We discuss the following equations:

$$\sum_{|\alpha|,|\beta|\leq m} D^\beta(a_{\alpha\beta}D^\alpha u) = \sum_{|\beta|\leq m} D^\beta f_\beta \tag{1}$$

Here $f_\beta \in L^2(\Omega)$, $\Omega \in \mathbb{R}^n$ bounded.

The weak formulation of the equation is as follows:

$$\int_\Omega \sum_{|\alpha|,|\beta|\leq m} (-1)^{|\beta|} a_{\alpha\beta} D^\alpha u D^\beta \zeta = \int_\Omega \sum_{|\beta|\leq m} (-1)^\beta f_\beta D^\beta \zeta, \forall \zeta \in C_0^\infty(\Omega) \tag{2}$$

And the uniformly ellipticity holds. That is:

$$\sum_{|\alpha|,|\beta|=m} a_{\alpha\beta}(x)\lambda_\alpha\lambda_\beta \geq \mu \sum_{|\alpha|,|\beta|=m} \lambda_\alpha^2, \forall x \in \Omega, \forall \{\lambda_\alpha\}_{\alpha=1}^n \subseteq \mathbb{R} \tag{3}$$

Lemma 1. *If $u \in W^{m,2}$ is the weak solution of the equation (1), (3) is satisfied by the equation (1), $|D^l a_{\alpha\beta}| \leq C_l, l \leq k$ and $f_\beta \in W^{k,2}$, then the following inequality holds:*

$$||u||_{W^{m+k,2}(B_{\theta R}(x_0))} \leq C(||u||_{W^{m-1,2}(B_R(x))} + \sum_{|\beta|\leq m} ||f_\beta||_{W^{k,2}(B_R(x_0))})$$

Corollary 1. *If $f_\beta, a_{\alpha\beta} \in C^\infty$, then $u \in C^\infty$.*

Now we begin to talk about the interior Schauder theory.

The simplest example is the following:

$$\Delta u = f$$

It is easy to verify that if $f \in C^{0,\alpha}$, then $u \in C^{2,\alpha}$.

The proof of the Schauder estimate can be divided into two steps: the first step is to prove the estimate for constant coefficient diferential operators; the second step is prove the estimate of free coefficient differential operators based on the observation that these operators are just perturbations of the constant coefficient ones.

Define the norm as follows:

Definition 1.

$$[u]_{\mu,\Omega} = \sup_{x,y\in\Omega, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\mu}$$

$$|u|_{k,\mu} = \sum_{j=0}^k |D_j u|_0 + [D^k u]_\mu$$

$$|D^k u|_{C^{0,\mu}} = [D^k u]_\mu = \sum_{|\beta|=k} [D^\beta u]_{\mu,\mathbb{R}^n}$$

(Worth noticing that the symbols of norms and seminorm might vary in this lecture notes.)

We have the following lemma:

Lemma 2. *If u is a weak solution of the following equation ON $\mathbb{R}^n$:*

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} D^{\alpha+\beta} u = \sum_{|\beta|=2m-k} D^\beta f_\beta$$

Here $|a_{\alpha\beta}| \leq \Lambda$ is a constant. Assume also that $[D^k u]_{\mu,\mathbb{R}^n} < \infty$. Then we will have the following estimate:

$$|D^k u|_{C^{0,\mu}(\mathbb{R}^n)} \leq C \sum_{|\beta|=2m-k} |f_\beta|_{C^{0,\beta}(\mathbb{R}^n)}$$

with $C = C(\Lambda, \lambda_k, k, n, \mu)$.

Proof. Suppose the inequality doesn't hold, $\exists a_{\alpha\beta}^{(j)}, f_\beta^{(j)}, u_j$ such that:

$$|f_\beta^{(j)}|_{C^{0,\mu}(\mathbb{R}^n)} < \frac{1}{j} |D^n u_j|_{C^{0,\mu}(\mathbb{R}^n)} \tag{4}$$

And there exists $x_j, y_j \in \mathbb{R}^n$ such that:

$$\frac{|D^k u_j(y_j) - D^k u_j(x_j)|}{|x_j - y_j|^p} \geq C |D^k u_j|_{C^{0,\mu}(\mathbb{R}^n)}$$

Let's define $|D^k u_j|_{C^{0,\mu}(\mathbb{R}^n)} := \lambda_k, \sigma_j := |y_j - x_j|$ respectively. Now let $\widetilde{u}_j(x) = \sigma^{-\mu-k} u_j(x_j + \sigma_j x)$, let $x_j \to 0, y_j \to \partial B(0,1)$.

