Math 742: Geometric Analysis

Lecture 5 and 6 Notes

Jacky Chong

jwchong@math.umd.edu

The following notes are based upon Professor Yanir Rubenstein's lectures with reference to Variational Methods 4^{th} edition by Struwe and A User's Guide to Optimal Transport by Ambrosio and Gigli.

1 Direct Method in the Calculus of Variations

THEOREM 1. *Let X be a topological Hausdorff space, and suppose $E : X \to \mathbb{R} \cup \{+\infty\}$ satisfies the* ***bounded compactness condition****: For any $\alpha \in \mathbb{R}$ the sublevel set, $K_\alpha = \{u \in X : E(u) \leq \alpha\}$ is compact. Then there exists a uniform real constant C_X on X such that $E \geq C_X$ on X and E attains its infimum.*

Proof. If $E \equiv +\infty$ on X, then there is nothing to prove. So, assume $E \not\equiv +\infty$. Denote $\alpha = \inf_{u \in X} E(u) \geq -\infty$ and choose a strictly decreasing sequence (α_m) such that $\alpha_m \searrow \alpha$ as $m \to \infty$. Let $K_{\alpha_m} = \{u \in X : E(u) \leq \alpha_m\}$, then by assumption K_{α_m} is compact. Moreover, it's clear that $K_{\alpha_{m+1}} \subset K_{\alpha_m}$ for all $m \in \mathbb{N}$, i.e. we have a decreasing nesting of compact sets. By Cantor's Intersection Theorem, we know that

$$K = \bigcap_{m=1}^{\infty} K_{\alpha_m} \neq \emptyset.$$

Let $u^* \in K \subset X$, then we observe (i) $C_X := E(u^*) \neq -\infty$ and (ii)

$$E(u^*) \leq \alpha_m \qquad \text{for all } m.$$

Hence it follows $E(u^*) \leq \alpha$ when passing through the limit which means $E(u^*) = \alpha$. Thus E attains its infimum in X. □

DEFINITION 1. *Let $(V, \|\cdot\|_V)$ be a reflexive Banach space and $X \subset V$ a subspace. Then $E : X \to \mathbb{R} \cup \{\pm\infty\}$ is* ***coercive*** *provided $E(u) \to \infty$ if $\|u\|_V \to \infty$.*

DEFINITION 2. *$E : X \to \mathbb{R} \cup +\infty$ is said to be* ***sequentially weakly lower semicontinuous*** *over X provided for every sequence $(u_n)_{n=1}^\infty \subset X$ and $(u_n)_{n=1}^\infty$ converges weakly V, $u_n \rightharpoonup u$ then*

$$E(u) \leq \liminf_{n\to\infty} E(u_n).$$

EXAMPLE 1. Consider the function $E : \mathbb{R} \to \mathbb{R}$ given by

$$E(x) = \begin{cases} \dfrac{1}{1+x^2} & \text{if } x \neq 0 \\ 2 & \text{if } x = 0. \end{cases}$$

It's clear that E is not coercive since $E(x) \to 0$ when $|x| \to \infty$. Moreover, it is also clear that E is not lower semi-continuous (in particular, not weakly l.s.c.) since

$$2 = E(0) \not\leq \liminf_{n\to\infty} E(x_n) = 1$$

where $x_n \to 0$ as $n \to \infty$. Also, note that E does not attain its minimium.

THEOREM 2 (Special Case of Theorem 1)**.** *Let $(V, \|\cdot\|_V)$ be a reflexive Banach Space and let $X \subset V$ be a weakly closed subset of V. Assuming*

(i) $E : X \to \mathbb{R} \cup \{+\infty\}$ is coercive,

(ii) E is sequentially weakly lower semi-continuous.

Then E is bounded from below on X and attains its infimum in X.

Proof. Let $\alpha := \inf_{u\in X} E(u) \geq -\infty$. Similar to the proof of Theorem 1, we begin by choose a minimizing sequence $(\alpha_m) \subset X$ such that $E(u_m) \to \alpha$ as $m \to \infty$. Now, let us first make two claims:

(i) the sequence (u_m) is uniformly bounded in V,

(ii) there exists a subsequence such that $u_{m_k} \rightharpoonup u$ for some $u \in V$.

Since $u_{m_k} \rightharpoonup u$ by (ii) and X is weakly closed, then we have that $u \in X$. Applying the fact that E is weakly l.s.c., we have that

$$\alpha \leq E(u) \leq \liminf_{k\to\infty} E(u_{m_k}) = \alpha$$

Hence E attains its minimium in X. Thus, to complete the proof we need to justify (i) and (ii).
Proof of Claim (i): This is immediate from the definition of coerciveness of E: $E(u_m) \not\to \infty$ implies $\| u_m \|_V \not\to \infty$.
Proof of Claim (ii): Since V is reflexive, then by Theorem 8 in the Appendix we know that any closed ball, $\overline{B(0,R)}$, of V is weakly compact. Since (u_m) is bounded in V by (i), then $(u_m) \subset \overline{B(0,R)}$ for some R. Now, apply Theorem 6 in the Appendix, we see that (u_m) has a subsequence which converges weakly to some $u \in \overline{B(0,R)} \subset V$.

□

2 Some Examples and Applications

2.1 p-Laplacian in $\mathbb{R}^n$

Let $u \in W_0^{1,p}(\mathbb{R}^n)$, we informly defined the p-Laplacian as following

$$\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u). \tag{1}$$

Observe, we have that $\Delta_p = \Delta$ the usual Laplacian on $\mathbb{R}^n$ when $p = 2$.

