times s$ matrix over S, where $q \geq s$ and S is a UFD. Recall that $\operatorname{coker}(A) = S^q/\langle A \rangle$, where $\langle A \rangle$ denotes the S–module generated by the columns of A. We have:

Lemma 3.1. *Assume that A has maximal rank s. Then coker(A) is torsion free if and only if the $s \times s$ minors of A are relatively prime.*

Proof. We denote the columns of A by $\mathbf{a}_1, \ldots, \mathbf{a}_s$. For a vector $\mathbf{a}$ in any S^q, we denote by $\gcd(\mathbf{a})$ a greatest common divisor of the coordinates of $\mathbf{a}$ (if the coordinates are relatively prime, we assume that $\gcd(\mathbf{a}) = 1$).

It is convenient to use the notation and some basic tools from exterior algebra (see, for example [9]). We consider the vector $\beta = \mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_s$, let $d = \gcd(\beta)$, and write $\beta = d\alpha$. So we are to prove that $\operatorname{coker}(A)$ is torsion free if and only if $d = 1$.

Let us first assume that $d = 1$. Let $\mathbf{a} \in S^q$ and assume that $c \in S$ is such that $c \neq 0$ and $c\mathbf{a} \in \langle A \rangle$. Choose $c_1, \ldots c_s \in S$ such that

$$c\mathbf{a} = c_1\mathbf{a}_1 + \cdots + c_s\mathbf{a}_s.$$

Then for all $j = 1, \ldots, s$, we have

$$c\left(\mathbf{a} \wedge \bigwedge_{i \neq j} \mathbf{a}_i\right) = c\mathbf{a} \wedge \bigwedge_{i \neq j} \mathbf{a}_i = \pm c_j\beta = \pm c_j\alpha.$$

Thus c divides c_j, since $\gcd(\alpha) = 1$. Therefore $\mathbf{a} \in \langle A \rangle$, and so $\operatorname{coker}(A)$ is torsion free.

Conversely assume that d has a prime divisor p. We use $\bar{v}$ to denote reduction mod p whenever v is a vector in any S^q or an element of S. Since $\mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_s = d\alpha$, we have that

$$\bar{\mathbf{a}}_1 \wedge \ldots \wedge \bar{\mathbf{a}}_s = \bar{0}.$$

Therefore the vectors $\bar{\mathbf{a}}_1, \ldots, \bar{\mathbf{a}}_s$ are linearly dependent over the quotient field of S/pS. Thus there exist $c_1, \ldots, c_s \in S$, not all divisible by p, such that

$$\bar{c}_1 \bar{\mathbf{a}}_1 + \cdots + \bar{c}_s \bar{\mathbf{a}}_s = \bar{0}.$$

So there is a vector $\mathbf{a} \in S^q$ such that

$$p\mathbf{a} = c_1 \mathbf{a}_1 + \cdots + c_s \mathbf{a}_s \in \langle A \rangle.$$

However $\mathbf{a} \notin \langle A \rangle$, since at least one c_i is not divisible by p and the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_s$ are linearly independent over the quotient field of S. Therefore $\mathrm{coker}(A)$ has torsion elements.

$\diamondsuit$

4. Removability of Compact Singularities

Let $q = (q_1, \ldots, q_n)$ be the variable in $\mathbb{H}^n$, and let us write

$$q_t = x_{0t} + \mathbf{i}x_{1t} + \mathbf{j}x_{2t} + \mathbf{k}x_{3t}, \qquad t = 1, \ldots, n$$

with $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ the usual imaginary units in the sense of quaternions. The Cauchy–Fueter operator, whose kernel on distributions gives the space of regular functions, is defined by

$$(\frac{\partial}{\partial \bar{q}_1}, \ldots, \frac{\partial}{\partial \bar{q}_n})$$

where, for $t = 1, \ldots, n$,

$$\frac{\partial}{\partial \bar{q}_t} = \frac{\partial}{\partial x_{0t}} + \mathbf{i}\frac{\partial}{\partial x_{1t}} + \mathbf{j}\frac{\partial}{\partial x_{2t}} + \mathbf{k}\frac{\partial}{\partial x_{3t}}.$$

This operator is usually known as the left Cauchy–Fueter operator, and its kernel is the space of left regular functions; a similarly defined right Cauchy–Fueter operator gives rise to right regular functions, but since the theories are completely symmetric, we will restrict our attention to the case of left regular functions throughout the rest of the paper.

If $f : \mathbb{H}^n \longrightarrow \mathbb{H}$ is a regular function (in $\mathbb{H}^n$ or in a subset of it), then, by definition we have that

$$\frac{\partial f}{\partial \bar{q}_t} = 0 \qquad t = 1, \dots, n,$$

wherever f is regular. If we write

$$f = f_0 + \mathbf{i} f_1 + \mathbf{j} f_2 + \mathbf{k} f_3$$

(with f_0, f_1, f_2, f_3 defined on $\mathbb{R}^{4n}$ and taking values in $\mathbb{R}$), the equation

$$\frac{\partial f}{\partial \bar{q}_t} = 0$$

becomes (for $x_k = x_{kt}$, $k = 0, \; 1, \; 2, \; 3$)

