# APPLICATIONS OF COMMUTATIVE AND COMPUTATIONAL ALGEBRA TO PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We propose a computational algebra approach to some classical problems in the theory of linear partial differential equations. Our approach is based on the developments of new tools in computational algebra, as well as on the algebraization process which has occured in the theory of partial differential equations over the last 30 years. We describe several problems in well-known areas (calculation of Ext modules, free resolutions, Koszul complexes, Noetherian operators, etc.) but we also describe some new problems such as the development of a possible theory for weakly commutative systems.

In this paper we propose a new approach to some problems in partial differential equations based on computational algebra. We list some fundamental problems whose solutions would validate our approach. We hope that this paper will pave the way towards research in computational algebraic analysis. A more detailed version of this work will be published subsequently. We also refer to [2, 3, 4], where we show how some of our ideas can be effectively implemented in some concrete cases.

Let us first describe the general set up for our approach. Let $R = \mathbb{C}[z_1, \ldots, z_n]$, and let $P = [P_{ij}]$ be an $r_1 \times r_0$ matrix of polynomials in $R$ so that $P(D) = [P_{ij}(D)]$ is a matrix of linear constant coefficients differential operators, where $D = \left(-i\frac{\partial}{\partial x_1}, \ldots, -i\frac{\partial}{\partial x_n}\right)$. If $\mathcal{S}$ is a suitable sheaf of generalized functions ($\mathcal{S}$ could be either the sheaf $\mathcal{A}$ of real analytic functions or $\mathcal{B}$ of hyperfunctions or $\mathcal{D}'$ of distributions or $\mathcal{E}$ of infinitely differentiable functions, etc.) we denote by $\mathcal{S}^P$ the kernel of the map $\mathcal{S}^{r_0} \xrightarrow{P(D)} \mathcal{S}^{r_1}$, i.e. the sheaf of $\mathcal{S}$-solutions to the system $P(D)u = 0$.

Now the matrix $P^t = [P_{ji}]$, the transpose of $P$, can be considered as an $R$-homomorphism $R^{r_1} \xrightarrow{P^t} R^{r_0}$ and if $M = R^{r_0}/P^tR^{r_1}$ is its cokernel, then the Hilbert syzygy theorem shows that there is a finite free resolution

$$0 \longrightarrow R^{r_m} \xrightarrow{P_m^t} R^{r_{m-1}} \longrightarrow \cdots \longrightarrow R^{r_2} \xrightarrow{P_2^t} R^{r_1} \xrightarrow{P^t} R^{r_0} \longrightarrow M \longrightarrow 0, \tag{1}$$

with $m \leq n$.

Sequence (1), together with its transpose (which clearly forms a complex)

$$0 \longrightarrow R^{r_0} \xrightarrow{P} R^{r_1} \xrightarrow{P_2} R^{r_2} \longrightarrow \cdots \longrightarrow R^{r_{m-1}} \xrightarrow{P_m} R^{r_m} \longrightarrow 0 \tag{2}$$

are extremely important in the algebraic treatment of the system $P(D)u = 0$. In fact, one can say that much of its analysis is hidden in the algebra of (1) and (2).

Even though the free resolution (1) is far from unique, one still has that the $p$-th cohomology groups of (2), which will be denoted by $\mathrm{Ext}^p(M, R)$, are uniquely determined by $M$; they are actually the objects which carry the relevant analytic information.

In a recent series of papers [2, 3, 4], we have shown how to use the computer algebra package CoCoA[1] to compute Resolution (1) and its corresponding Ext-modules for a special case of the Cauchy–Fueter system (a

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[1]CoCoA is a special purpose system for doing computations in commutative algebra. It is the ongoing product of a research team in Computer Algebra at the University of Genova, Italy. It is freely available, and more information can be obtained by sending an e-mail message to `cocoa@dima.unige.it`.

generalization of the Cauchy-Riemann system, whose kernel sheaf is the sheaf $\mathcal{R}$ of regular functions); in [4] we have also indicated a more general approach which allows us to compute all the Ext-modules associated to any Cauchy–Fueter system. In this paper we explain the problems which arise when dealing with more general systems.

Using Gröbner bases techniques, see [1], one can compute two objects which are of great importance. First one can compute syzygies, that is generators for the kernel of any homomorphism $\phi : R^r \longrightarrow R^s$. Therefore one can compute resolutions such as (1). In fact, if the $R$-module $M$ is graded, then one can compute a minimal free resolution of $M$. Second, one can also compute the module of homomorphisms between two modules, and more generally, one can compute Ext and Tor modules. So, for a specific system of partial differential equations, one can compute, in principle, the free resolution given in (1), and if the matrix which arises from the system has homogeneous columns, then one can compute a minimal free resolution. One might think that this says that the question of the computation of Resolution (1) and of the cohomology groups of (2) is answered. In fact, there are three difficulties which need to be addressed.