Note: We must check $\Delta_p u$ is indeed well-defined on $W_0^{1,p}(\mathbb{R}^n)$ since the p-Laplacian "requires" second order (weak) derivatives of u. Observe for any $\phi \in C_0^\infty(\mathbb{R}^n)$ we have that

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \Delta_p u(x)\phi(x) \, dx \right| &:= \left| \int_{\mathbb{R}^n} |\nabla u(x)|^{p-2} \langle \nabla u(x), \nabla \phi(x) \rangle_{\mathbb{R}^n} \, dx \right| \\ &\leq \int_{\mathbb{R}^n} |\nabla u(x)|^{p-1} |\nabla \phi(x)| \, dx \\ &\leq \| \nabla \phi \|_{L^\infty} \int_{\mathbb{R}^n} |\nabla u(x)|^{p-1} \chi_{\overline{\text{supp}(\phi)}}(x) \, dx \\ &\leq C_{\text{supp}(\phi)} \| \nabla \phi \|_{L^\infty} \| \nabla u \|_{L^p}^{p-1} < \infty. \end{aligned}$$

Thus, to define the $\Delta_p u$ in the distributional sense, we only need that $\nabla u \in L^p(\mathbb{R}^n)$. Hence it is clear that $\Delta_p u$ is well-defined on $W_0^{1,p}(\mathbb{R}^n)$.

Assuming $\Omega \subset \mathbb{R}^n$ is a bounded domain and consider the Dirichlet problem:

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Claim: We claim $-\Delta_p u = f$ is the Euler-Lagrange equation of the function

$$E(u) = \int_\Omega \frac{1}{p} |\nabla u(x)|^p - u(x) f(x) \, dx$$

where the Lagrangian, $L : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$, is given by

$$L(x, y, z) = \frac{1}{p} |y|^p - z f(x).$$

Since the Euler-Lagrange equation to a function $E[u] = \int_\Omega L(x, Du(x), u(x)) \, dx$ is given by

$$L_z(x, Du, u) - \sum_{i=1}^n (L_{y_i}(x, Du, u))_{x_i} = 0 \tag{3}$$

then indeed we have that

$$-f(x) - \nabla \cdot (|\nabla u|^{p-2} \nabla u) = -f(x) - \sum_{i=1}^n \left(|\nabla u|^{p-1} \frac{u_{x_i}}{|\nabla u|} \right)_{x_i} = 0.$$

Thus, informly, solving the Dirichlet problem (2) is similar to finding a minimizer for E in $W_0^{1,p}(\Omega)$.

THEOREM 3. *Let $p \in [2, \infty)$ with conjugate q satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Assume also $f \in W^{-1,q}(\Omega)(:= W^{1,p}(\Omega)^*)$. Then there exists a weak solution $u \in W_0^{1,p}(\Omega)$ solving the Dirichlet problem (2).*

Proof. Base upon the above discussion, let us show that E attains its minimium in $W_0^{1,p}(\Omega)$. We shall invoke Theorem 2, by checking that E satisfies coercivity and sequentially weakly l.s.c. over $W_0^{1,p}(\Omega)$.
Coercivity of E: Observe by corollary 1 and the definition of the dual norm

$$\begin{aligned} E(u) &= \frac{1}{p} \| u \|^p_{W_0^{1,p}(\Omega)} - \int_\Omega f(x)u(x)\ dx \\ &\geq \frac{1}{p} \| u \|^p_{W_0^{1,p}(\Omega)} - \| f \|_{W^{-1,q}(\Omega)} \| u \|_{W_0^{1,p}(\Omega)} \\ \text{“Young's Inequality”} &\geq \frac{1}{p} \| u \|^p_{W_0^{1,p}(\Omega)} - \frac{1}{2^p p} \| u \|^p_{W_0^{1,p}(\Omega)} - \frac{2^q}{q} \| f \|^q_{W^{-1,q}(\Omega)} \\ &= C \| u \|^p_{W_0^{1,p}(\Omega)} - \frac{2^q}{q} \| f \|^q_{W^{-1,q}(\Omega)} \end{aligned}$$

where $C > 0$ and $\frac{2^q}{q} \| f \|^q_{W^{-1,q}(\Omega)}$ is independent of u. This shows that E is coercive over $W_0^{1,p}(\Omega)$.
Sequentially Weakly L.S.C. of E: Consider a sequence $(u_m) \subset W_0^{1,p}(\Omega)$ such that $u_m \rightharpoonup u \in W_0^{1,p}(\Omega)$. Applying proposition 2(d) in the appendix with $\phi(x) = \chi_\Omega(x)$ and the fact that $f \in W^{-1,q}(\Omega)$, then it follows

$$E(u) \leq \liminf_{n \to \infty} E(u_m).$$

Thus, it is now clear that E is both coercive and weakly l.s.c. which means E attains its infimum in $W_0^{1,p}(\Omega)$. Lastly, we need to check that a minimum of E, say u, is a weak solution to problem (2). Let $\phi \in C_0^\infty(\Omega)$ and u a minimizer of E, since

$$\begin{aligned} &\frac{d}{d\epsilon} \left(\frac{1}{p} |\nabla u(x) + \epsilon \nabla \phi(x)|^p - (u(x) + \epsilon\phi(x))f(x) \right) \\ =& |\nabla u(x) + \epsilon \nabla \phi(x)|^{p-2} \langle \nabla u(x) + \epsilon \nabla \phi(x), \nabla \phi(x) \rangle_{\mathbb{R}^n} - \phi(x) f(x) \end{aligned}$$

is bounded by some integrable function then by some variant of the Lebesgue Dominated Convergence Theorem we have that

$$\begin{aligned} \frac{d}{d\epsilon}\Big|_{\epsilon=0} E(u + \epsilon\phi) =& \frac{d}{d\epsilon} \int_\Omega \frac{1}{p} |\nabla u(x) + \epsilon \nabla \phi(x)|^p - (u(x) + \epsilon \phi(x)) f(x)\ dx \Big|_{\epsilon=0} \\ =& \int_\Omega |\nabla u(x) + \epsilon \nabla \phi(x)|^{p-2} \langle \nabla u(x) + \epsilon \nabla \phi(x), \nabla \phi(x) \rangle_{\mathbb{R}^n} - \phi(x) f(x)\ dx \Big|_{\epsilon=0} \\ =& \int_\Omega |\nabla u(x)|^{p-2} \langle \nabla u(x), \nabla \phi(x) \rangle_{\mathbb{R}^n} - \phi(x) f(x)\ dx = 0 \end{aligned}$$