$$\left(\frac{\partial}{\partial x_0} + \mathbf{i}\frac{\partial}{\partial x_1} + \mathbf{j}\frac{\partial}{\partial x_2} + \mathbf{k}\frac{\partial}{\partial x_3}\right)(f_0 + \mathbf{i} f_1 + \mathbf{j} f_2 + \mathbf{k} f_3) = 0,$$

i.e. (separating the various coefficients)

$$\begin{cases} \dfrac{\partial f_0}{\partial x_0} - \dfrac{\partial f_1}{\partial x_1} - \dfrac{\partial f_2}{\partial x_2} - \dfrac{\partial f_3}{\partial x_3} = 0 \\ \dfrac{\partial f_0}{\partial x_1} + \dfrac{\partial f_1}{\partial x_0} - \dfrac{\partial f_2}{\partial x_3} + \dfrac{\partial f_3}{\partial x_2} = 0 \\ \dfrac{\partial f_0}{\partial x_2} + \dfrac{\partial f_1}{\partial x_3} + \dfrac{\partial f_2}{\partial x_0} - \dfrac{\partial f_3}{\partial x_1} = 0 \\ \dfrac{\partial f_0}{\partial x_3} - \dfrac{\partial f_1}{\partial x_2} + \dfrac{\partial f_2}{\partial x_1} + \dfrac{\partial f_3}{\partial x_0} = 0. \end{cases}$$

When we consider that f must satisfy $\frac{\partial f}{\partial \bar{q}_t} = 0$ for $t = 1, \dots, n$, we actually obtain the following matrix of differential operators:

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}$$

where each A_t is the 4×4 matrix given by

$$A_t = \begin{bmatrix} \partial/\partial x_{0t} & -\partial/\partial x_{1t} & -\partial/\partial x_{2t} & -\partial/\partial x_{3t} \\ \partial/\partial x_{1t} & \partial/\partial x_{0t} & -\partial/\partial x_{3t} & \partial/\partial x_{2t} \\ \partial/\partial x_{2t} & \partial/\partial x_{3t} & \partial/\partial x_{0t} & -\partial/\partial x_{1t} \\ \partial/\partial x_{3t} & -\partial/\partial x_{2t} & \partial/\partial x_{1t} & \partial/\partial x_{0t} \end{bmatrix} \tag{1}$$

and A is therefore a $4n \times 4$ matrix which maps $[\mathcal{E}(\mathbb{R}^{4n})]^4$ to $[\mathcal{E}(\mathbb{R}^{4n})]^{4n}$. According to the algebraic theory which we have described in Section 2, it is now sufficient to take the Fourier transform of A and consider the map induced by

$$\hat{A} = \begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \\ \vdots \\ \hat{A}_n \end{bmatrix}$$

on S^4, where

$$S = \mathbb{C}[x_{01}, x_{11}, x_{21}, x_{31}, x_{02}, x_{12}, x_{22}, x_{32}, \ldots, x_{0t}, x_{1t}, x_{2t}, x_{3t}, \ldots, x_{0n}, x_{1n}, x_{2n}, x_{3n}],$$

and

$$\hat{A}_t = \begin{bmatrix} x_{0t} & -x_{1t} & -x_{2t} & -x_{3t} \\ x_{1t} & x_{0t} & -x_{3t} & x_{2t} \\ x_{2t} & x_{3t} & x_{0t} & -x_{1t} \\ x_{3t} & -x_{2t} & x_{1t} & x_{0t} \end{bmatrix}.$$

The condition given in [11] for the removability of singularities involves the regularity of a suitable sequence of 3×3 minors of A (see [11] for details). Unfortunately, a quick verification (using CoCoA) even in the case of $n = 2$ shows that such condition is not verified, as the codimension of the variety generated by four of such minors is only 3. As we have discussed earlier, however, the condition in [11] is too strong, and it really suffices to verify that the cokernel of $\hat{A} : S^4 \longrightarrow S^8$ is torsion free.

We now use the results of Sections 2 and 3 to prove

Theorem 4.1. *Let K be a compact convex subset of $\mathbb{H}^n$, $n > 1$, such that $\mathbb{H}^n \backslash K$ is connected, and let f be a (left) regular function on $\mathbb{H}^n \backslash K$. Then f extends uniquely to an entire (left) regular function.*

Proof. In view of Theorem 2.1, it is sufficient to prove that coker($\hat{A}$) is torsion free. It is immediately verified that $\hat{A}$ is of maximal rank. Because of its special form, on the other hand, it is obvious that its 4×4 minors are relatively prime, so we can apply Lemma 3.1 to conclude the proof.

$\diamondsuit$

Now we want to show that an analogue of this result holds for monogenic functions with values in a Clifford Algebra.

