Geometry, Fall 2012

University of Maryland, Department of Mathematics course 430

HW7: (due in class November 6)

Please write-up your own solutions to problems in an organized and neat fashion and staple your sheets. If collaborating in the problem solving process please write the names of the people with whom you collaborated next to each solution.

0. *Review of proof writing:*
Denote by $\mathbf{x} = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$ a point in $\mathbb{R}^{n+1}$ and by

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \, : \, x_1^2 + \ldots x_{n+1}^2 = 1\}$$

the n-sphere.
In class we showed the following two theorems when $n = 2$:

Theorem 1. Let f be an isometry of $\mathbb{R}^{n+1}$ that satisfies $f(\mathbf{0}) = \mathbf{0}$, where $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^{n+1}$ denotes the origin. Denote by $f_{S^n} : S^n \to \mathbb{R}^{n+1}$ the map induced from f by restriction to S^n, namely,

$$f_{S^n}(\mathbf{x}) := f(\mathbf{x}), \quad \forall \mathbf{x} \in S^n.$$

Then f_{S^n} is an isometry of S^n.

Theorem 2. Let f be an isometry of S^n. Then there exists a unique isometry $\tilde{f}$ of $\mathbb{R}^{n+1}$ such that $\tilde{f}(\mathbf{0}) = \mathbf{0}$ and such that $\tilde{f}(\mathbf{x}) = f(\mathbf{x})$, $\forall \mathbf{x} \in S^n$.

Give complete, detailled, and organized proofs of Theorems 1 and 2 (for all n), justifying all steps.

1. Denote as before $\mathbf{x} = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$. Let A be an $(n+1) \times (n+1)$ matrix, and define a map $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$f(\mathbf{x}) = A\mathbf{x},$$

where the latter denotes matrix multiplication of A with $\mathbf{x}$.
Suppose that f is an isometry of $\mathbb{R}^{n+1}$.
(i) Prove that $A^T A = I$, where I denotes the identity matrix, and where A^T denotes the matrix transpose of A.
(ii) Show that f induces by restriction also an isometry of S^n.

2. *Clifford translations:*
Recall that an isometry f of S^n is called a Clifford translation if there exists some number $C \geq 0$ such that

$$d_{S^n}(p, f(p)) = C, \quad \forall p \in S^n.$$

Denote $\mathbf{x} = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$. Let $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ denote the map defined by

$$f(\mathbf{x}) = A\mathbf{x},$$

where A is the $(n+1) \times (n+1)$ matrix given by

$$A = \begin{pmatrix} \sin\theta_1 & \cos\theta_1 & 0 & \ldots & & & 0 \\ -\cos\theta_1 & \sin\theta_1 & 0 & \ldots & & & 0 \\ 0 & \ldots & \vdots & & & & \\ 0 & \ldots & \sin\theta_k & \cos\theta_k & 0 & \ldots & 0 \\ 0 & \ldots & -\cos\theta_k & \sin\theta_k & 0 & \ldots & 0 \\ 0 & \ldots & & & \pm 1 & 0\ldots & 0 \\ 0 & \ldots & \vdots & & & & \\ 0 & \ldots & & & & 0 & \pm 1 \end{pmatrix}.$$

Here, of course, $2k \leq n+1$, and ± 1 represent a row consisting of zeros except the the diagonal element that is either 1 or -1.
In class we showed that when $n = 2$, if f is a Clifford translation then either $A = I$ or $A = -I$, where I denotes the identity matrix.

(i) Prove that the same conclusion holds more generally also whenever n is even.

(ii) What are all the Clifford translations when n is odd?

(iii) Give an explicit example of a non-trivial (i.e., not equal to $\pm I$) Clifford translation when $n = 3$.

3. (i) Let A and B be two spaces. Give a definition of a local isometry between A and B.

(ii) Prove that there is no local isometry between S^2 and $\mathbb{R}^2$.