which indeed solves the weak formulation of problem (2). □

2.2 Harmonic Maps into $\mathbb{R}^n$ -Generalizing Geodesics

Another important class of problem is finding minimal hypersurfaces in a Riemannian manifold (M, g) (geodesics, etc) which has a variational formulation. Let Ω be a bounded domain in $\mathbb{R}^d$, and M be a compact subset of $\mathbb{R}^n$ with a symmetric positive-definite Riemannian metric $g = (g_{ij})_{1 \leq i,j \leq N}$. Consider the $u \in H^1(\Omega, \mathbb{R}^n)$ and the functional

$$E(u) = \frac{1}{2} \int_\Omega g_{ij}(u(x)) \nabla u^i(x) \nabla u^j(x) \, dx$$

we shall compute its Euler-Lagrange equation.

PROPOSITION 1. *The Euler-Lagrange equation of $E(u)$ is equivalent to*

$$\Delta u^k + \Gamma^j_{ik} \nabla u^i \nabla u^j = 0 \tag{4}$$

in Ω.

Proof. First, observe

$$L_{p^k_l} = g_{ik} \frac{\partial u^i}{\partial x^l} \quad \Rightarrow \quad (L_{p^k_l})_{x^l} = \partial_j g_{ik} \frac{\partial u^j}{\partial x^l} \frac{\partial u^i}{\partial x^l} + g_{ik} \frac{\partial^2 u^i}{(\partial x^l)^2}$$

and

$$L_{z^k} = \frac{1}{2} \partial_k g_{ij} \nabla u^i \nabla u^j.$$

Using the Euler-Lagrange equations for systems

$$\sum_{l=1}^n (L_{p^k_l}(x, u, Du))_{x_l} - L_{z^k}(x, u, Du) = 0$$

we get that

$$-\frac{1}{2} \partial_k g_{ij} \nabla u^i \nabla u^j + \partial_j g_{ik} \nabla u^i \nabla u^j + g_{ik} \Delta u^i = 0$$

or

$$\Delta u^k + g^{ik} \left(\partial_j g_{ik} - \frac{1}{2} \partial_k g_{ij} \right) \nabla u^i \nabla u^j = \Delta u^k + \Gamma^j_{ik} \nabla u^i \nabla u^j = 0$$

for $k = 1, \ldots, n$. □

EXAMPLE 2. If u is a solution to (4), then we say that u is a *harmonic mapping* from $\Omega \subset \mathbb{R}^d$ to (M, g). In particular, if $d = 1$, then we have that (4) is just the geodesic equations

$$\ddot{u}^k + \Gamma^j_{ik} \dot{u}^i \dot{u}^j = 0$$

where $u : (0, 1) \to M \subset \mathbb{R}^n$.

2.3 Optimal Transport in $\mathbb{R}^n$

The subject of Optimal Transport will be "explain" in more details in subsequential lecture notes, we shall only introduce the different formulations of the subject in this section.

DEFINITION 3. *Given a Polish space (X, d) (i.e. a complete and separable metric space), we shall denote $\mathscr{P}(X)$ the set of Borel probability measures on X. The **support** $\operatorname{supp}(\mu)$ of a measure $\mu \in \mathscr{P}(X)$ is the smallest closed set, C on which $\int_C d\mu = 1$.*

DEFINITION 4. *If X, Y are two Polish spaces, $T : X \to Y$ is a Borel measurable map, and $\mu \in \mathscr{P}(X)$ a measure, the measure $T_{\#}\mu \in \mathscr{P}(Y)$, called the **push forward** of μ through T is defined by*

$$T_{\#}\mu(E) = \mu(T^{-1}(E)), \quad \forall\ E \subset Y,\ \textit{Borel.}$$

In particular, we have following characterization of $T_{\#}\mu$:

$$\int_E f\ dT_{\#}\mu = \int_{T^{-1}(E)} f \circ T\ d\mu$$

for every Borel function $f : Y \to \mathbb{R} \cup \{\pm\infty\}$ that is left hand side equals the right hand side provided one of the integrals exists.

2.3.1 Monge's Formulation of Optimal Transport Problem

Assume $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ bounded and let $\mu, \nu \in \mathscr{P}(\mathbb{R}^n)$ such that $\text{supp}(\mu) \subseteq \Omega_1$ and $\text{supp}(\nu) \subseteq \Omega_2$. Also fix a Borel function $c : \Omega_1 \times \Omega_2 \to \mathbb{R} \cup \{+\infty\}$ which we shall call the *cost* function.
Monge's Problem: Find $T : \Omega_1 \to \Omega_2$ such that

$$T \mapsto \int_{\Omega_1} c(x, T(x))\ d\mu$$

is minimize among all T where $T_{\#}\mu = \nu$, i.e. minimize the functional

$$E(T) = \int_{\Omega_1} c(x, T(x))\ d\mu$$

over the set $\{T : \Omega_1 \to \Omega_2 \text{ Borel} : T_{\#}\mu = \nu\}$.
EXAMPLE 3. Assume $c(x, y) = |x - y|^2$. Consider $\Omega_1 = B(\mathbf{0}, 1) \subset \mathbb{R}^2$ and $\Omega_2 = B((2, 0), 1) \subset \mathbb{R}^2$ where μ and ν are uniform measures over Ω_1 and Ω_2 respectively. Hence the Monge's problem is asking one to find a map T to recenter the unit ball which minimizes the total square distance traveled by each point $x \in B(0, 1)$ to $T(x) \in B((0, 2), 1)$. It's not hard to see that T is just the regular translation map $T(x) = x + (2, 0)$.
Despite the intuitive formulation,Monge's problem can be ill-posed because:

(i) No admissible T exists.

(ii) The constraints $T_{\#}\mu = \nu$ is not weakly sequentially closed, with respect to any reasonable weak topology.