By our definition, $|D^k \widetilde{u_j}|_{C^{0,\mu}} = 1$, and $\widetilde{f}_j, \widetilde{u_j}$ satisfy (1) w.r.t $\widetilde{f}_\beta^{(j)}$ and $\tilde{a}_{\alpha\beta}^{(j)}$. By (4), we have that $|\widetilde{f}_\beta^{(j)}|_{C^{0,\mu}} \leq \frac{1}{j}$.

And the following inequality will hold for the sequence $\eta_j := \sigma_j^{-1}(y_j - x_j)$ (Notice that $|\eta_j| = 1$):

$$|D^k \tilde{u}_j(\eta_j) - D^k \tilde{u}_j(0)| \geq C \tag{5}$$

Now we can rewrite the $\tilde{u_j}$ as $\widetilde{u}_j = P_j(x) + R_j$. P_j here is the degree k Taylor expansion at the point $x = 0$. The P_j term here will not matter because you will quotient out by $|x_j - y_j|^\alpha$ here. So $|D^\alpha(\widetilde{u_j} - P_j)|_{C^{0,\mu}} = |D^\alpha \widetilde{u_j}|_{C^{0,\mu}} = 1$.

Now we can define a new variable $v_j := \tilde{u}_j - P_j$. And it is easy to see that $D^k v_j(0) = 0, |D^k v_j(\eta_j)| \geq C$, C is a constant (by (5)). We have $\frac{|D^k v_j(0) - D^k v_j(\eta_j)|}{|0-\eta_j|^\mu} \geq$

C. We know that $|D^k v_j|_{C^{0,\mu}(\mathbb{R}^n)} = 1$, and we have the following inequality (Exercise 1):

$$|D^l v_j|_{C^0(B_R(0))} \leq CR^{k-l+\mu}$$

Proof. The repeated application of the fundamental theorem of calculus tell us the following:

$$\sup_{B_R(0)} |v_j| \leq R^k \sup_{x \in B_R(0)} |D^k \tilde{u}_j(x) - D^k \tilde{u}_j(0)|$$

Now the definition of the $|D^k \tilde{u}_j|_{C^{0,\mu}} = [D^k \tilde{u}_j]_\mu$ norm tell us that the right hand side is smaller than $R^\mu |D^k \tilde{u}_j|_{C^{0,\mu}}$. Combining these observations, we have the following

$$\sup_{B_R(0)} |v_j| \leq R^{k+\mu}$$

Now simply apply the above bound to the function $D^l v_j$, we can easily get the estimate needed. $\square$

So $||v_j||_{C^l(B_R(0))}$ is bounded. By Arzela-Ascoli Thworem, we can find the converging subsequence in $C^{k,\delta}(B_R(0)), \forall \delta < \mu$ to a $v \in C^{k,\mu}(B_R(0)), v \not\equiv 0$.

(*)

$$\begin{aligned} |D^k(v(\eta))| &\neq 0 \\ |D^k(v(0))| &= 0 \\ |D^k v|_{C^{0,\mu}} = [D^k v]_{0,\mu} &\leq 1 \end{aligned}$$

Here $\eta = \lim \eta_{j'}$. The last equality holds because $|D^k v_j|_{C^{0,\mu}} = [D^k v_j]_{0,\mu} = 1$ and the fact that the norm here is lower semicontinuous. Furthermore, we will have that v satisfy the following equation weakly (Exercise 2):

$$\sum_{|\alpha|=|\beta|=m} a^0_{\alpha\beta} D^{\alpha+\beta} v = 0$$

Here $a^0_{\alpha\beta} = lim_{j' \to \infty} a^{j'}_{\alpha\beta}$.

Proof. Because of the construction, $|\widetilde{f^{(j)}_\mu}|_{C^{0,\mu}} \leq \frac{1}{j}$. So the limit $[f_\beta]_{C^{0,\mu}} = 0$, it tells us that f_β is a constant.

Now all the $\tilde{v}_j$ satisfy the following integration equation (weak form):

$$\sum_{|\alpha|=|\beta|=m} (-1)^\beta \int \tilde{a}_{\alpha\beta} D^\alpha \tilde{v}_j D^\beta w dx = \sum_{|\beta|=2m-k} (-1)^\beta \int \tilde{f}^{(j)}_\beta D^\beta w dx, w \in H^m$$

By Lebesgue Dominated convergence theorem, the right hand side goes to zero, and the left hand side become $\sum_{|\alpha|=|\beta|=m}(-1)^{\beta}\int {a^0}_{\alpha\beta}D^{\alpha}vD^{\beta}wdx$

So we have that:

$$\sum_{|\alpha|=|\beta|=m}(-1)^{\beta}\int {a^0}_{\alpha\beta}D^{\alpha}vD^{\beta}wdx=0,\forall w\in H^m$$