First, in general, we do not have a specific system of partial differential equations in mind, but rather a family of systems (e.g., we may not have a specific number $n$ of variables in mind). So we want to analyze the free resolution (1) for a family of systems (e.g. a generic number of variables $n$). For example, in [4], we analyze the minimal free resolution of the module $R^4/\langle P^t\rangle$, where $R = \mathbb{C}[x_{ij}], 1 \leq i \leq n, 0 \leq j \leq 3$, and $P^t$ is the matrix which arises from the Cauchy–Fueter system. We actually compute the free resolution for this system in the cases $n = 1, 2, 3$ in [2], and in [4] we give a general formula for the projective dimension of $R^4/\langle P^t\rangle$ for a general $n$. To generalize this result requires theoretical results from commutative and computational algebra. So we propose the following.

**Problem 1.** For a given family of systems of partial differential equations, give a formula for the projective dimension of $R^{r_0}/\langle P^t\rangle$, where $P^t$ is the matrix which arises from the given system.

The families of systems we have in mind include, in addition to the Cauchy–Fueter system, mixed systems where, for example, one set of variables appears in a partial Cauchy–Riemann system and another set of variables appears in a partial de Rham system. Other systems arise from the study of generalized Jacobi structures (see [6, 10, 14, 15]).

The algebraic question raised in Problem 1 has an analytic counterpart which would be interesting to investigate. For example, when the system $P$ is elliptic, one can actually try to obtain a bound on the flabby dimension of the sheaf ${\mathcal{D}'}^P$ of distribution solutions to $P$. Or more generally we have

**Problem 2.** For a given family of systems of partial differential equations, give a formula for the flabby dimension of the sheaf $\mathcal{S}^P$, where $\mathcal{S}$ is a suitable sheaf of generalized functions.

To get even more information about the system of partial differential equations we need to solve the following problem.

**Problem 3.** For a given family of systems of partial differential equations, give information about the matrices which define the maps in the free resolution of $R^{r_0}/\langle P^t\rangle$. The information we have in mind includes: the size and degrees of these matrices, but also the algebraic variety defined by the maximal minors of these matrices (or at least of the last matrix), or more generally the kind of system of differential equation these matrices define (e.g. elliptic systems).

The second difficulty we need to address is that even though we can, in principle, compute a free resolution such as (1), for any specific system of PDE's, it is very unlikely that this will actually be possible, using the existing algorithms, for large systems (e.g. with large number of variables). There might be an algorithmic way to circumvent this complexity issue. We make the following observation. The modules we are dealing with are not just any kind of modules. Indeed, they all arise from matrices with a certain structure which originates in the analytic problem they model.

**Problem 4.** For a given family of systems of partial differential equations, use the structure of the matrix $P^t$ which arises from the system to improve the algorithm to compute the free resolution of $R^{r_0}/\langle P^t\rangle$.

In general, it is of interest to obtain, when possible, other complexes determined by a given module. In [5] the authors discuss several complexes and, in particular, the one given by the so-called *generalized Koszul complex*. Such a complex plays a very important role in many issues related to systems of differential and convolution equations, [6, 11, 16], but even in very simple cases it may be extremely hard to compute. Nevertheless, its explicit knowledge may be quite important as indicated in [10].

**Problem 5.** Construct an effective algorithm which has a polynomial matrix $P$ as its input and its generalized Koszul complex as its output, and determine its exactness.

When one looks at the differential operators defining regular functions of several quaternionic variables, one has to deal with an essentially non-commutative situation (since $\frac{\partial^2}{\partial \bar{q}_j \partial \bar{q}_k} \neq \frac{\partial^2}{\partial \bar{q}_k \partial \bar{q}_j}$, where $q_j$ and $q_k$ are different quaternionic variables). This difficulty is resolved by looking at these operators as a single $4n \times 4$ matrix and applying the methods discussed previously. There is however an interesting problem which arises when complexes of more than one matrix are considered. This is the case, for example, when one tries to characterize theta functions as solutions to a specific system of differential equations (i.e. when one is essentially studying generalized Jacobi structures). In this case (see [6, 10]) one has to deal with a non-commutative situation in which the polynomials $P_1, \ldots, P_r$ are replaced by matrices of polynomials $Q_1, \ldots, Q_r$; several difficulties immediately arise; to begin with we lack any commutativity structure, but also we lack the algebraic results needed to deal with the situation. In order to construct the necessary resolution, Kawai and Takei have used a very ingenious procedure in [10], but their procedure can probably be improved with the use of a suitable generalized Koszul complex. In any case, it becomes important to be able to build up resolutions (not necessarily minimal ones) for such a situation, and one can therefore look at the following problem:

**Problem 6.** Construct an effective algorithm which takes $r$ matrices as input and produces a resolution of the associated complex as output.