Let $f : \Omega \subseteq \mathbb{R}^{m+1} \longrightarrow \mathcal{C}_n$, $m \leq n$, where $\mathcal{C}_n$ is a real Clifford Algebra constructed by means of an orthonormal basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^n$. A basis for $\mathcal{C}_n$ is given by $\{\mathbf{e}_A = \mathbf{e}_{j_1} \ldots \mathbf{e}_{j_n} | A = (j_1, \ldots, j_n), 1 \leq j_1 \leq \ldots \leq j_n \leq n\}$. Let $D = \sum_{j=1}^{m} \mathbf{e}_j \frac{\partial}{\partial x_j}$, where $\mathbf{e}_0, \mathbf{e}_1 \ldots, \mathbf{e}_m$ is a basis of $\mathbb{R}^{m+1}$. A real differentiable function f is said (left) monogenic if $Df = 0$ in Ω. Let us denote by $(\mathbb{R}^{m+1})^r$ the space $\mathbb{R}^{m+1} \times \cdots \times \mathbb{R}^{m+1}$, r times, $r \geq 1$, and by $(x_{0t}, x_{1t}, \ldots, x_{mt})$, $t = 1, \ldots, r$, each group of variables in $\mathbb{R}^{m+1}$.

A real differentiable function $f : \Omega \subseteq (\mathbb{R}^{m+1})^r \longrightarrow \mathcal{C}_n$ is said to be (left) monogenic if

$$D_t f = \sum_{j=0}^{m} \mathbf{e}_j \frac{\partial f}{\partial x_{jt}} = 0, \quad t = 1, \ldots, r.$$

Let us remark that a (left) monogenic function $f : \mathbb{R}^{m+1} \longrightarrow \mathcal{C}_n$ can be seen as a (left) regular function $f : \mathcal{C}_n \longrightarrow \mathcal{C}_n$. In fact such a function f is said to be (left) regular if

$$\sum_{A} \mathbf{e}_A \frac{\partial f}{\partial x_A} = 0.$$

It is known, (see [14]), that $\mathcal{C}_n$ is a direct sum of 2^{n-2} algebras isomorphic to the algebra of quaternions. After a suitable change of variables, the matrix T expressing the regularity is

$$T = \begin{bmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ 0 & \ldots & \ldots & A_{n-2} \end{bmatrix}$$

where A_t is the matrix (1). It is an obvious fact that T has maximal rank.

Theorem 4.2. *Let K be a compact convex subset of $(\mathbb{R}^{m+1})^r$, such that $(\mathbb{R}^{m+1})^r \backslash K$ is connected, and let f be a (left) monogenic function on $(\mathbb{R}^{m+1})^r \backslash K$. Then f extends uniquely to an entire (left) monogenic function.*

Proof. Once we fix the Clifford Algebra $\mathcal{C}_n$, it is easy to explicitely write the matrix of differential operators

$$A = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_r \end{bmatrix}$$

associated with the system

$$\begin{cases} D_1 f = 0 \\ \quad \vdots \\ D_t f = 0. \end{cases}$$

We can take the Fourier transform of A and consider the map induced by

$$\hat{A} = \begin{bmatrix} \hat{T}_1 \\ \hat{T}_2 \\ \vdots \\ \hat{T}_r \end{bmatrix}$$

on $S = \mathbb{C}[x_{01}, \ldots, x_{m1}, x_{02}, \ldots, x_{m2}, \ldots, x_{0r}, \ldots x_{mr}]$. Each $\hat{T}_t$, for $t = 1, \ldots, r$ is a particular case of a matrix of the type T, so it has maximal rank; moreover its $2^n \times 2^n$ minors are relatively prime (since they involve different sets of variables), and therefore, by Lemma 3.1 and Theorem 2.1, we obtain the result.

$\diamondsuit$

We conclude this section by pointing out a simple byproduct of our considerations. Let us restrict our attention to two quaternionic variables, say q_1 and q_2, and let us consider the map $S : \mathbb{H}^2 \longrightarrow \mathbb{H}$ defined by

$$S(\alpha, \beta) = q_1\alpha + q_2\beta.$$

The question of computing the syzygies of S (or of similar maps) comes up quite naturally in several settings (the first being the Hartogs' phenomenon we just discussed) and is certainly of interest if one tries to develop a theory leading to a Nullstellensatz in the space of regular functions. In this simple case, however, syzygies can be easily computed, using the matrix $\hat{S}$ which is a real version of the multiplication map S. By applying CoCoA to the eight rows of $\hat{S}$, which are

$$[x_0, x_1, x_2, x_3], [-x_1, x_0, x_3, -x_2], [-x_2, -x_3, x_0, x_1], [-x_3, x_2, -x_1, x_0],$$

$$[y_0, y_1, y_2, y_3], [-y_1, y_0, y_3, -y_2], [-y_2, -y_3, y_0, y_1], [-y_3, y_2, -y_1, y_0]$$

one can compute the syzygies of $\hat{S}$, which are given in Appendix I.

These eight vectors now provide (if we consider each of them as an element of $\mathbb{H}^2$) the generators for the syzygies of S itself. The argument we used, has clearly nothing to do with the simple definition of the map A. In fact, we can apply it to any polynomial from $\mathbb{H}^n$ to $\mathbb{H}$, to obtain the following obvious consequence of Hilbert's syzygies theorem:

Proposition 4.3. *Let R be the ring of polynomials in $n \geq 1$ variables over $\mathbb{H}$. For any R–homomorphism*

$$A : R^{s_1} \longrightarrow R^{s_0}$$

we can construct a finite exact sequence

$$0 \longrightarrow R^{s_d} \longrightarrow R^{s_{d-1}} \longrightarrow \cdots \longrightarrow R^{s_1} \xrightarrow{A} R^{s_0}$$

of length $d \leq 4n+1$. Moreover, the sequence can be explicitely constructed.