That is the weak form we are looking for. □

By Exercise 1, $\sup_{B_R}|v|_{B_R}\leq CR^{k+\mu}$. By the L^2 estimate (Theorem 2 below), we have that for any q:

$$\sup_{B_{R/2}}|D^q v|\leq \frac{C}{R^q}\sup_{B_R}|v|\leq CR^{k+\mu-q}$$

Here the constant C doesn't depend on anything. Take $q=k+1$ let $R\to\infty$, we will have that $D^{k+1}v=0$ on $\mathbb{R}^n$. So $D^k v$ is a constant. But correspond to (*) above, we have that $|D^k v(\eta)|\neq 0$ and $D^k v(0)=0$, which is a contradiction. This completes the proof. □

Theorem 1. *$u\in C^{m,\mu}(B_R(x_0))$ is the solution to the equation. Then*

$$|u|_{m,\mu,B_{\theta R}(x_0)}\leq C(|u|_{0,B_R(x_0)}+\textstyle\sum_{|\beta|\leq m}|f_\beta|_{0,\mu,B_R(x_0)}),$$
$$C=C(R,n,m,\theta,\gamma,\mu,\Lambda)$$

Proof. Let $R=1,x_0=0,B_\delta(y)\subseteq B_1(0)$. $x\in B_\sigma(y)$, so we will have $z_n=\frac{x-y}{\sigma}\in B_1(0)$. Now let's define the $\widetilde{u}(z):=u(y+\sigma z)$ on $B_1(0)$, and $\widetilde{f}_\beta(z)=f_\beta(y+\sigma z),\widetilde{a}_{\alpha\beta}(z)=a_{\alpha\beta}(y+\sigma z)$ And we have the following pull back equation:

$$\sum_{|\alpha|,|\beta|\leq m}D^{\beta}(\widetilde{a}_{\alpha\beta}D^{\alpha}\widetilde{u})=\sum_{|\beta|\leq m}D^{\beta}\widetilde{f}_\beta$$

on $B_1(0)$.

Here we will have:

$$|\widetilde{a}_{\alpha\beta}|_{C^0(B_1)}\leq\Lambda$$
$$|\widetilde{a}_{\alpha\beta}|_{C^{0,\mu}(B_1)}\leq\Lambda\sigma^{\mu}$$

Now with $\widetilde{a}_{\alpha\beta}(0)=a_{\alpha\beta}(y)$, we have the following:

$$\sum_{|\alpha|,|\beta|\leq m}D^{\beta}(\widetilde{a}_{\alpha\beta}(0)D^{\alpha}\widetilde{u})=\sum D^{\beta}((\widetilde{a}_{\alpha\beta}(0)-\widetilde{a}_{\alpha\beta})D^{\alpha}\widetilde{u})+\sum_{|\beta|\leq m}D^{\beta}\widetilde{f}_\beta$$

Now extend $\widetilde{u}$ to $\mathbb{R}^n$. $\varphi\in C_0^\infty(\mathbb{R}^n),supp(\varphi)\subseteq B_1(0),\varphi=1$ on $B_{1/2}(0)$. $\widetilde{u}\to\varphi\widetilde{u},f_n$ on $\mathbb{R}^n$

(*)

$$
\begin{aligned}
&\sum D^\beta(\widetilde{a}_{\alpha\beta}(0)D^\alpha(\varphi\widetilde{u}))\\
=&\sum_{|\beta,|\gamma|,|\delta|\le m} D^\gamma(c_{\beta\gamma\delta}D^\delta\varphi\widetilde{f}_\beta)+\sum_{|\alpha|,|\beta|,|\gamma|,|\delta|\le m} D^\gamma(d_{\alpha\beta\gamma\delta}D^\delta\varphi(\widetilde{a}_{\alpha\beta}(0)-\widetilde{a}_{\alpha\beta})D^\alpha\widetilde{u})\\
&+\sum_{|\alpha|<m,|\beta|,|\delta|\le m}(b_{\alpha\beta\gamma}D^\gamma\varphi D^\alpha\widetilde{u})
\end{aligned}
$$

The following statment is useful at this stage (which follows easily from $D^2(\varphi f)=\varphi D^2 f+D(2D\varphi f)-(D^2\varphi)f$ and mathematical induction):

$$
\varphi D^\beta\widetilde{f}_\beta=\sum_{|\gamma|,|\delta|\le m} D^\gamma(c_{\beta\gamma\delta}D^\delta\varphi\widetilde{f}_\beta) \tag{6}
$$

$c_{\beta\gamma\delta}$ is a constant.

