Geometry, Fall 2012

University of Maryland, Department of Mathematics course 430

HW8: (due in class November 13)

Please write-up your own solutions to problems in an organized and neat fashion and staple your sheets. If collaborating in the problem solving process please write the names of the people with whom you collaborated next to each solution.

0.

Definition. A metric space (X, d) is a pair consisting of a space X together with a function $d : X \times X \to \mathbb{R}_+$ (here $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$) satisfying: (i) $d(x, y) = 0$ if and only if $x = y$ ("non-degeneracy"), (ii) $d(x, y) + d(y, z) \geq d(x, z)$ (the "triangle inequality"), and (iii) $d(x, y) = d(y, x)$ ("symmetry") for all $x, y \in X$ (d is then called a *distance function on* X).

Using polar coordinates (r, θ) where $r \geq 0$ is distance to the origin and $\theta \in [0, 2\pi)$ is the angle made with the x-axis, define a function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$d((r_1, \theta_1), (r_2, \theta_2)) = \begin{cases} |r_2 - r_1| & \text{if } \theta_2 = \theta_1, \\ r_1 + r_2 & \text{if } \theta_2 \neq \theta_1. \end{cases}$$

Is d a metric? Prove or disprove.

1. (i) Show that $d(x, y) = |\log y/x|$ defines a metric on $\mathbb{R}_{>0} := \mathbb{R}_+ \setminus \{0\}$.
(ii) Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a function defined by $f(x) = 2x$. Prove or disprove that f is an isometry.
(iii) Let Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a function defined by $g(x) = x + 1$. Prove or disprove that g is an isometry.
(iv) Let Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a function defined by $h(x) = e^x$. Prove or disprove that h is an isometry.

2.
Let (X, d) be a metric space. Suppose $f : X \to X$ is an isometry and $g : X \to X$ is an isometry. Prove that the composition $f \circ g : X \to X$ is an isometry.

3.
Let (X, d) and (Y, δ) be metric spaces.
(i) Prove that that if $f : X \to Y$ is an isometry, then f is one-to-one (another term for one-to-one: injective). That is, f cannot map two distinct points to the same point. Yet another way of saying this is that if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

A function $f : X \to Y$ is onto if every point of Y is hit by a point in X. That is, for every $y \in Y$, there is some x such that $f(x) = y$. Other ways to say that f is onto is that f maps X onto Y, the image of X is all of Y , or that f is surjective.
If $f : X \to Y$ is an isometry, then f is not necessarily onto (another term for onto: surjective).
(ii) Write down an isometry $f : \mathbb{R} \to \mathbb{R}^2$ with the standard metrics that is not onto.

(iii) What if f is a map from (X, d) to itself? Is it true that every isometry $f : X \to X$ is onto? Answer this question for the following two examples (justify your answers) (Z, d_{discrete}), and $(\mathbb{R}^2, d_{\mathbb{R}^2})$. Here d_{discrete} is defined by $d_{\text{discrete}}(x, y) = 1$ if $x \neq y$, and $d_{\text{discrete}}(x, x) = 0$, while $d_{\mathbb{R}^2}$ is the standard Euclidean distance on $\mathbb{R}^2$.