First, lets consider the problem of (i). Observe if $\mu = \delta_{x_0} \in \mathscr{P}(0, 1)$ a delta measure and $\nu \in \mathscr{P}(0, 1)$ is a uniform measure, then we see that

$$T_{\#}\mu \neq \nu$$

because $T_{\#}\mu$ will also be a delta measure for all T, Borel.
The problem of (ii) can be readily seen in the following example

EXAMPLE 4. Consider the 1-periodic function $T:\mathbb{R}\to\mathbb{R}$ given by

$$T(x)=\begin{cases}1 & \text{if } x\in[0,\frac{1}{2})\\ -1 & \text{if } x\in[\frac{1}{2},1)\end{cases}$$

define the sequence $T_n(x) := T(nx)$ for $n\geq 1$. Assume $\mu := \mathcal{L}\mid_{[0,1]}$ the restricted Lebesgue measure and $\nu := \dfrac{\delta_{-1}+\delta_1}{2}$. To check that $(T_n)_{\#}\mu = \nu$ for all n, it suffices to check it for open intervals. Suppose (a,b) contains 1 and not -1, then

$$(T_n)_{\#}\mu((a,b)) = \mathcal{L}\mid_{[0,1]}(T_n^{-1}(a,b)) = \mathcal{L}\mid_{[0,1]}\left(\bigcup_{k=1}^{n-1}\left[\frac{k}{n},\frac{2k+1}{2n}\right)\cup\left[0,\frac{1}{n}\right)\right) = \frac{1}{2}$$

and likewise when (a,b) contains -1 and not 1. If (a,b) contains both 1 and -1, then it's clear that $(T_n)_{\#}\mu((a,b)) = 1$. Lastly, if (a,b) contains neither 1 nor -1, then $(T_n)_{\#}\mu((a,b)) = 0$. This shows that $(T_n)_{\#}\mu = \nu$ for all open interval. Thus $(T_n)_{\#}\mu = \nu$ for all Borel sets. Let f be a continuous function on $\mathbb{R}$, then consider

$$\int_{\mathbb{R}} T_n(x)f(x)\;d\mathcal{L}\mid_{[0,1]} = \int_{\mathbb{R}} T(x)f\left(\frac{x}{n}\right)\frac{d\mathcal{L}\mid_{[0,1]}}{n} = \int_0^1 T(x)f\left(\frac{x}{n}\right)\frac{dx}{n}\to 0$$

as $n\to\infty$ since T and $f(x)$ is bounded in $[0,1]$, i.e. $T_n \rightharpoonup 0$. Hence it follows for any g Borel, we have that

$$\int_{\mathbb{R}} g\;d(0)_{\#}\mathcal{L}\mid_{[0,1]} = \int_0^1 g(0)\;dx = g(0)$$

which means $(0)_{\#}\mu = \delta_0 \neq \nu$.

2.3.2 Kantorovich's formulation

To overcome the difficulties in the previous examples, Kantorovich proposed an alternative formulation of the optimal transport problem.

DEFINITION 5. *We define the set of* ***transport plan*** $\mathrm{ADM}(\mu,\nu)$ *to be the set of Borel Probability measures* γ *on* $X\times Y$, *i.e.* $\gamma\in\mathscr{P}(X\times Y)$ *such that*

$$\gamma(A\times Y)=\mu(A)\quad\forall A\in\mathscr{B}(X),\qquad \gamma(X\times B)=\nu(B)\quad\forall B\in\mathscr{B}(Y)$$

Equivalently: $\pi^X_{\#}\gamma=\mu, \pi^Y_{\#}\gamma=\nu$, *where* π^X,π^Y *are natural projection from* $X\times Y$ *onto* X *and* Y *respectively.*

Kantorovich's Problem We minimize

$$\gamma\mapsto\int_{X\times Y} c(x,y)\;d\gamma(x,y)$$

in $\mathrm{ADM}(\mu,\nu)$ where $c(x,y)$ is the cost function.

Remark: We shall see in later lectures how Kantorovich's formulation puts the optimal transport problem in the framework of calculus of variation.

3 Constraints and Variational Problems

3.1 Introducing Constraints to Variational Problems

Let us begin by look at the following Yamabe type problem: Assume $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Choose $p > 2$ and if $n \geq 3$ then assume p also satisfies the condition $p < \frac{2n}{n-2}$. Fix $\lambda \in \mathbb{R}$ and consider the problem

$$\begin{cases} -\Delta u + \lambda u = |u|^{p-2}u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u \equiv 0 & \text{on } \partial\Omega \end{cases} . \tag{5}$$

We shall begin our studies of problem (5) similar to how we did for the p-Laplacian Dirichlet problem.
Claim: $-\Delta u + \lambda u = |u|^{p-2}u$ is the Euler-Lagrange equation of the following functional

$$E(u) = \int_\Omega \frac{1}{2}(|\nabla u(x)|^2 + \lambda |u(x)|^2) - \frac{1}{p}|u(x)|^p \, dx$$

where the Lagrangian is given by

$$L(x, y, z) = \frac{1}{2}(|y|^2 + \lambda z^2) - \frac{1}{p}|z|^p.$$

Indeed, we have

$$\begin{aligned} L_z(x, Du, u) - \sum_{i=1}^n (L_{y_i}(x, Du, u))_{x_i} &= \lambda u - |u|^{p-1}\operatorname{sgn}(u) - \nabla \cdot \left(|\nabla u| \frac{\nabla u}{|\nabla u|}\right) \\ &= \Delta u + \lambda u - |u|^{p-2}u. \end{aligned}$$