With the same satement, we have the following:

$$
\varphi D^\beta((\widetilde{a}_{\alpha\beta}(0)-\widetilde{a}_{\alpha\beta}(x))D^\alpha\widetilde{u})=\sum_{|\gamma|,|\delta|\le m} D^\gamma(d_{\alpha\beta\gamma\delta}D^\delta\varphi(\widetilde{a}_{\alpha\beta}(0)-\widetilde{a}_{\alpha\beta}(x)D^\alpha\widetilde{u}))
$$

For suitable constants $d_{\alpha\beta\gamma\delta}$ with $|d_{\alpha\beta\gamma\delta}|\le C, C=C(n,m)$. With these, the equation above (*) can be rewritten:

(**)

$$
\begin{aligned}
&\sum_{|\alpha|=|\beta|=m} D^\beta(\widetilde{a}_{\alpha\beta}(0)D^\alpha v)\\
=&\sum_{|\beta|,|\gamma|,|\delta|\le m} D^\gamma(c_{\beta\gamma\delta}D^\delta\varphi\widetilde{f}_\beta)+\sum_{|\alpha|,|\beta|,|\gamma|,|\delta|\le m} D^\gamma(d_{\alpha\beta\gamma\delta}D^\delta\varphi(\widetilde{a}_{\alpha\beta}(0)-\widetilde{a}_{\alpha\beta}(x))D^\alpha\widetilde{u})\\
&+\sum_{|\alpha|\le m,|\beta|,|\gamma|\le m} D^\beta(b_{\alpha\beta\gamma}D^\gamma\varphi D^\alpha\widetilde{u})
\end{aligned}
$$

Here the constants $b_{\alpha\beta\gamma}, c_{\beta\gamma\delta}, d_{\alpha\beta\gamma\delta}$ are constants with $|c_{\beta\gamma\delta}|, |d_{\alpha\beta\gamma\delta}|\le C$ and $|b_{\alpha\beta\gamma}|\le C\Lambda, C=C(m,n)$.

Because v now has compact support, we can extend the function v to all of $\mathbb{R}^n$ by simply setting it to be zero elsewhere. And if $|\beta|<m$, we can rewrite $D^\beta g=D^{\widetilde{\beta}}\widetilde{g}$ for suitable $\widetilde{\beta}$ with $|\beta|=m$ and for suitable $\widetilde{g}$ such that $[\widetilde{g}]_\mu\le C(|g|_0+[g]_\mu)$ (This is just a simple exercise in FTC, one has to be carefull of the seminorm of the function they get.). Now we can apply the Lemma 1 to (**), getting the following function:

$$
\begin{aligned}
[D^m v]_\mu \leq & C(\sum_{|\alpha|,|\beta| \leq m} |(\widetilde{a}_{\alpha\beta}(0) - \tilde{a}_{\alpha\beta}(x)) D^\alpha \widetilde{u}|_0 \\
& + [(\widetilde{a}_{\alpha\beta}(0) - \tilde{a}_{\alpha\beta}(x)) D^\alpha \widetilde{u}]_\mu + \sum_{|\alpha|<m} (|D^\alpha \widetilde{u}|_0 + [D^\alpha \widetilde{u}]_\mu) \\
& + \sum_{|\beta| \leq m} (|\widetilde{f}_\beta|_0 + [\widetilde{f}_\beta]_\mu))
\end{aligned}
$$

Here all the norms and seminorms are over $B_1(0)$. In order to get rid of the φ factor, we have applied the simple relations of the seminorms, i.e., $[f+g]_\mu \leq [f]_\mu + [g]_\mu, [fg]_\mu \leq |f|_0[g]_\mu + [f]_\mu |g|_0$ to estimate the Holder norm of the terms $b_{\beta\gamma\delta} D^\gamma \varphi D^\delta u$. The same technique can be apply to the first two terms on the right (the terms involving $(\widetilde{a}_{\alpha\beta}(0) - \tilde{a}_{\alpha\beta}(x))$), we will have:

(***)

$$
\begin{aligned}
[D^m \widetilde{u}]_\mu \leq & C(\sum_{|\alpha|,|\beta| \leq m} |\widetilde{a}_{\alpha\beta}(0) - \widetilde{a}_{\alpha\beta}(y)|_0 [D^\alpha \widetilde{u}]_\mu + \sum_{|\alpha|,|\beta| \leq m} [\widetilde{a}_{\alpha\beta}]_\mu |D^\alpha \widetilde{u}|_0 \\
& + \sum_{|\alpha|<m} (|D^\alpha \widetilde{u}|_0 + [D^\alpha \widetilde{u}]_\mu) + \sum_{|\beta| \leq m} (|\widetilde{f}_\beta|_0 + [\widetilde{f}_\beta]_\mu))
\end{aligned}
$$

Here all the norms and semi-norms are over $B_1(0)$ and C depends only on $\theta, \Lambda, \gamma, \mu$, m and n.