Specific systems of differential equations which should be studied in this framework are, for example, mixed systems where one set of variables appears in a partial Cauchy–Riemann system and another set of variables appears in a partial de Rham system. Other examples, as we said, arise from the study of generalized Jacobi structures; more generally, we would like to extend some ideas from [10] to the case of pairs of systems which we could call *weakly commuting* in the following sense. Let $Q_1$ and $Q_2$ be two $p \times p$ square matrices of differential operators. In general, $Q_1$ and $Q_2$ will not commute, and therefore it is not possible to construct the Koszul complex associated to the map from $R^{p^2} \times R^{p^2}$ to $R^{p^2}$ defined by

$$(P_1, P_2) \longmapsto P_1 Q_1 + P_2 Q_2.$$

The construction of such a complex, indeed, heavily depends on the commutativity of the operators which define the map. Let us then consider the Lie brackets of $Q_1$ and $Q_2$ and then their iterated Lie brackets, i.e.

$$R_0 = [Q_1, Q_2], \quad R_1 = [Q_1, R_0], \quad R_2 = [Q_2, R_0].$$

Now we note that the system associated to $(Q_1, Q_2, R_0, R_1, R_2)$ is equivalent to the original system associated to the pair $(Q_1, Q_2)$ since any solution to

$$Q_1(D)f = Q_2(D)f = 0$$

is a solution of

$$Q_1(D)f = Q_2(D)f = R_0(D)f = R_1(D)f = R_2(D)f = 0,$$

and the converse is true as well. We might therefore be interested in studying a resolution for the quintuple $(Q_1, Q_2, R_0, R_1, R_2)$ rather than the original pair. Define now $L_j = Q_j$ for $j = 1, 2$ and $L_j = R_{j-3}$ for

$2 < j \leq 5$. We say that the pair $(Q_1, Q_2)$ is weakly commuting if there are polynomials $c_{jk\ell}$ such that, for all $j$ and $k$, with $1 \leq j, k \leq 5$,

$$[L_j, L_k] = \sum_{1 \leq \ell \leq 5} c_{jk\ell} L_\ell. \tag{3}$$

In this case it might be possible to construct a resolution for the quintuple defined above since, even though we don't have complete commutativity, we can take advantage of this particular form of weak commutativity. In this case the first step of the resolution can be constructed in the form of a $10 \times 5$ matrix whose entries are $p \times p$ matrices which can be computed by using the relation (3). We note also that this argument can be generalized to consider also the case in which (3) does not hold, but a similar relationship can be established when considering higher order Lie brackets (this complex is studied, for example, in the theory of Lie algebra cohomology as given in [7]). A first remarkable example of this approach is given in [10], where however the operators which are involved are quite complicated because they are infinite order operators. It would therefore be of great interest to answer to Problem 6 at least for the case in which $r = 2$ and the matrices involved are weakly commuting.

Let us now get to a final fascinating problem. Consider the so-called *Fundamental Principle* of Ehrenpreis and Palamodov for systems of linear differential equations with constant coefficients [8, 9, 12, 13]. This is a very important result in the theory of linear constant coefficients partial differential operators which is, however, a consequence of the following algebraic result.

Let $P_1, \ldots, P_r$ be polynomials in $R = \mathbb{C}[z_1, \ldots, z_n]$, and let $I = \langle P_1, \ldots, P_r \rangle$ be the ideal they generate in $R$. We denote by $V = V(I)$ the zero variety of $I$, i.e. we set

$$V(I) = \{z \in \mathbb{C}^n \,|\, P(z) = 0 \text{ for all } P \in I\}.$$

In [9, 13], Ehrenpreis and Palamodov independently prove the following deep refinement of Hilbert's Nullstellensatz:

**Theorem 0.1** (Ehrenpreis-Palamodov Quotient Theorem). *Let $P_1, \ldots, P_r \in R$, let $I$ be the ideal they generate in $R$, and let $V = V(I)$. Then there exist subvarieties $V_1, \ldots, V_t \subseteq V$ and differential operators with polynomial coefficients $\partial_1, \ldots, \partial_t$ on $V_1, \ldots, V_t$ such that, for any given polynomial $P \in R$, the following conditions are equivalent:*

1. $P \in I$;
2. $\partial_j P\big|_{V_j} = 0, j = 1, \ldots, t$.

If one follows the literature published immediately after the discovery of the Fundamental Principle, it is easy to notice that many of its interesting applications do not require the actual knowledge of the operators $\partial_j$'s, nor of the splitting of $V$ into its subvarieties $V_j$, but rather rely on the geometric properties of $V$ (see, e.g., [9], Chapter XI, or the final notes in [13]). There are, however, several instances where a concrete description of noetherian operators is of great use. We therefore state our last problem:

**Problem 7.** Give an algorithm which inputs polynomials $P_1, \ldots, P_r \in R$ and outputs their multiplicity variety

$$V_m = \{(V_j, \partial_j), j = 1, \ldots, t\}.$$

More generally, the input may be a matrix $P = [P_{ij}]$ of polynomials and the multiplicity variety is then defined by the maximal minors of the matrix (i.e. the $k \times k$ minors, where $k$ is the rank of $P$). In this case the variety (as a set) is just the set of points where the rank of $P$ evaluated at this point is $< k$.

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