Hence we have a found ourselves a functional in which we hope we can find it's minimizer and use it to show that problem (5) has a weak solution.
Note: However, the functional we have here is fundamentally different from the functional given in problem (2). Here E is not coercive over $H_0^1(\Omega)$! Let $u \in H_0^1(\Omega)$ be a smooth bump function about a ball $B(x, r) \subset \Omega$ such that $u \equiv 1$ on $B(x, r)$ and vanish outside of Ω. Now, consider $v_n = \alpha_n u$ where $\alpha_n > 0$ such that $\alpha_n \to \infty$ as $n \to \infty$. Hence $\| v_n \|_{H_0^1(\Omega)} \to \infty$, but

$$\begin{aligned} \lim_{n\to\infty} E(v_n) &= \lim_{n\to\infty} \int_\Omega \frac{\alpha_n^2}{2}(|\nabla u(x)|^2 + \lambda |u(x)|^2) - \frac{\alpha_n^p}{p}|u(x)|^p \, dx \\ &= \lim_{n\to\infty} \alpha_n \int_\Omega \frac{1}{2}(|\nabla u(x)|^2 + \lambda |u(x)|^2) - \frac{\alpha_n^{p-2}}{p}|u(x)|^p \, dx = \quad -\infty \end{aligned}$$

since $p > 2$ which means the integrand will eventually be negative. Due to the lack of coercivity, we can not directly apply Theorem 2 to guarantee E has a minimium in $H_0^1(\Omega)$. However, a solution to problem

(5) can be obtained via solving a constrained minimization problem for the functional

$$J(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda |u|^2)\; dx$$

over $H_0^1(\Omega)$ restricted to the set $X = \{u \in H_0^1(\Omega) : \int_\Omega |u|^p\; dx = 1\}$.

THEOREM 4. *Suppose $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$ are the eigenvalues of the operator $-\Delta$ on $H_0^1(\Omega)$. Then for any $\lambda > -\lambda_1$ there exists a positive solution $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ to (5).*

Proof. Consider the above discussion. Let us begin by showing that J satisfies both coercivity and weakly lower semi-continuity over $X = \{u \in H_0^1(\Omega) : \int_\Omega |u|^p\; dx = 1\}$.

Proof of weakly lower semi-continuity: Apply proposition 2 with $\phi(x) = \chi_\Omega(x)$, then we see that J is indeed weakly lower semi-continuous.

Proof of coercivity: Using the Rayleigh formula

$$\lambda_1 = \inf_{\substack{u \in H_0^1(\Omega)\\ u \not\equiv 0}} \frac{\int_\Omega |\nabla u|^2\; dx}{\int_\Omega |u|^2\; dx}$$

we get that

$$\begin{aligned} J(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\lambda\lambda_1}{\lambda_1}|u|^2\; dx \\ &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\lambda}{\lambda_1}|\nabla u|^2\; dx \\ &= \frac{1}{2}\left(1 + \frac{\lambda}{\lambda_1}\right) \| u \|^2_{H_0^1(\Omega)} \end{aligned}$$

provided $\lambda < 0$ and $J(u) \geq \frac{1}{2}\| \nabla u \|^2_{H_0^1(\Omega)}$ if $\lambda \geq 0$. Thus, this proves coercivity.

However, to apply Theorem 2, we still need to check that X is weakly closed in $H_0^1(\Omega)$.

Proof of Weakly Closedness of X: Applying Rellich-Kondrakov theorem, we see the injection $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ is a compact embedding (in particular, a continuous embedding) for $p < \frac{2n}{n-2}$, if $n \geq 3$. For the case $n = 1, 2$, we can apply Morrey's inequality and show that $H_0^1(\Omega) \hookrightarrow C^{0,\gamma}(\Omega)$ is a continuous embedding. Since X is closed in both $L^p(\Omega)$ and $C^{0,\gamma}(\Omega)$ accordingly with the respective dimensions, then the preimage of the inclusion map shows that indeed X is closed with respect to the weak topology of $H_0^1(\Omega)$. Hence Theorem 2 guarantee the existence of a minimizer for J in X, say u^*.

Since $E(u) = E(|u|)$, we may assume $u^* \geq 0$. Note that J is continuously Frechet-differentiable and

$$\begin{aligned} \frac{d}{d\epsilon}\Big|_{\epsilon=0} J(u + \epsilon v) &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} \frac{1}{2}\int_\Omega |\nabla u + \epsilon \nabla v|^2 + \lambda |u + \epsilon v|^2\; dx \\ \text{“Dominated Conv. Theorem”} \quad &= \int_\Omega \langle \nabla u + \epsilon \nabla v, \nabla v\rangle + \lambda \langle u + \epsilon v, v\rangle\; dx \Big|_{\epsilon=0} \\ &= \int_\Omega \nabla u \nabla v + \lambda u v\; dx \\ &=: \; \langle DJ(u), v\rangle \end{aligned}$$

Moreover, consider the function $G : H_0^1(\Omega) \to \mathbb{R}$ given by

$$G(u) = \int_\Omega |u(x)|^p \; dx - 1$$

where $p \leq \frac{2n}{n-2}$. It is also clear that G is continuously Frechet-differentiable with

$$\langle DG(u), v\rangle := p \int_\Omega |u|^{p-2} uv \; dx.$$

In particular, for any $u \in X$ we have

$$\langle DG(u), u\rangle = p \int_\Omega |u|^p \; dx = p \neq 0.$$

Hence 0 is a regular value the G. Then by the Implicit Function Theorem for Banach Spaces, we have $X = G^{-1}(0)$ is a C^1-submanifold of $H_0^1(\Omega)$. Now applying Lagrange Multipler rule, there exists $\mu \in \mathbb{R}$ such that

$$\langle DJ(u^*) - \mu DG(u^*), v\rangle = \int_\Omega (\nabla u^* \nabla v + \lambda u^* v - \mu u^* |u^*|^{p-2} v) \; dx = 0$$

for all $v \in H_0^1(\Omega)$. Therefore, if we set $v = u^*$, then we have that

$$\begin{aligned} \langle DJ(u^*) - \mu DG(u^*), u^*\rangle &= \int_\Omega (|\nabla u^*|^2 + \lambda |u^*|^2) \; dx - \mu \int_\Omega |u^*|^p \; dx \\ &= 2J(u^*) - \mu = 0. \end{aligned}$$

Then it follows from the inequality of the above coercivity argument that

$$\mu = 2J(u^*) \geq \min\left\{1, 1 + \frac{\lambda}{\lambda_1}\right\} \| u^* \|_{H_0^1(\Omega)} > 0$$

since $u^* \in X$, i.e. $u^* \not\equiv 0$ on Ω.