Now by assumption on $\tilde{a}_{\alpha\beta}$, $\max_{|\alpha|=|\beta|=m} |\widetilde{a}_{\alpha\beta} - \widetilde{a}_{\alpha\beta}(0)|_{0,B_1(0)} \leq \Lambda \sigma^\mu$ and by easy calculus argument we have:

$$
|D^k \widetilde{u}(x) - D^k \widetilde{u}(y)| \leq |D^{k+1} \widetilde{u}|_0 |x-y|, k \geq 0
$$

In other words,

$$
[D^k \widetilde{u}]_\mu \leq 2 |D^{k+1} \widetilde{u}|_0, m-1 \geq k \geq 0
$$

Then using the above and the interpolation inequality (Lemma 3 below), we have the following:

$$
[D^\alpha \tilde{u}]_\mu \leq |D^{\alpha+1} \tilde{u}|_0 \leq \epsilon [D^m u]_\mu + C|u|_0
$$

Here the radius of the ball is one, so we don't have the extra R^l factor. Do the same thing to all the terms in (***), we will have that:

$$
[D^m \widetilde{u}]_{\mu, B_{1/2}(0)} \leq C(\epsilon + \sigma^\mu [D^m \widetilde{u}]_\mu + |\widetilde{u}|_0 + \sum_{|\beta| \leq m} (|\widetilde{f}_\beta|_0 + [\tilde{f}_\beta]_\mu), \tag{7}
$$

Where C here depends only on $\mu, \theta, n, m, \Lambda, \gamma$, and where all norms and seminorms are still over $B_1(0)$. The σ^μ term comes from the $\tilde{a}_{\alpha\beta}$.

Now by scaling and translation to get back to the original function u on the original ball $B_\sigma(y) \subseteq B_1(0)$, we have proved:

$$\sigma^{m+\mu}[D^m u]_{\mu,B_{\sigma/2}(y)} \leq C((\epsilon+\sigma^\mu)\sigma^{m+\mu}[D^m u]_{\mu,B_\sigma(y)}+|\widetilde{u}|_{0,B_\sigma(y)}+\sum_{|\beta|\leq m}|f_\beta|_{0,B_\sigma(y)}+\sigma^\mu[f_\beta]_{\mu,B_\sigma(y)}) \tag{8}$$

which implies,

$$\sigma^{m+\mu}[D^m u]_{\mu,B_{\sigma/2}(y)} \leq C(\epsilon+\sigma^\mu)\sigma^{m+\mu}[D^m u]_{\mu B_\sigma(y)} + \gamma$$

for every $B_\sigma(y) \subset B_1(0)$, $\gamma := C(|u|_{0,B_1(0)} + \sum_{|\beta|\leq m}|f_\beta|_{0,\mu,B_1(0)})$. Notice that here we expand the norm of u and f_β to the whole ball $B_1(0)$ in order to make γ a fixed number.

The proof is now completed by setting $\sigma = \epsilon$, with ϵ small enough, and applying the Absorption Lemma 4with $l = m+\mu$, $\epsilon_0 = \epsilon^\mu + \epsilon$, and with $S(A) = [D^m u]_{\mu,A}$:

$$[D^m u]_{\mu,B_\theta(0)} \leq C\gamma = C(|u|_{0,B_1(0)} + \sum_{|\beta|\leq m}|f_\beta|_{0,\mu,B_1(0)})$$

and hence, by the interpolation inequality, $|u|_{m,\mu,B_\theta(0)} \leq C\gamma$ as required. The proof is now complete. □

In the above proof, we have applied the following two Lemma:

Lemma 3. *(Interpolation Lemma) For any $u \in C^{k,\mu}(\bar{B}_R(x_0))$, we have the following interpolation inequatlity:*

$$R^l|D^l u|_0 \leq \epsilon R^{k+\mu}[D^k u]_\mu + C|u|_0$$

for eache $\epsilon > 0$ and $1 \leq l \leq k$, where $C = C(\epsilon, \mu, k, n)$.

Proof. It is easy to observe that:

$$|D_j u(y) - D_j u(x)| \leq |y-x|^\beta [D_j u]_{\beta,B_R(x_0)}$$

So we can easily derive the following

$$|D_j u(y)| \leq |D_j u(x)| + |y-x|^\beta [D_j u]_{\beta,B_R(x_0)}$$

Because $D_j u$ is continuous, we can choose σ and y such that $x_{max} \in B_\sigma(y)$, x_{max} is the point in $B_R(x_0)$ where the $D_j u$ reaches its maximum value. Now it is obvious that $|x_{max} - z|^\beta \leq \sigma^\beta$ for $\forall z \in B_\sigma(y)$. So we have the following:

$$|D_j u|_{0,B_R(x_0)} = |D_j u(x_{max})| \leq \inf_{B_\sigma(y)} |D_j u| + (2\sigma)^\beta [D_j u]_{\beta,B_R(x_0)}$$

