Notes on Maximal Principles for Second Order Equations and Green's function

November 17, 2013

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1 Maximal Principal

We consider the general form of elliptic operator of the form

$$Lu = \sum_{i,j=1}^{n} a_{ij} D_i D_j + \sum_{j=1} b_j D_j u + cu, \tag{1}$$

where $a_{i,j}, b_j, c$ are bouded founctions on $\Omega(\Omega$ is a bounded domain in $\mathbb{R}^n$, $a_{i,j} = a_{j,i}$, also we have strictly ellipticity assumption that there is a constant $\mu > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \mu|\xi|^2 \tag{2}$$

for all$x \in \Omega, \xi \in \mathbb{R}^n$.

Theorem 1 (Weak Maximum Principle). *Under the assumption above and let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy $Lu \geq 0$ in Ω. Assume also $c(x) \leq 0$, then*

$$\max_{\Omega} u \leq \max_{\partial\Omega} \max\{u, 0\} \tag{3}$$

If $c \equiv 0$, the inequality (3) holds with u in place of $\max\{u, 0\}$.

Remark. 1. When $Lu \leq 0$, we can apply the above theorem to $-u$, thus giving

$$\min_{\Omega} u \geq -\max_{\partial\Omega} \max\{-u, 0\}. \tag{4}$$

2. In the case $Lu = 0$, (In particular $\triangle u = 0$) we have

$$\max_{\Omega} |u| = \max_{\partial\Omega} |u| \tag{5}$$

Proof of the Weak Maximum Principle. Consider an auxiliary function $v = u + \epsilon e^{rx_1}$, where $\epsilon > 0, r > 0$, so

$$Lv = Lu + \epsilon e^{rx_1}(a_{11}r^2 + b_1 r + c) \tag{6}$$

By the ellipticity condition (2)(implies $a_{11} \geq \mu$) and the boundedness condition of coefficients b_1 and c, let r be sufficiently large(independent of ϵ), we can get $Lv > 0$. We claim v cannot attains maximum in Ω. If not, suppose that v attains maximum at $x_0 \in \Omega$ *with* $v(x_0) \geq 0$, then $D_i v(x_0) = 0$ *for* $i = 1, 2, \ldots, n$, and $D^2 v(x_0) \leq 0$. So $Lv(x_0) = \sum_{i,j} a_{ij} D_i D_j v(x_0) + c(x_0)v(x_0) \leq 0$, Which contradicts the fact that $Lv > 0$ in Ω. Let ϵ goes to 0, we can get the required result.

Theorem 2 (Weak Maximum Principle in Non-homogeneous Case). *Suppose $Lu = f$, where $f \in L^\infty(\Omega)$, any other condition are the same as Theorem 1. Then*

$$\max_{\Omega} |u| \leq \max_{\partial\Omega} |u| + \mu^{-1} e^{2(1+\beta)} d^2 \sup_{\Omega} |f|, \tag{7}$$

where β is any upper bound for $\mu^{-1}(d|b_1| + d^2|c|)$, d is any constant such that $\Omega \subset \{x = (x^1, \ldots, x^n) : |x^1| < d\}$. (We can take $d = diam(\Omega)$ since we can translate so that $0 \in \Omega$.)

Proof of Theorem 2. Define the auxiliary function $v = u+\mu^{-1}d^2\, e^{r(1+\frac{x^1}{d})} sup_\Omega |f|$, where $r \geq 1$ is constant to be chosen. By direct computation, we get

$$\begin{aligned} Lv &= Lu + \sup_\Omega |f|\mu^{-1}(a_{11}r^2 + b_1 dr + cd^2)e^{r(1+\frac{x^1}{d})} \\ &\geq f + \sup_\Omega |f|\mu^{-1}(\mu r^2 - |b_1|dr - |c|d^2)e^{r(1+\frac{x^1}{d})} \\ &\geq f + \sup_\Omega |f| r(r-\beta)e^{r(1+\frac{x^1}{d})}. \end{aligned} \tag{8}$$

So that if we choose that $r = 1 + \beta$, then we have $Lv \geq 0$and thus we can apply the weak maximum principle to v. Thus

$$\max_\Omega u \leq \max_\Omega v \leq \max_{\partial\Omega} \max\{v, 0\} \leq \max_{\partial\Omega} |u| + \mu^{-1}e^{2(1+\beta)}d^2 \sup_\Omega |f|. \tag{9}$$

This completes the proof.