Rescaling u^* to Obtain Weak Solution: Consider $u = \alpha u^*$, then we see that

$$\int_\Omega \nabla u \nabla v + \lambda uv - u|u|^{p-2} v \; dx = \alpha \int_\Omega (\nabla u^* \nabla v + \lambda u^* v - \alpha^{p-2} u^* |u^*|^{p-2} v) \; dx$$

for all $v \in H_0^1(\Omega)$. Hence it follows that if $\alpha = \mu^{\frac{1}{p-2}}$, then

$$\int_\Omega \nabla u \nabla v + \lambda uv - u|u|^{p-2} v \; dx = \alpha \int_\Omega (\nabla u^* \nabla v + \lambda u^* v - \mu u^* |u^*|^{p-2} v) \; dx = 0$$

for all $v \in H_0^1(\Omega)$. Thus $u = \mu^{\frac{1}{p-2}} u^*$ is a candidate for a weak solution to problem (5). To show that u is indeed a solution to problem (5), we must verify that $u > 0$ in Ω.

Regularity of u: Since $u \in H_0^1(\Omega)$, then by the Sobolev Embedding Theorem, we have that $u \in L^{\frac{2n}{n-2}}(\Omega)$. Now observe

$$\left| u|u|^{p-2} - \lambda u \right| \leq C(1 + |u|^{p-1})$$

for some $C > 0$ where $p-1 > 1$ and $p-1 < \frac{n+2}{n-2}$ whenever $n \geq 3$. Then it follows

$$\left| u|u|^{p-2} - \lambda u \right| \leq C\frac{1+|u|^{p-1}}{1+|u|}(1+|u|)$$

where

$$\begin{aligned}\int_\Omega \left(\frac{1+|u|^{p-1}}{1+|u|}\right)^{n/2} dx &= \int_{\Omega\cap\{x:|u|\geq 1\}} + \int_{\Omega\cap\{x:|u|<1\}} \left(\frac{1+|u|^{p-1}}{1+|u|}\right)^{n/2} dx \\ &\leq 2\int_{\Omega\cap\{x:|u|\geq 1\}} |u|^{n(p-2)/2}\, dx + 2m(\Omega) < \infty\end{aligned}$$

since $u \in L^{\frac{2n}{n-2}}(\Omega)$ and $L^{\frac{n(p-2)}{2}}(\Omega) \subset L^{\frac{2n}{n-2}}(\Omega)$. Thus, we are now in the position to apply Lemma to get $u \in L^q(\Omega)$ for all $q < \infty$. Moreover, it also follows that $u|u|^{p-2} - \lambda u \in L^q(\Omega)$ by the growth bound given above. Thus by the Calderon-Zygmund inequality $u \in W^{2,q}$ which then by Sobolev Embedding we get that $u \in W^{2,q} \cap H_0^1(\Omega) \hookrightarrow C^{1,\alpha}(\overline{\Omega})$. Using Schauder estimate, we conclude $u \in C^2(\Omega)$. Finally, apply strong maximum principle to conclude $u > 0$ in Ω. **Remark**: Schauder Theory and Maximum Principle will be explained in details in subsequential lectures. □

3.2 Inequality Constraints and Weak Sub/Super Solution

DEFINITION 6. *A function $g : X \times \mathbb{R} \to \mathbb{R}$ is called a **Caratheodory function** provided $g(x,u)$ is continuous with respect to x and measurable with respect to u.*

Now, let us look at the problem: Suppose $\Omega \subset \mathbb{R}^n$, and let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function with the property that there exists a measurable function h such that $|g(x,u)| \leq h(R)$ whenever $|u(x)| \leq R$ almost everywhere. Given $f \in H_0^1(\Omega)$, then consider the problem

$$\begin{cases} -\Delta u = g(\cdot, u) & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}. \tag{6}$$

DEFINITION 7. *Assume $u \in H_0^1(\Omega)$, then u is a weak **subsolution (supersolution)** to (6) if $u \leq f$ ($u \geq f$) on $\partial\Omega$ and*

$$\int_\Omega \nabla u \nabla \varphi\, dx - \int_\Omega g(\cdot,u)\varphi\, dx \leq 0 \ (\geq 0) \qquad \textit{for all } \varphi \in C_0^\infty(\Omega), \varphi \geq 0.$$

THEOREM 5. *Assume (6) has weak sub and super solution $\underline{u}, \overline{u} \in H^1(\Omega)$ and assume there exists $C_1, C_2 \in \mathbb{R}$ such that $-\infty < C_1 \leq \underline{u} \leq \overline{u} \leq C_2 < \infty$ holds a.e. in Ω. Then there exists a weak solution $u \in H^1(\Omega)$ to (6), satisfying the condition $\underline{u} \leq u \leq \overline{u}$ a.e. in Ω.*

Proof. Since $f \in H_0^1(\Omega)$, we may assume $f \equiv 0$. Let $G(x,u) = \int_0^u g(x,v)\, dv$ denote a primitive of g. Then it's clear that problem (6) is actually the Euler-Lagrange equations of the functional

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \int_\Omega G(x,u)\, dx.$$

Like problem (5), we need to look at a subspace in which we can apply Theorem 2. Consider E restricted to

$$M = \{u \in H_0^1(\Omega) : \underline{u} \le u \le \overline{u} \text{ almost everywhere}\}.$$

Since $\underline{u}, \overline{u} \in L^\infty$ by assumption, then it follows $M \subset L^\infty$. Let $\overline{u} \le K$. Hence by our conditions on g, we have that

$$\left| \int_0^{u(x)} g(x,v)\ dv \right| \le \int_0^K |g(x,v)|\ dv \le K h(K) =: c$$

for almost every $x \in \Omega$ and $h(K)$ is a constant the depends only on K. Let us now check for weakly closedness of M, coercivity and weakly l.s.c. of E.