For the $\inf_{B_\sigma(y)} |D_j u|$, we have the following inequality based on elementary calculus:

$$|u(x) - u(y)| \geq (\inf_{B_\sigma(y)} |D_j u|)|y - x|$$

Here without loss of generality, you $\inf_{B_\sigma(y)} |D_j u|$ can simply choose $|y-x| = 2\sigma$, and the j-th coordinate of y and x to be the same. And by the triangular inequality, we will have:

$$\begin{aligned} 2|u|_{0,B_\sigma(y)} &\geq (\inf_{B_\sigma(y)} |D_j u|)2\sigma \\ \sigma^{-1}|u|_{0,B_\sigma(y)} &\geq \inf_{B_\sigma(y)} |D_j u| \end{aligned}$$

Now we are ready to sum up through j and come to the following result:

$$|Du|_{0,B_R(x_0)} \leq n\sigma^{-1}|u|_{0,B_R x_0} + 2\sigma^\beta [Du]_{\beta,B_R(x_0)} \tag{9}$$

Now we are ready to prove the general interpolation theorem. Based on the above estimate, we have the following (choose β to be 1):

$$|Du|_{0,B_R(x_0)} \leq n\sigma^{-1}|u|_{0,B_R(x_0)} + 2\sigma[Du]_{1,B_R}$$

Notice that $[Du]_{1,B_R}$ is bounded by $|D^2u|_{0,B_R}$, we have the following:

$$|Du|_{0,B_R(x_0)} \leq n\sigma^{-1}|u|_{0,B_R(x_0)} + 2\sigma|D^2u|_{0,B_R}$$

Now we can also replace the u in the above inequality by the function $D_j u$, so we end up with the following estimate:

$$|D(D_j u)|_{0,B_R(x_0)} \leq n\epsilon^{-1}|D_j u|_{0,B_R(x_0)} + 2\epsilon|D^3u|_{0,B_R}$$

Combine the above two, and sum up all the j th components we will have:

$$|D^2u|_{0,B_R(x_0)} \leq n^3\epsilon^{-1}\sigma^{-1}|u|_{0,B_R(x_0)} + 2n^2\epsilon^{-1}\sigma|D^2u|_{0,B_R} + 2n\epsilon|D^3u|_{0,B_R}$$

Now we can choose $\sigma = \frac{\epsilon}{4n^2}$:

$$(1-\frac{1}{2})|D^2u)|_{0,B_R(x_0)} \le \frac{n}{4}\epsilon^{-2}|u|_{0,B_R(x_0)} + 2n^2\epsilon|D^3u|_{0,B_R}$$

Doing the same thing to other higher order derivatives and choose the corresponding σ carefully, we will end up with the following estimate (through induction):

$$|D^k u|_0 \le C(\sigma^k |u|_0 + \sigma^{m-k}|D^m u|_0)$$

Then use the inequality (6) with $D^m u$ in place of u and with $\beta = \mu$ give us the final estimate shown in the lemma.

□

The second lemma is the following:

Theorem 2. *If The differential operator $Lu = \sum_{|\alpha|\le m} a_\alpha D^\alpha u$ satisfy the following two conditions:*

$$|a_\alpha| \le M, |\alpha| \le 2m \tag{10}$$

$$\sum_{|\alpha|=2m} a_\alpha \xi^\alpha \ge \mu|\xi|^{2m}, \xi \in \mathbb{R}^n \tag{11}$$

and $u \in L^2_{loc}(B_R(x_0))$ is a weak solution of the equation $Lu = \sum_{|\beta|\le 2m-k} D^\beta f_\beta$ on $B_R(x_0)$, where $||f_\beta||_{l,B_R(x_0)} < \infty$. Then $u \in H^k_{loc}(B_R(x_0))$, and in fact:

$$||u||_{k+l,B_{\theta R}(x_0)} \le C(||u||_{0,B_R(x_0)} + \sum_{|\beta|\le 2m-k} ||f_\beta||_{l,B_R(x_0)}) \tag{12}$$

for each $\theta \in (0,1)$, where C depends only on n, M, μ, θ, R. (In particular, if $u \in C^\infty(B_R(x_0))$ if all the $f_\beta \in C^\infty(B_R(x_0))$)