Corollary (Exercise). If $\epsilon \in (0, 1)$ and the hypotheses are as in Theorem 2, except that the hypothesis $c \leq 0$ is replaced by the condition $\mu^{-1}e^{2(1+\beta)}d^2 \sup_\Omega c_+ \leq \epsilon$, then

$$\max_\Omega |u| \leq (1-\epsilon)^{-1} \max_{\partial\Omega} |u| + \mu^{-1}(1-\epsilon)^{-1}e^{2(1+\beta)}d^2 \sup_\Omega |f|. \tag{10}$$

Proof of Corollary. Note that $c = c_+ - c_-$,so the equation $Lu = f$ can be written as $L_1 u = \tilde{f}$, where $\tilde{f} = f - c_+ u$, and where L_1 is the same as L with $-c_{-1}$ in place of c. It is easy to get this result by applying Theorem 2 directly.

Our main goal is to prove the strong (or Hopf) maximum principle.Let us first state it here.

Theorem 3 (Strong Maximum Principle). *Suppose $u \in C^2(\Omega), Lu \geq 0$, $c \leq 0$, (1),(2) above holds and Ω is connected and $\partial\Omega$ is smooth. Then if u attains a non-negative maximum in Ω,then u is constant in Ω.*

In the theorem 3, the assumption that $c \leq 0$ is essential. It cannot be dropped. For this we first need to establish the following Hopf boundary point lemma.

Lemma 1(Hopf boundary point lemma). *Assume $B = B(y, \rho)$ is an open ball in $\mathbb{R}^n$, $u \in C^2(B), Lu \geq 0$ in $B, x_0 \in \partial B$, u is continuous at x_0, $u(x_0) \geq 0$, and $u(x) < u(x_0)$ for each $x \in B$.Then, if D_η denotes directional derivative in the direction of the inward pointing unit normal η of ∂B, we have*

$$D_\eta u(x_0) < 0, \tag{11}$$

if this derivative exists and in any case

$$\limsup_{h\downarrow 0} \frac{u(x_0 + h\eta) - u(x_0)}{h} < 0. \tag{12}$$

Proof of Theorem 3 by using Lemma 1. Proof by contradiction. Assume that us is non-constant and achieves its maximum M.Let $S = x \in \Omega : u(x) = M$. By continuity, S is relatively closed in Ω. It remains to show that B is open. Take any point $x_0 \in S, x_0 \notin Int(S)$, there exists $y \in \Omega \setminus S$ with $dist(y, S) < dist(y, \partial\Omega)$. Let x_0 be the closest point of S to y and let $\rho = |x_0 - y|$. By Lemma 1 we can conclude that $Du(x_0) \neq 0$, contradicting the fact that u has a maximum at x_0, i.e., $Du(x_0)$ should be 0.

Proof of Lemma. Without loss of generality, we can assume that u is continuous on $\bar{B}$, otherwise replace B with by a small ball $\tilde{B}$, with its closure contained in $B \cup \{x_0\}$ and $x_0 \in \partial\tilde{B}$.Let $r = |x - y|$, and consider the auxiliary function $w = e^{-\alpha r^2} - e^{-\alpha\rho^2}$ with $\alpha > 0$ a constant to be chosen. Then by direct computation

$$\begin{aligned} Lw &= e^{-\alpha r^2}(4\alpha^2 \sum_{i,j=1}^{n} a_{ij}(x^i - x^i)(x^j - y^j) - 2\alpha(\sum_{i=1}^{n} a_{ii} + \sum_{i}^{n} b_i(x^i - y^i))) + cw \\ &\geq e^{-\alpha r^2}(4\alpha^2\mu r^2 - 2\alpha(\sum_{i=1}^{n} a_{ii} + \sum_{i}^{n} |b_i|r) + c) \end{aligned} \tag{13}$$