5. The Cauchy–Fueter Complex

In this section we apply our techniques to deal with some related problems for regular functions of several quaternionic variables. An important result in the theory of several complex variables is the solvability of the inhomogeneous Cauchy–Riemann system under the natural compatibility conditions. If we consider the case of $\mathbb{C}^2$ (the general case of $\mathbb{C}^n$, $n \geq 2$, is treated similarly) we have that the system

$$\begin{cases} \dfrac{\partial f}{\partial \bar{z}_1} = g_1 \\ \dfrac{\partial f}{\partial \bar{z}_2} = g_2 \end{cases} \tag{2}$$

has a $\mathcal{C}^\infty$ solutions $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ for any $\mathcal{C}^\infty$ pair (g_1, g_2) such that

$$\frac{\partial g_1}{\partial \bar{z}_2} = \frac{\partial g_2}{\partial \bar{z}_1}. \tag{3}$$

The same question may now be asked in $\mathbb{H}^2$, by considering the system

$$\begin{cases} \dfrac{\partial f}{\partial \bar{q}_1} = g_1 \\ \dfrac{\partial f}{\partial \bar{q}_2} = g_2 \end{cases} \tag{4}$$

The analog of (3), i.e. condition

$$\frac{\partial g_1}{\partial \bar{q}_2} = \frac{\partial g_2}{\partial \bar{q}_1},$$

has clearly no reason to exist since $\frac{\partial^2}{\partial \bar{q}_1 \partial \bar{q}_2} \neq \frac{\partial^2}{\partial \bar{q}_2 \partial \bar{q}_1}$, but we can prove the following;

Theorem 5.1. *Let Ω be a convex open set in $\mathbb{H}^2$. Then system (4) has a $\mathcal{C}^\infty$ solution f if and only if*

$$B \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = 0$$

where B is the 8×8 matrix given in Appendix II.

Proof. As before, we consider (4) as an 8×4 system of differential equations acting on $\mathcal{C}^\infty(\Omega)$. Since Ω is convex, $\mathcal{C}^\infty(\Omega)$ is a LAU–space (in the sense of Ehrenpreis [5]) and therefore by the calculations of the syzygies of

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

together with Theorem 6.1 of [5] (but see also [12]), we immediately obtain the result.

$\diamondsuit$

Remark. If Ω is not convex, $\mathcal{C}^\infty(\Omega)$ is not a LAU–space any longer, and we do not have an analog of Theorem 6.1 from [5]. It is however known from the general theory of linear constant coefficients partial differential operators (essentially due to Malgrange [10] and Ehrenpreis [5]) that Theorem 5.1 holds if Ω is A–convex for open sets in $\mathbb{R}^8$, for A the Cauchy–Fueter system. We will hopefully come back, in a later paper, to the problem of providing a geometric characterization of A–convex sets.

In the literature, [6], [15], some of us have developed a theory of quaternionic hyperfunctions in one variable. Without going into details, we recall that if $\mathcal{R}$ is the sheaf of (left) regular functions on $\mathbb{H}$, one defines, for any open set $\Omega \subseteq \tilde{\mathbb{H}} = \{q \in \mathbb{H} : x_0 = 0\}$, the space of $\mathbb{H}$–hyperfunctions by the equation

$$\mathcal{F}(\Omega) = H^1(\mathbb{H}, \mathbb{H}\backslash\Omega; \mathcal{R})$$

or, equivalently,

$$\mathcal{F}(\Omega) = \frac{\mathcal{R}(\mathbb{H}\backslash\Omega)}{\mathcal{R}(\mathbb{H})}.$$

If we restrict our attention to the space $\mathcal{F}_K$ of compactly supported $\mathbb{H}$–hyperfunctions (such a notion is well-defined, since $\mathcal{F}$ turns out to be a sheaf), we have that

$$\mathcal{F}_K = H^1(\mathbb{H}, \mathbb{H}\backslash K; \mathcal{R}).$$

An interesting question would be the possibility of extending this notion to two (or several) quaternionic variables. In this endeavor however, many obstacles arise and it is not clear which definition would be an appropriate one.

Our description of regular functions as solutions of the overdetermined Cauchy–Fueter system, however, allows us to prove a simple duality theorem (well known in general, [8],

[16], but not exploited in this specific case), which will provide a good candidate for the space of compactly supported hyperfunctions in two variables.

Lemma 5.2. *Let* $S = \mathbb{C}[x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3]$. *Then the matrix* $\hat{A}$ *given in Section 4 has a finite resolution as follows:*

$$0 \longrightarrow S^4 \xrightarrow{\hat{A}} S^8 \xrightarrow{\hat{B}} S^8 \xrightarrow{\hat{C}} S^4 \longrightarrow 0 \tag{5}$$

where $\hat{B}$ *is given in Appendix III, and*

$$\hat{C} = \begin{bmatrix} -2x_0 & -2x_1 & 2x_2 & 2x_3 & 2y_0 & 2y_1 & -2y_2 & -2y_3 \\ 2x_1 & -2x_0 & 2x_3 & -2x_2 & -2y_1 & 2y_0 & -2y_3 & 2y_2 \\ 2x_2 & 2x_3 & 2x_0 & 2x_1 & -2y_2 & -2y_3 & -2y_0 & -2y_1 \\ 2x_3 & -2x_2 & -2x_1 & 2x_0 & -2y_3 & 2y_2 & 2y_1 & -2y_0 \end{bmatrix}$$

Proof. The resolution (5) is immediately constructed using CoCoA.