Weakly Closedness of M: Observe $H_0^1(\Omega)$ is reflexive (moreover, $H_0^1(\Omega)$ is a Hilbert space). By a general fact in functional analysis: if E is a convex subset of a Hilbert space V such that E is closed with respect to norm of V, then E is weakly closed. Therefore, it's sufficient to prove that M is closed and convex. First, observe M is convex since

$$t\underline{u} \le tu \le t\overline{u} \quad \text{and} \quad (1-t)\underline{u} \le v(1-t) \le (1-t)\overline{u} \quad \Rightarrow \quad \underline{u} \le ut + v(1-t) \le \overline{u}.$$

Next, let us prove that M is closed with respect to the $H_0^1(\Omega)$ norm. Take $\{u_n\} \subset M$ such that $u_n \to u$ in $H_0^1(\Omega)$. Since $|u_n| \le \max\{|C_1|, |C_2|\}$ a.e. on Ω, which has finite measure, then it's clear that $\{u_n\}$ is dominated by a L^2 function. Thus there exists $u_{n_k} \to u$ a.e such that $u(x) = \lim_{k\to\infty} u_{n_k}(x) \le \overline{u}(x)$ a.e. and likewise $u(x) \ge \underline{u}(x)$ for almost every x. Therefore, M is closed with respect to $H_0^1(\Omega)$ which then follows M is weakly closed.

Coercivity of E: Using the boundedness of $G(x, u(x))$, we have that

$$E(u) \ge \frac{1}{2} \| u \|^2_{H_0^1(\Omega)} - c.$$

Hence coercivity holds.

Weakly L.S.C. of E: We shall first show that

$$\int_\Omega G(x, u_m)\ dx \to \int_\Omega G(x,u)\ dx$$

whenever $u_m \rightharpoonup u$ in $H_0^1(\Omega)$. Then we can pass it to a subsequence, so WLOG assume $u_m \to u$ pointwise almost everywhere. Since $|G(x, u_m(x))| \le c$, then we can apply Lebesgue Dominated Convergence Theorem to get the desired results. Combining with proposition 2 where $\phi(x) = \chi_\Omega(x)$, then we see that E is indeed weakly lower semi-continuous. Hence by Theorem 2 E has a minimizer in M. We shall leave the verification of that u is a solution for problem (6) to the reader or consult [Struwe] pp. 17-18. □

4 Appendix

4.1 Functional Analysis

THEOREM 6 (Eberlin-Smulian)**.** *Let X be a subset of a Banach Space $(V, \|\cdot\|_V)$. Then the following statements are equivalent:*

(i) X is weakly sequentially compact;

(ii) Every infinite subset of X has a weak limit point in V;

(iii) Closure of X is weakly compact.

THEOREM 7 (Banach-Alaoglu)**.** *The closed unit ball of the dual space of V, V^* is compact in V^{**}.*

THEOREM 8. *A Banach space V is reflexive if and only if its closed unit ball is weakly compact.*

PROPOSITION 2. *Define for any $k > 1$*

$$F(u) = \int_{\mathbb{R}} |u(x)|^k \phi(x)\ dx,$$

for some $\phi \in L^\infty(\mathbb{R}^n)$ with $\phi \geq 0$. Then the following holds:

(i) F is convex.

(ii) F is continuous for the L^k norm.

(iii) F is sequentially lower semi-continuous for any L^p norm.

(iv) If u_n converges weakly to u in L^p then F is sequentially weakly lower semi-continuous.

Proof. (i) First, consider the function $\psi(t) = t^k$ for $k > 1$ and $t \geq 0$. Observe that $\psi(t)$ is $C^\infty((0,\infty))$ and $\psi''(t) = k(k-1)t^{k-2} > 0$ on $(0,\infty)$. Thus, $\psi(t)$ is a convex function on $(0,\infty)$. In particular, we have that

$$|\lambda|u(x)| + (1-\lambda)|v(x)||^k \leq \lambda|u(x)|^k + (1-\lambda)|v(x)|^k$$

for $\lambda \in (0,1)$. Then it follows that

$$\begin{aligned} F(\lambda u + (1-\lambda)v) &= \int_{\mathbb{R}} |\lambda u(x) + (1-\lambda)v(x)|^k \phi(x)\ dx \\ &\leq \int_{\mathbb{R}} \lambda|u(x)|^k \phi(x) + (1-\lambda)|v(x)|^k \phi(x)\ dx \\ &= \lambda \int_{\mathbb{R}} |u(x)|^k \phi(x)\ dx + (1-\lambda)\int_{\mathbb{R}} |v(x)|^k \phi(x)\ dx \\ &= \lambda F(u) + (1-\lambda)F(v). \end{aligned}$$

Therefore, indeed F is a convex functional (Remark: convexity is trivial if everything blows up).

(ii) Let us begin by proving the following inequality

$$|x^k - y^k| \leq k|x-y|(x^{k-1} + y^{k-1})$$

for $k > 1$ and $x, y \geq 0$. The inequality holds trivially in the case where either $x = 0$ or $y = 0$. Therefore, assume neither x nor y are zero. Consider the function $\phi(t) = k(1-t)(1+t^{k-1}) + t^k - 1$ on $[0,1]$ and observe that $\phi(0) = k - 1 > 0$.