Proof. After mollification, the mollified function u_σ satisfies the classical equation:

$$Lu_\sigma = \sum_{|\beta|\le 2m-k} D^\beta (f_\beta)_\sigma \tag{13}$$

on $B_{R-\sigma}(x_0)$

So assume $\theta \in (0,1), \sigma < (1-\theta)R/2$, and let ψ be an arbitrary $C^\infty_c(B_R(x_0))$ function. Notice that by repeatedly using the Leibniz formula, we can rewrite the equation 13 in the form:

$$\sum_{|\alpha|=2m} D^\alpha(a_\alpha \psi u_\sigma) = \sum_{|\gamma|+|\delta|\le 2m, |\delta|\le 2m-1} D^\delta(b_{\gamma\delta}(D^\gamma\psi)u_\sigma)$$
$$+\sum_{|\delta|+|\gamma|\le 2m-k, |\delta|\le 2m-k-1} D^\delta(d_{\beta\gamma\delta}(D^\gamma\psi)(f_\beta)_\sigma) + \sum_{|\beta|=2m-k} D^\beta(\psi f_\beta)$$

for some constant $b_{\gamma\delta}, d_{\beta\gamma\delta}$ with $|b_{\gamma\delta}| \le CM, |d_{\beta\gamma\delta}| \le C$. Now we take the Fourier transform of the above equation, then we will have:

$$\sum_{|\alpha|=2m} a_\alpha(-i\xi)^\alpha \widehat{\psi u_\sigma} = \sum_{|\gamma|+|\delta|\le 2m, |\delta|\le 2m-1} (i)^{|\delta|}\xi^\delta b_{\gamma\delta}\widehat{(D^\gamma\psi)u_\sigma}$$
$$+\sum_{|\delta|+|\gamma|\le 2m-k, |\delta|\le 2m-k-1} (i)^{|\delta|} d_{\beta\gamma\delta}(D^\gamma\widehat{\psi)(f_\beta})_\sigma) + \sum_{|\beta|=2m-k} (i)^{2m-k}\xi^\beta \widehat{\psi f_\beta}$$

Now we can apply the ellipticity condition 11 to the left hand side and get $\mu|\xi|^{2m}$. Also we want to get something equivalent to the H^l norm on the left hand side, so we add extra lower order term such that the l.h.s become $(1+|\xi|)^{2m}|\widehat{\psi u_\sigma}|$, now our euqation turn into the inequality:

$$(1+|\xi|)^{2m}|\widehat{\psi u_\sigma}| \le C(1+|\xi|)^{2m-1}\sum_{|\gamma|\le 2m} |\widehat{(D^\gamma\psi)u_\sigma}|$$
$$C(1+|\xi|)^{2m-k-1}\sum_{|\gamma|\le 2m-k} |(D^\gamma\widehat{\psi)(f_\beta})_\sigma| + C(1+|\psi|)^{2m-k}|\widehat{\psi f_\beta}|$$

We like the term $(1+|\xi|)^{2m}|\widehat{\psi u_\sigma}|$ because we have the following relation:

$$||\sum_{|\alpha|\le 2m} |\xi|^\alpha \hat{u}||_{L^2} \simeq ||u||_{H^{2m}}$$
$$\sum_{|\alpha|\le 2m} (|\xi|^\alpha) \simeq (1+|\xi|)^{2m}$$

The first is based on the observation that $\widehat{\partial_j f} = (-i\xi_j)\hat{f}$. So we have that $||(1+|\xi|)^{2m}|\widehat{\psi u_\sigma}|||_{L^2} \simeq ||u||_{H^{2m}}$.

Now we can multiply both side by $(1+|\xi|)^{k+l-2m}$, where $k+l \ge 1$. So we will have (applying the Plancheral theorem)

$$||\psi u_\sigma||_{k+l, B_R(x_0)} \le \sum_{|\gamma|\le 2m} ||(D^\gamma\psi)u_\sigma||_{k+l-1, B_R(x_0)} +$$
$$\sum_{|\gamma|\le 2m-k} ||(D^\gamma\psi)(f_\beta)_\sigma||_{(l-1)_+, B_R(x_0)} + C||\psi(f_\gamma)_\sigma||_{l_+, B_R(x_0)}$$

where j_+ means $\max(j,0)$. Keeping in mind that ψ was an arbitrary $C_c^\infty(B_R(x_0))$ function (so that if $k+l \ge 2$ the argument can be repeated with $D^\gamma\psi$ in place of ψ), we conclude by induction on l that

$$||\psi u_\sigma||_{k+l, B_R(x_0))} \le \sum_{|\gamma|\le 2(k+l)m} ||(D^\gamma\psi)u_\sigma||_{0, B_R(x_0)} + \sum_{|\gamma|\le 2(k+l)m} ||(D^\gamma\psi)(f_\beta)_\sigma||_{l, B_R(x_0)}$$