$\diamondsuit$

Remark 5.1. We have already seen in Section 3 that Im($\hat{A}$)= Ker($\hat{B}$). Using CoCoA, we have also been able to verify that Im($\hat{B}$)= Ker($\hat{C}$). We therefore know that, for $M = \frac{S^8}{\hat{A}S^4}$, $\mathrm{Ext}^0(M,S) = \mathrm{Ext}^1(M,S) = \mathrm{Ext}^2(M,S) = 0$ (this is essentially Theorem 2.1).

As Professor Palamodov has kindly pointed out to one of us, the vanishing of the second Ext module allows us to obtain an interesting result on the removability of some particular non-compact singularities.

Corollary 5.1. *Let* K *be a compact subset of* $\mathbb{H}^2$ *such that* $\mathbb{H}^2 \setminus K$ *is connected. Let* $\sum_1$ *and* $\sum_2$ *be two closed half-spaces in* $\mathbb{H}^2$. *Then if* f *is defined and regular in* $\mathbb{H}^2 \setminus (K \cup \sum_1 \cup \sum_2)$, f *can be extended to* $\tilde{f} \in \mathcal{R}(\tilde{\mathbb{H}} \setminus (\sum_1 \cup \sum_2))$.

Proof. As we have shown in Remark 5.1, $\mathrm{Ext}^1(M,S) = \mathrm{Ext}^2(M,S) = 0$, and so the result is an immediate consequence of a result of Palamodov's [12] (Theorem 4, p. 405).

$\diamondsuit$

Let $\mathcal{E}$ denote the sheaf of differentiable functions, and let $\mathcal{E}^{C^t}$ denote the sheaf of differentiable solutions of the system associated to the matrix C^t; the above Lemma allows us now to prove the following

Theorem 5.3. *Let K be a compact convex set in $\mathbb{H}^2$ such that*

$$\dim H^j(K, \mathcal{E}^{C^t}) < +\infty, \quad j = 1, 2.$$

Then

$$H^3(\mathbb{H}^2, \mathbb{H}^2 \backslash K; \mathcal{R}) \cong [H^0(K, \mathcal{E}^{C^t})]'.$$

Proof. This is an immediate consequence of the ellipticity of the Cauchy–Fueter system (see [3]), of Lemma 5.2, and of Theorem 4.7 in [16] (see also [8]).

$\diamondsuit$

Remark 5.2. $\hat{A}$, $\hat{B}$, $\hat{C}$ act on $\mathcal{R}$, while A, B, C are matrices of differential operators.

Remark 5.3. C^t is, up to some signs, once again a Cauchy–Fueter system. This parallels the situations when $\mathbb{H}^2$ is replaced by $\mathbb{C}^2$. In that case we have the classical Sato's result

$$\mathcal{B}_K = H^2(\mathbb{C}^2, \mathbb{C}^2 \backslash K; \mathcal{O}) \cong [\mathcal{O}(K)]'.$$

On the other hand, if one really wants to characterize the space

$$[\mathcal{R}(K)]',$$

one can simply dualize Sequence (5) and consider the following finite resolution of $\hat{C}^t$:

$$0 \longrightarrow S^4 \xrightarrow{\hat{C}^t} S^8 \xrightarrow{\hat{B}^t} S^8 \xrightarrow{\hat{A}^t} S^4 \longrightarrow 0. \tag{6}$$

It is easy to verify the exactness of (6) at the level of S^8, and so the same argument as in Theorem 5.3 gives the following result:

Theorem 5.4. Let $\mathcal{S}$ be the sheaf of distribution solutions of the system $\hat{C}^t$, then

$$[\mathcal{R}(K)]' \cong H^3(\mathbb{H}^2, \mathbb{H}^2 \backslash K; \mathcal{S}).$$

Remark 5.5. This last duality theorem provides a possible definition for the notion of compactly supported hyperfunctions of two variables. We will come back in the future to the development of this concept.

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Appendix I

$\mathrm{Syz}(\hat{S})$=Module($[\frac{1}{2}y_0^2 + \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + \frac{1}{2}y_3^2, 0, 0, 0, -\frac{1}{2}x_0y_0 - \frac{1}{2}x_1y_1 - \frac{1}{2}x_2y_2 - \frac{1}{2}x_3y_3, -\frac{1}{2}x_1y_0 + \frac{1}{2}x_0y_1 + \frac{1}{2}x_3y_2 - \frac{1}{2}x_2y_3, -\frac{1}{2}x_2y_0 - \frac{1}{2}x_3y_1 + \frac{1}{2}x_0y_2 + \frac{1}{2}x_1y_3, -\frac{1}{2}x_3y_0 + \frac{1}{2}x_2y_1 - \frac{1}{2}x_1y_2 + \frac{1}{2}x_0y_3]$,