By condition (1)and (2)we get that for α large enough, we have

$$Lw > 0 \ in \ A, A = B(y, \rho) \setminus B(y, \rho/2). \tag{14}$$

Let $v = u - u(x_0) + \epsilon w$, so $Lv = Lu - cu(x_0) + \epsilon Lw > 0$.Since $w \equiv 0$ on ∂B and $u < u(x_0)$ on $\partial B(y, \rho/2)$, we can choose ϵ small enough such than $v \leq 0$ on ∂A. By the weak maximum principle, we have $v \leq 0$ in A; i.e.

$$u(x) - u(x_0) \leq -\epsilon w(x), \qquad x \in A. \tag{15}$$

Hence we have

$$\limsup_{h \downarrow 0} \frac{u(x_0 + h\eta) - u(x_0)}{h} \leq -\epsilon D_\eta w(x_0) < 0. \tag{16}$$

This completes the proof.

Please refer to [1] for different version of maximal principle and some direct applications.

2 Green's function

Please refer to [2] for more background and more application. We can consider Green's function in more general setting. First we consider the simplest case, i.e. $\mathbb{R}^n$, then we proceed to find Green's function on the domain in $\mathbb{R}^n$, later we can generalize to the manifold case (compact one, or non-compact one). For the later two cases, please refer to the next lecture notes.

Here we define the Laplace operator as $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$. The Laplace equation is given by $\Delta u = 0$, where $n = 2, 3, \ldots$. We define the fundamental solution of Laplace's equation as follows:

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log|x| & \text{if } n = 2, \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & \text{if } n \geq 3. \end{cases} \tag{17}$$

where $\alpha(n)$ denotes the volume of the unit ball in $\mathbb{R}^n$. Let us state some basic facts about $\Phi(x)$:

1. $\Phi(x) = \Phi(|x|)$, i.e.it is radial;
2. $|D^k \Phi(x)| \leq \frac{C_k}{|x|^{n-2+k}}, \ x \neq 0$;
3. Φ is harmonic on $\mathbb{R}^n \setminus \{0\}$. (Exercise)

Let's prove the property 3 above. It is easy to compute that

$$D_i \Phi(x) = \frac{-1}{n\alpha(n)} x_i |x|^{-n} \tag{18}$$

and

$$D_{ij} \Phi(x) = \frac{-1}{n\alpha(n)} \left(|x|^2 \delta_{ij} - n x_i x_j \right) |x|^{-n-2} \tag{19}$$

So

$$\Delta \Phi(x) = \sum_{i=i}^{n} D_{ii} \Phi(x) = \frac{-1}{n\alpha(n)} \sum_{i=1}^{n} \left(|x|^2 - n x_i^2 \right) |x|^{-n-2} = 0$$

if $x \neq 0$.
Consequently, if $y \in \mathbb{R}^n$, $\Phi(x-y)$ is also harmonic away from y. Generally the linear combination of $\Phi(x - y_i)$, where $i = 1, 2, \ldots K$, is also harmonic away from $\{y_1, y_2, \ldots, y_K\}$. That is to say, function

$$\tilde{\Phi}(x) = \sum_{i=1}^{K} \Phi(x - y_i) f(y_i) \tag{20}$$

is always harmonic except for finite points.

But this is not true for the convolution

$$\int_{\mathbb{R}^n} \Phi(x-y) f(y) \mathrm{d}y. \tag{21}$$

Actually, we have the following conclusion:

Claim 1(Solving Poisson's equation. *For $f \in C_c^2(\mathbb{R}^n)$,define u by (21),then $u \in C^2(\mathbb{R}^n)$ and $\Delta(-u) = (-\Delta)u = f$ in $\mathbb{R}^n$.*

In [1], the fundamental solutions defined there have different signs with Φ. In that case, i.e, all Φ is replaced by $-\Phi$, then $\Delta u = f$. That's why I write $\Delta(-u) = f$ in the claim. In the chapter 4 of [1],we can see the it's enough to assume f is bounded and locally Hölder continuous.

