Differential Geometry I, Autumn 2013

University of Maryland, Department of Mathematics course 436

HW 3: (due by October 31, in class.)

Please write-up your own solutions to problems in an organized and neat fashion. Collaborating in the problem solving process is not allowed for this homework.

1.
(a) Let $F : \mathbb{R} \to \mathbb{R}^2$ be given by $F(x_1) = (x_1^2, e^{x_1})$. Compute $dF\big(\frac{\partial}{\partial x_1}\big)$.

(b) Let $F : \mathbb{R}^2 \to \mathbb{R}$ be given by $F(x_1, x_2) = x_2$. Compute $dF\big(\frac{\partial}{\partial x_i}\big)$ for $i = 1, 2$.

(c) Let $F : \mathbb{R}^3 \to \mathbb{R}^4$ be given by $F(x_1, x_2, x_3) = (x_1^3, x_2^3, x_3^3, x_1x_2x_3)$. Compute dF.

(d) Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $F(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3, x_3 + x_1)$. Prove or disprove: F a diffeomorphism of $\mathbb{R}^3$.

2. For each of the maps in Problem 1 (a)–(d) compute the pull-back of the Euclidean metric. For example, for part (c) compute $F^\star(dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + dx_3 \otimes dx_3 + dx_4 \otimes dx_4)$.

3. Consider the setting of Problem 1 (b). Let $h = F^\star(dx_1^2)$. Let $\gamma : [0, 1] \to \mathbb{R}^2$ be a smooth path and write $\gamma(t) = (\gamma^1(t), \gamma^2(t))$. Suppose that $\dot\gamma^2(t) > 0$ and define $L(\gamma) = \int_0^1 \sqrt{h(\dot\gamma(t), \dot\gamma(t))}dt$. Compute $L(\gamma)$ solely in terms of $\gamma(0)$ and $\gamma(1)$.

4. In HW2 and in class we proved that the Riemannian volume form dV_g associated to a Riemannian manifold (M, g) can be expressed, with respect to local coordinates $x_1, \ldots, x_n$ valid in some open set U of M containing the point $p \in M$, as

$$dV_g(x) = \sqrt{\det[g_{ij}(x)]}dx^1 \wedge \cdots \wedge dx^n.$$

Prove that the right hand side is indeed independent of the choice of coordinates by going through the following steps.

(a) Choose another coordinate chart $y_1, \ldots, y_n$ valid in some open set V of M containing p. Set $W := U \cap V$. Find a formula, valid on W, for dy^j in terms of $\{dx^i\}_{i=1}^n$.

(b) Find a formula, valid for each $q \in W$, for $g_{ij}(x(q))$ in terms of $g_{ij}(y(q))$ and the Jacobian matrix for the identity map (expressed with respect to the two sets of coordinates, i.e., the map $x(q) \mapsto y(q)$). Here we denote $g_{ij}(y(q)) = g(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j})(y(q))$, while $g_{ij}(x(q)) = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})(x(q))$.

(c) Combine (a) and (b) to show that

$$\sqrt{\det[g_{ij}(x(q))]}dx^1 \wedge \cdots \wedge dx^n = \sqrt{\det[g_{ij}(y(q))]}dy^1 \wedge \cdots \wedge dy^n.$$