$[0, \frac{1}{2}y_0^2 + \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + \frac{1}{2}y_3^2, 0, 0, \frac{1}{2}x_1y_0 - \frac{1}{2}x_0y_1 - \frac{1}{2}x_3y_2 + \frac{1}{2}x_2y_3, -\frac{1}{2}x_0y_0 - \frac{1}{2}x_1y_1 - \frac{1}{2}x_2y_2 - \frac{1}{2}x_3y_3, -\frac{1}{2}x_3y_0 - \frac{1}{2}x_2y_1 - \frac{1}{2}x_1y_2 + \frac{1}{2}x_0y_3, \frac{1}{2}x_2y_0 + \frac{1}{2}x_3y_1 - \frac{1}{2}x_0y_2 - \frac{1}{2}x_1y_3]$,

$[0, 0, -\frac{1}{2}y_0^2 - \frac{1}{2}y_1^2 - \frac{1}{2}y_2^2 - \frac{1}{2}y_3^2, 0, -\frac{1}{2}x_2y_0 - \frac{1}{2}x_3y_1 + \frac{1}{2}x_0y_2 + \frac{1}{2}x_1y_3, -\frac{1}{2}x_3y_0 + \frac{1}{2}x_2y_1 - \frac{1}{2}x_1y_2 + \frac{1}{2}x_0y_3, \frac{1}{2}x_0y_0 + \frac{1}{2}x_1y_1 + \frac{1}{2}x_2y_2 + \frac{1}{2}x_3y_3, \frac{1}{2}x_1y_0 - \frac{1}{2}x_0y_1 - \frac{1}{2}x_3y_2 + \frac{1}{2}x_2y_3]$,

$[0, 0, 0, -\frac{1}{2}y_0^2 - \frac{1}{2}y_1^2 - \frac{1}{2}y_2^2 - \frac{1}{2}y_3^2, -\frac{1}{2}x_3y_0 + \frac{1}{2}x_2y_1 - \frac{1}{2}x_1y_2 + \frac{1}{2}x_0y_3, \frac{1}{2}x_2y_0 + \frac{1}{2}x_3y_1 - \frac{1}{2}x_0y_2 - \frac{1}{2}x_1y_3, -\frac{1}{2}x_1y_0 - \frac{1}{2}x_0y_1 + \frac{1}{2}x_3y_2 + \frac{1}{2}x_2y_3, \frac{1}{2}x_0y_0 + \frac{1}{2}x_1y_1 + \frac{1}{2}x_2y_2 + \frac{1}{2}x_3y_3]$,

$[\frac{1}{2}x_0y_0 + \frac{1}{2}x_1y_1 + \frac{1}{2}x_2y_2 + \frac{1}{2}x_3y_3, -\frac{1}{2}x_1y_0 + \frac{1}{2}x_0y_1 + \frac{1}{2}x_3y_2 - \frac{1}{2}x_2y_3, -\frac{1}{2}x_2y_0 - \frac{1}{2}x_3y_1 + \frac{1}{2}x_0y_2 + \frac{1}{2}x_1y_3, -\frac{1}{2}x_3y_0 + \frac{1}{2}x_2y_1 - \frac{1}{2}x_1y_2 + \frac{1}{2}x_0y_3, -\frac{1}{2}x_0^2 - \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_3^2, 0, 0, 0]$,

$[\frac{1}{2}x_1y_0 - \frac{1}{2}x_0y_1 - \frac{1}{2}x_3y_2 + \frac{1}{2}x_2y_3, \frac{1}{2}x_0y_0 + \frac{1}{2}x_1y_1 + \frac{1}{2}x_2y_2 + \frac{1}{2}x_3y_3, -\frac{1}{2}x_3y_0 + \frac{1}{2}x_2y_1 - \frac{1}{2}x_1y_2 + \frac{1}{2}x_0y_3, \frac{1}{2}x_2y_0 + \frac{1}{2}x_3y_1 - \frac{1}{2}x_0y_2 - \frac{1}{2}x_1y_3, 0, -\frac{1}{2}x_0^2 - \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_3^2, 0, 0]$,

$[-\frac{1}{2}x_2y_0 - \frac{1}{2}x_3y_1 + \frac{1}{2}x_0y_2 + \frac{1}{2}x_1y_3, -\frac{1}{2}x_3y_0 + \frac{1}{2}x_2y_1 - \frac{1}{2}x_1y_2 + \frac{1}{2}x_0y_3, -\frac{1}{2}x_0y_0 - \frac{1}{2}x_1y_1 - \frac{1}{2}x_2y_2 - \frac{1}{2}x_3y_3, \frac{1}{2}x_1y_0 - \frac{1}{2}x_0y_1 - \frac{1}{2}x_3y_2 + \frac{1}{2}x_2y_3, 0, 0, \frac{1}{2}x_0^2 + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2, 0]$,

$[-\frac{1}{2}x_3y_0 + \frac{1}{2}x_2y_1 - \frac{1}{2}x_1y_2 + \frac{1}{2}x_0y_3, \frac{1}{2}x_2y_0 + \frac{1}{2}x_3y_1 - \frac{1}{2}x_0y_2 - \frac{1}{2}x_1y_3, -\frac{1}{2}x_1y_0 + \frac{1}{2}x_0y_1 + \frac{1}{2}x_3y_2 - \frac{1}{2}x_2y_3, -\frac{1}{2}x_0y_0 - \frac{1}{2}x_1y_1 - \frac{1}{2}x_2y_2 - \frac{1}{2}x_3y_3, 0, 0, 0, \frac{1}{2}x_0^2 + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2]$)