Now we select the function ψ such that $\psi = 1$ in $B_{\theta R}(x_0)$, $\psi = 0$ outside $B_{(1+\theta)R/2}(x_0)$ and

$$|D^\alpha \psi| \le C(1-\theta)^{-|\alpha|} R^{-|\alpha|}$$

for each multi-index α, with $C = C(n, \alpha)$. Then the above inequality gives the required inequality. □

The last lemma we need is the following:

Lemma 4. *Let S be a real valued monotone sub-additive function on the class of all convex subsets of $B_R(x_0)$ (i.e. $S(A) \le \sum_{j=1}^N S(A_j)$ when ever $A, A_1, A_2 ... A_N$ are convex sets with $A \subset \cup_{j=1}^N A_j \subset B_R(x_0)$), and suppose that $\theta \in (0,1), \mu \in (0,1], \gamma \ge 0$, and $l \ge 0$ are given constants. There is $\epsilon_0 = \epsilon(l, \theta, n) > 0$ such that if (*)*

$$\rho^l S(B_{\theta\rho}(y)) \le \epsilon_0 \rho^l S(B_\rho(y)) + \gamma$$

whenever $B_\rho(y) \subset B_R(x_0)$ and $\rho \le \mu R$, then

$$R^l S(B_{\theta R}(x_0)) \le C\gamma$$

where $C = C(\theta, \mu, l, n)$.

Proof. Basically we want to consume the $\epsilon_0 \rho^l S(B_\rho(y))$ term and get a "global" estimate of the $S(B_{\theta R})$. We can think of the S here as measures on the space or the norms (the H^k norms, for example) of a fixed funtion over varying sets.

Let's first define the following quantity:

$$Q = \sup_{B_\rho(y) \subset B_R(x_0), \rho \le \mu R} \rho^l S(B_{\theta\rho}(y)).$$

Then from (*) we can easily find that (replace ρ by $\theta\rho$ and take sup on the right hand side):

(**)

$$(\theta\rho)^l S(B_{\theta^2\rho}(y)) \le \epsilon_0 Q + \gamma$$

for each ball $B_\rho(y) \subset B_R(x_0)$ with $\rho \le \mu R$. Take any ball $B_\rho(y) \subset B_R(x_0)$ with $\rho \le \mu R$. Then we can sellect balls $\{B_{(1-\theta)}(y_i)\}_{j=1,2,3,...N}$ with centers $y_j \in B_{\theta\rho}(y)$ s.t. $B_{\theta\rho}(y) \subset \cup_{j=1}^N B_{\theta^2(1-\theta)\rho}(y_j)$ and with $N \le C$, where C is a constant depending only on θ, n. (Be careful of the radius of the balls here. One set of balls are of radius $(1-\theta)\rho$, the other set of balls are of radius $\theta^2(1-\theta)$. The balls with smaller raidus cover the ball $B_\theta \rho$.) We introduce $(1-\theta)$ factor in the radius here because we want the ball constructed to lie inside the ball $B_\rho(y)$ Since each $B_{(1-\theta)\rho}(y_j) \subset B_R(x_0)$, we can apply (**) with $B_{(1-\theta)\rho}$

$$(\theta(1-\theta)\rho)^l S(B_{\theta^2(1-\theta)}(y)) \leq \epsilon_0 Q + \gamma$$

Now by the subadditibity of the function S, and the construction that $B_{\theta\rho}(y) \subset \cup_{j=1}^{N} B_{\theta^2(1-\theta)\rho}(y_j)$, we will have the following:

$$\begin{aligned}
&(\theta(1-\theta))^l \rho^l S(B_{\theta\rho}(y)) \\
\leq&(\theta(1-\theta))^l \rho^l S(\cup_{j=1}^{N} B_{\theta^2(1-\theta)\rho}(y_i)) \\
\leq&(\theta(1-\theta))^l \rho^l \sum_{j=1}^{N} S(B_{\theta^2(1-\theta)\rho}(y_i)) \\
\leq&C(\epsilon_0 Q + \gamma), \\
&C = C(n, l)
\end{aligned}$$

We can devide the $(1-\theta)^l\theta^l$ on both side, and take sup to get the following:

$$Q \leq C\epsilon_0 Q + C\gamma, C = C(n, l, \theta)$$

and hence if $\epsilon_0 \leq \frac{1}{2}C^{-1}$, we getting

$$Q \leq 2C\gamma$$

So that:

$$\rho^l S(B_{\theta\rho}(y)) \leq 2C\gamma$$

for every ball $B_\rho(y) \subset B_R(x_0)$ with $\rho \leq \mu R$. Since we can cover $B_{\theta R}(x_0)$ by at most $C = C(\theta, \mu, n)$ balls $B_{\theta\mu R}(y_j)$ with $B_{\mu R}(y_j) \subset B_R x_0$, we can again use the given subadditivity of S to conclude the stated inequality.

$\square$