Proof of Claim 1. First, note that

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) \,\mathrm{d}y = \int_{\mathbb{R}^n} \Phi(y) f(x-y) \,\mathrm{d}y; \tag{22}$$

Therefore

$$\frac{u(x+he_i) - u(x)}{h} = \int_{\mathbb{R}^n} \Phi(y) \left[\frac{f(x+he_i-y) - f(x-y)}{h} \right] \mathrm{d}y, \tag{23}$$

where $h \neq 0$ and $e_i = (0, \ldots, 1, \ldots, 0)$,the 1 in the i^{th}-slot. But

$$\frac{f(x+he_i-y) - f(x-y)}{h} \to f_{x_i}(x-y) \tag{24}$$

uniformly on $\mathbb{R}^n$ as $h \to 0$, and thus

$$D_i u(x) = u_{x_i}(x) = \int_{\mathbb{R}^n} \Phi(y) f_{x_i}(x-y) \,\mathrm{d}y \qquad (i = 1, \ldots, n). \tag{25}$$

Similarly,

$$D_{ij} u(x) = \int_{\mathbb{R}^n} \Phi(y) f_{x_i x_j}(x-y) \,\mathrm{d}y \qquad (i, j = 1, \ldots, n). \tag{26}$$

$D_{ij}u(x)$ is continuous, that means $u \in C^2(\mathbb{R}^n)$.

Remark: If f is only assumed to be bounded and continuous, of course this method doesn't work. In that case ,we try to use the cutoff function to eliminate the singularity of Φ, see [1] Chapter 4 for details.

Now, let's prove the second part. The key point is to use Green's formula to break the whole space into two parts or more which are easily to compute and use the property 2 of fundamental solution to get the desired estimates.

Since Φ blows up at 0, we need isolate this singularity inside a small ball. So fix $\epsilon > 0$. Then

$$\begin{aligned} \Delta u(x) &= \int_{B(0,\epsilon)} \Phi(y) \Delta_x f(x-y) \,\mathrm{d}y + \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \Phi(y) \Delta_x f(x-y) \,\mathrm{d}y \\ &= I_\epsilon + J_\epsilon. \end{aligned} \tag{27}$$

Now

$$|I_\epsilon| \leq C \|D^2 f\|_{L^\infty(\mathbb{R}^n)} \int_{B(0,\epsilon)} |\Phi(y)| \,\mathrm{d}y \leq \begin{cases} C\epsilon^2 |\log \epsilon| & (n = 2) \\ C\epsilon^2 & (n \geq 3). \end{cases} \tag{28}$$

By integration by parts,

$$
\begin{aligned}
J_\epsilon &= \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \Phi(y) \Delta_y f(x-y)\,\mathrm{d}y \\
&= -\int_{\mathbb{R}^n \setminus B(0,\epsilon)} D\Phi(y) \cdot D_y f(x-y)\,\mathrm{d}y + \int_{\partial B(0,\epsilon)} \Phi(y) \frac{\partial f}{\partial \nu}(x-y)\,\mathrm{d}S(y) \\
&=: K_\epsilon + L_\epsilon,
\end{aligned}
\tag{29}
$$

where ν denotes the inward pointing unit normal along $\partial B(0,\epsilon)$. And

$$
|L_\epsilon| \le \|Df\|_{L^\infty(\mathbb{R}^n)} \int_{\partial B(0,\epsilon)} |\Phi(y)|\,\mathrm{d}S(y) \le \begin{cases} C\epsilon|\log \epsilon| & (n=2) \\ C\epsilon & (n \ge 3). \end{cases} \tag{30}
$$

For K_ϵ, do integration by parts again, we get

$$
\begin{aligned}
K_\epsilon &= \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \Delta\Phi(y) f(x-y)\,\mathrm{d}y - \int_{\partial B(0,\epsilon)} \frac{\partial \Phi}{\partial \nu}(y) f(x-y)\,\mathrm{d}S(y) \\
&= -\int_{\partial B(0,\epsilon)} \frac{\partial \Phi}{\partial \nu}(y) f(x-y)\,\mathrm{d}S(y).
\end{aligned}
\tag{31}
$$