Appendix II

$B = [(\partial_{\bar{q}_2} + \partial_{q_2})^2 + (\partial_{\bar{q}_2}\mathbf{i} + \mathbf{i}\partial_{q_2})^2 + (\partial_{\bar{q}_2}\mathbf{j} + \mathbf{j}\partial_{q_2})^2 + (\partial_{\bar{q}_2}\mathbf{k} + \mathbf{k}\partial_{q_2})^2, 0, 0, 0, -(\partial_{\bar{q}_1} - \partial_{q_1})(\partial_{\bar{q}_2} - \partial_{q_2}) - (\partial_{\bar{q}_1}\mathbf{i} - \mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i} - \mathbf{i}\partial_{q_2}) - (\partial_{\bar{q}_1}\mathbf{j} - \mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j} - \mathbf{j}\partial_{q_2}) - (\partial_{\bar{q}_1}\mathbf{k} - \mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k} = \mathbf{k}\partial_{q_2}), (\partial_{\bar{q}_1}\mathbf{i} -$

$\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})-(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}-\mathbf{i}\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}-\mathbf{j}\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}-\mathbf{k}\partial_{q_2}),(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}-\mathbf{i}\partial_{q_2})-(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}-\mathbf{j}\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}-\mathbf{k}\partial_{q_2}),(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}-\mathbf{i}\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}-\mathbf{j}\partial_{q_2})-(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}-\mathbf{k}\partial_{q_2})],$

$[0,(\partial_{\bar{q}_2}+\partial_{q_2})^2+(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})^2+(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})^2+(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2})^2,0,0,(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})-(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2}),-(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2}),-(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})+(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2}),(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})-(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2})],$

$[0,0,-(\partial_{\bar{q}_2}+\partial_{q_2})^2-(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})^2-(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})^2-(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2})^2,0,-(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})+(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2}),-(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})+(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2}),(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2}),(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})-(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2})],$

$[0,0,0,-(\partial_{\bar{q}_2}+\partial_{q_2})^2-(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})^2-(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})^2-(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2})^2,-(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})+(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2}),(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})-(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2}),-(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})-(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2}),(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2})],$

$[(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{k}-$

$\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2}), -(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})+(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2}), -(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})+(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})+\frac{1}{2}x_1y_(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2}), -(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})+(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2}), -(\partial_{\bar{q}_1}+\partial_{q_1})^2-(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})^2-(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})^2-(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})^2, 0, 0, 0],$

$[(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})-(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2}), (\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2}), -(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})+(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2}), (\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})-(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2}), 0, -(\partial_{\bar{q}_1}+\partial_{q_1})^2-(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})^2-(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})^2-(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})^2, 0, 0],$

$[-(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})+(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2}), -(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})+(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2}), -(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2}), (\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})-(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2}), 0, 0, (\partial_{\bar{q}_1}+\partial_{q_1})^2+(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})^2+(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})^2+(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})^2, 0],$

$[-(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})+(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2}), (\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})-(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2}), -(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})+(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})+(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2}), -(\partial_{\bar{q}_1}+\partial_{q_1})(\partial_{\bar{q}_2}+\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{i}+\mathbf{i}\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{j}+\mathbf{j}\partial_{q_2})-(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})(\partial_{\bar{q}_2}\mathbf{k}+\mathbf{k}\partial_{q_2}), 0, 0, 0, (\partial_{\bar{q}_1}+\partial_{q_1})^2+(\partial_{\bar{q}_1}\mathbf{i}-\mathbf{i}\partial_{q_1})^2+(\partial_{\bar{q}_1}\mathbf{j}-\mathbf{j}\partial_{q_1})^2+(\partial_{\bar{q}_1}\mathbf{k}-\mathbf{k}\partial_{q_1})^2])$

Appendix III

$\hat{B} = [\frac{1}{2}y_0^2 + \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + \frac{1}{2}y_3^2, 0, 0, 0, -\frac{1}{2}x_0y_0 - \frac{1}{2}x_1y_1 - \frac{1}{2}x_2y_2 - \frac{1}{2}x_3y_3,$
$\frac{1}{2}x_1y_0 - \frac{1}{2}x_0y_1 + \frac{1}{2}x_3y_2 - \frac{1}{2}x_2y_3, \frac{1}{2}x_2y_0 - \frac{1}{2}x_3y_1 - \frac{1}{2}x_0y_2 + \frac{1}{2}x_1y_3, \frac{1}{2}x_3y_0 + \frac{1}{2}x_2y_1 - \frac{1}{2}x_1y_2 -$
$\frac{1}{2}x_0y_3],$

$[0, \frac{1}{2}y_0^2 + \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + \frac{1}{2}y_3^2, 0, 0, -\frac{1}{2}x_1y_0 + \frac{1}{2}x_0y_1 - \frac{1}{2}x_3y_2 + \frac{1}{2}x_2y_3, -\frac{1}{2}x_0y_0 - \frac{1}{2}x_1y_1 - \frac{1}{2}x_2y_2 -$
$\frac{1}{2}x_3y_3, \frac{1}{2}x_3y_0 + \frac{1}{2}x_2y_1 - \frac{1}{2}x_1y_2 - \frac{1}{2}x_0y_3, -\frac{1}{2}x_2y_0 + \frac{1}{2}x_3y_1 + \frac{1}{2}x_0y_2 - \frac{1}{2}x_1y_3],$

$[0, 0, -\frac{1}{2}y_0^2 - \frac{1}{2}y_1^2 - \frac{1}{2}y_2^2 - \frac{1}{2}y_3^2, 0, \frac{1}{2}x_2y_0 - \frac{1}{2}x_3y_1 - \frac{1}{2}x_0y_2 + \frac{1}{2}x_1y_3, \frac{1}{2}x_3y_0 + \frac{1}{2}x_2y_1 - \frac{1}{2}x_1y_2 -$
$\frac{1}{2}x_0y_3, \frac{1}{2}x_0y_0 + \frac{1}{2}x_1y_1 + \frac{1}{2}x_2y_2 + \frac{1}{2}x_3y_3, -\frac{1}{2}x_1y_0 + \frac{1}{2}x_0y_1 - \frac{1}{2}x_3y_2 + \frac{1}{2}x_2y_3],$

$[0, 0, 0, -\frac{1}{2}y_0^2 - \frac{1}{2}y_1^2 - \frac{1}{2}y_2^2 - \frac{1}{2}y_3^2, \frac{1}{2}x_3y_0 + \frac{1}{2}x_2y_1 - \frac{1}{2}x_1y_2 - \frac{1}{2}x_0y_3, -\frac{1}{2}x_2y_0 + \frac{1}{2}x_3y_1 + \frac{1}{2}x_0y_2 -$
$\frac{1}{2}x_1y_3, \frac{1}{2}x_1y_0 - \frac{1}{2}x_0y_1 + \frac{1}{2}x_3y_2 - \frac{1}{2}x_2y_3, \frac{1}{2}x_0y_0 + \frac{1}{2}x_1y_1 + \frac{1}{2}x_2y_2 + \frac{1}{2}x_3y_3],$

$[\frac{1}{2}x_0y_0 + \frac{1}{2}x_1y_1 + \frac{1}{2}x_2y_2 + \frac{1}{2}x_3y_3, \frac{1}{2}x_1y_0 - \frac{1}{2}x_0y_1 + \frac{1}{2}x_3y_2 - \frac{1}{2}x_2y_3, \frac{1}{2}x_2y_0 - \frac{1}{2}x_3y_1 - \frac{1}{2}x_0y_2 +$
$\frac{1}{2}x_1y_3, \frac{1}{2}x_3y_0 + \frac{1}{2}x_2y_1 - \frac{1}{2}x_1y_2 - \frac{1}{2}x_0y_3, -\frac{1}{2}x_0^2 - \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_3^2, 0, 0, 0],$

$[-\frac{1}{2}x_1y_0 + \frac{1}{2}x_0y_1 - \frac{1}{2}x_3y_2 + \frac{1}{2}x_2y_3, \frac{1}{2}x_0y_0 + \frac{1}{2}x_1y_1 + \frac{1}{2}x_2y_2 + \frac{1}{2}x_3y_3, \frac{1}{2}x_3y_0 + \frac{1}{2}x_2y_1 - \frac{1}{2}x_1y_2 -$
$\frac{1}{2}x_0y_3, -\frac{1}{2}x_2y_0 + \frac{1}{2}x_3y_1 + \frac{1}{2}x_0y_2 - \frac{1}{2}x_1y_3, 0, -\frac{1}{2}x_0^2 - \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_3^2, 0, 0],$

$[\frac{1}{2}x_2y_0 - \frac{1}{2}x_3y_1 - \frac{1}{2}x_0y_2 + \frac{1}{2}x_1y_3, \frac{1}{2}x_3y_0 + \frac{1}{2}x_2y_1 - \frac{1}{2}x_1y_2 - \frac{1}{2}x_0y_3, -\frac{1}{2}x_0y_0 - \frac{1}{2}x_1y_1 - \frac{1}{2}x_2y_2 -$
$\frac{1}{2}x_3y_3, -\frac{1}{2}x_1y_0 + \frac{1}{2}x_0y_1 - \frac{1}{2}x_3y_2 + \frac{1}{2}x_2y_3, 0, 0, \frac{1}{2}x_0^2 + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2, 0],$

$[\frac{1}{2}x_3y_0 + \frac{1}{2}x_2y_1 - \frac{1}{2}x_1y_2 - \frac{1}{2}x_0y_3, -\frac{1}{2}x_2y_0 + \frac{1}{2}x_3y_1 + \frac{1}{2}x_0y_2 - \frac{1}{2}x_1y_3, \frac{1}{2}x_1y_0 - \frac{1}{2}x_0y_1 + \frac{1}{2}x_3y_2 -$
$\frac{1}{2}x_2y_3, -\frac{1}{2}x_0y_0 - \frac{1}{2}x_1y_1 - \frac{1}{2}x_2y_2 - \frac{1}{2}x_3y_3, 0, 0, 0, \frac{1}{2}x_0^2 + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2]$

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