By formula (18), we have $D\Phi(y) = \frac{-1}{n\alpha(n)} \frac{y}{|y|^n}$ $(y \neq 0)$ and $\nu = \frac{-y}{|y|} = -\frac{y}{\epsilon}$ on $\partial B(0,\epsilon)$. So

$$
\frac{\partial \Phi}{\partial \nu}(y) = \nu \cdot D\Phi(y) = \frac{1}{n\alpha(n)\epsilon^{n-1}}
$$

on $\partial B(0,\epsilon)$. Since $n\alpha(n)\epsilon^{n-1}$ is also the surface of the sphere $\partial B(0,\epsilon)$, we get

$$
\begin{aligned}
K_\epsilon &= -\frac{1}{n\alpha(n)\epsilon^{n-1}} \int_{\partial B(0,\epsilon)} f(x-y)\,\mathrm{d}S(y) \\
&= -\frac{1}{n\alpha(n)\epsilon^{n-1}} \int_{\partial B(x,\epsilon)} f(y)\,\mathrm{d}S(y) \to -f(x) \quad \textit{as } \epsilon \to 0.
\end{aligned}
\tag{32}
$$

Let $\epsilon \to 0$, we prove the claim.

Let $U \subsetneqq \mathbb{R}^n$ be a domain, and ∂U is smooth. And assume that $u \in C^2(\bar{U})$, then we have the general Green's representation formula

$$
u(x) = \int_{\partial U} \Phi(y-x) \frac{\partial u}{\partial \nu}(y) - u(y) \frac{\partial \Phi}{\partial \nu}(y-x)\,\mathrm{d}S(y) - \int_U \Phi(y-x)\Delta u(y)\,\mathrm{d}y, \tag{33}
$$

where ν denotes the outer unit normal vector on ∂U.

The Proof of Green's representation formula is left as exercise. Here we give a short hint. The details can be easily fit in. For the reader who are not familiar with this method, refer to [1] Chapter 2 or [2] Chapter 2 for details.

Fix $x \in U$, choose $\epsilon > 0$ so small such that $B(x,\epsilon) \subset U$, and apply the Green's formula on the region $V_\epsilon := U \setminus B(x,\epsilon)$ to $u(y)$ and $\Phi(y-x)$. We get

$$
\int_{V_\epsilon} u(y)\Delta\Phi(y-x) - \Phi(y-x)\Delta u(y)\,\mathrm{d}y = \int_{\partial V_\epsilon} u(y) \frac{\partial \Phi}{\partial \nu}(y-x) - \Phi(y-x) \frac{\partial u}{\partial \nu}(y)\,\mathrm{d}S(y) \tag{34}
$$

We know ∂V_ϵ has two parts, by examining the integral over $\partial B(x, \epsilon)$ (some strategy as in the proof of the claim), we can get formula (33).

We are interested in Dirichlet Problem and Neumann Problem. They are formulated as below:

$$\textit{Dirichlet Problem} \qquad \begin{cases} \Delta u = f & \textit{on } U \\ u = g & \textit{on } \partial U, \end{cases} \tag{35}$$

$$\textit{Neumann Problem} \qquad \begin{cases} \Delta u = f & \textit{on } U \\ \frac{\partial u}{\partial \nu} = g & \textit{on } \partial U. \end{cases} \tag{36}$$

Of course, Φ isn't the fundamental solution for Dirichlet Problem for Laplace equation.

Definition of Green's function. *Green's function for the domain U is*

$$G(x, y) := \Phi(y - x) - \phi^x(y) \ (x, y \in U, x \neq y), \tag{37}$$

where $\phi^x(y)$ is a corrector function satisfies

$$\begin{cases} \Delta \phi^x = 0 & \textit{in } U \\ \phi^x = \Phi(y - x) & \textit{on } \partial U \end{cases} \tag{38}$$

Apply Green's formula to $\phi^x(y)$ and $u(y)$, we get

$$-\int_U \phi^x(y) \Delta u(y) \, \mathrm{d}y = \int_{\partial U} u(y) \frac{\partial \phi^x}{\partial \nu}(y) - \Phi(y - x) \frac{\partial u}{\partial v}(y) \, \mathrm{d}S(y) \tag{39}$$

Adding (39) to (33), we get

$$u(x) = -\int_{\partial U} u(y) \frac{\partial G}{\partial \nu}(x, y) \, \mathrm{d}S(y) - \int_U G(x, y) \Delta u(y) \, \mathrm{d}y \qquad (x \in U) \tag{40}$$

For more details, refer to [1] and [2].

References

[1] David Gilbarg, Neil S. Trudinger *Elliptic Partial Differential Equations of Second Order* , Springer-Verlag, 1977

[2] Evans, L.C., *Partial Differential Equations* American Mathematical Society , Providence, RI, 1998