Consider the case $k \geq 2$ and observe that

$$\phi'(t) = k(k-1)(1-t)t^{k-2} - k$$

for $t \in (0, \infty)$. Now, consider the following function

$$\psi(t) = \frac{1}{t^{k-2}(1-t)}$$

on $(0,1)$. Observe

$$\psi'(t) = \frac{(k-1)t - (k-2)}{t^{k-1}(1-t)^2}$$

on $(0,1)$. Notice, that $\psi(t)$ as a critical point at $t_0 = \frac{k-2}{k-1}$ in $(0,1)$ which in this case is a minimum for $\psi(t)$. Hence it follows that

$$\psi(t) \geq \psi(t_0) = k - 1 \quad \text{equivalently} \quad (k-1)(1-t)t^{k-2} \leq 1$$

for $t \in (0,1)$. Thus it follows that $\phi'(t) \leq 0$ on $(0,1)$. Hence $\phi(1) \leq \phi(t)$ for all $t \in [0,1]$, i.e. $\phi(t) \geq 0$ on $[0,1]$.

Now consider the case $1 < k < 2$, then we have that

$$\phi''(t) = \frac{-k(k-1)((k-1)t + (2-k))}{t^{3-k}} \leq 0$$

on $(0,1)$. This means that $\phi(t)$ is a concave function which means

$$\phi(t) \geq (1-t)\phi(0) + t\phi(1) = (1-t)(k-1) \geq 0$$

for all $t \in (0,1)$. Hence we have established the fact that $\phi(t) \geq 0$ on $[0,1]$ for all $k > 1$.

Since $y > x > 0$ where $\dfrac{x}{y} \in (0,1)$, then it follows by plugging $\dfrac{x}{y}$ in to $\phi(t)$ that

$$k\,(x-y)\left(y^{k-1} + x^{k-1}\right) \leq x^k - y^k \leq y^k - x^k \leq k\,(y-x)\left(y^{k-1} + x^{k-1}\right)$$

or

$$|x^k - y^k| \leq k|x-y|(x^{k-1} + y^{k-1}).$$

With the above inequality, we are now ready to prove the continuity of F. Observe, if k' is the conjugate of k, $R = \max\{\| u \|_k, \| v \|_k\}$ and $\phi(x) \le M$, then it follows that

$$\begin{aligned}
|F(u) - F(v)| &= \left| \int_{\mathbb{R}} (|u(x)|^k - |v(x)|^k)\phi(x)\ dx \right| \\
&\le M \int_{\mathbb{R}} \left| |u(x)|^k - |v(x)|^k \right| \ dx \\
&\le kM \int_{\mathbb{R}} \left| |u(x)| - |v(x)| \right| (|u(x)|^{k-1} + |v(x)|^{k-1})\ dx \\
&\le kM \int_{\mathbb{R}} |u(x) - v(x)| (|u(x)|^{k-1} + |v(x)|^{k-1})\ dx \\
&\le kM \| u - v \|_k \| \, |u|^{k'} + |v|^{k-1} \, \|_{k'} \\
&\le kM \| u - v \|_k (\| u^{k-1} \|_{k'} + \| v^{k-1} \|_{k'}) \\
&= kM \| u - v \|_k (\| u \|_k^{k-1} + \| v \|_k^{k-1}) \\
&\le 2kMR^{k-1} \| u - v \|_k.
\end{aligned}$$

This shows that F is continuous with respect to the L^k-norm.

(iii) Consider sequence $\{u_n\}$ in L^p such that $u_n \to u$ in L^p. Let

$$\alpha = \liminf_{n \to \infty} F(u_n),$$

then we know that there exists a subsequence $\{F(u_{\sigma(n)})\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} F(u_{\sigma(n)}) = \alpha$. However, since $u_n \to u$ in L^p then we also have that $u_{\sigma(n)} \to u$. Hence it follows that there exists a sub-subsequence $\{u_{\tau \circ \sigma(n)}\}$ such that $u_{\tau\circ\sigma(n)} \to u$ a. e. Now, apply Fatou's lemma, we get that

$$\begin{aligned}
\int_{\mathbb{R}} |u(x)|^k \phi(x)\ dx &\le \liminf_{n\to\infty} \int_{\mathbb{R}} |u_{\tau\circ\sigma(n)}(x)|^k \phi(x)\ dx \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} |u_{\tau\circ\sigma(n)}(x)|^k \phi(x)\ dx \\
&= \lim_{n\to\infty} F(u_{\tau\circ\sigma(n)}) \\
&= \lim_{n\to\infty} F(u_{\sigma(n)}) = \alpha.
\end{aligned}$$

Hence we have that $F(u) \le \liminf_{n\to\infty} F(u_n)$, i.e. F is lower semi-continuous with respect to any L^p-norm.

(iv) Since $1 < p < \infty$, then $(L^p(\mathbb{R}^n))^* = L^q(\mathbb{R}^n)$ is separable by Theorem 7.4.11 in Royden-Fitzpatrick. Now, applying the fact that F is convex from (a) and F is lower semi-continuous with respect to L^p-norm for $1 < p < \infty$, from (b) then it follows that F is lower semi-continuous with respect to the weak-L^p topology, i.e., if $u_n \rightharpoonup u$ in L^p, then

$$F(u) \le \liminf_{n\to\infty} F(u_n).$$

Thus, we have our desired conclusion.

□

4.2 Partial Differential Equations

THEOREM 9 (Poincare's inequality). *Assume Ω is a bounded, open subset of $\mathbb{R}^n$. Suppose $u \in W_0^{1,p}(U)$ for some $1 \leq p < n$. Then we have the estimate*

$$\| u \|_{L^q(\Omega)} \leq C \| Du \|_{L^p(\Omega)}$$

for each $q \in [1, \frac{np}{n-p}]$, the constant C depending only on p, q, n and U. In particular, for all $1 \leq p \leq \infty$,

$$\| u \|_{L^p(\Omega)} \leq C \| Du \|_{L^p(\Omega)}.$$

COROLLARY 1. *Assume Ω is a bounded, open subset of $\mathbb{R}^n$, then $\| \cdot \|_{W_0^{1,p}(\Omega)}$ is equivalent to $\| \cdot \|_{L^p(\Omega)}$. In particular, we could replace $\| \cdot \|_{W_0^{1,p}(\Omega)}$ with*

$$\| u \| = \left(\int_\Omega |\nabla u|^p \, dx \right)^{1/p}$$

as the norm for $W_0^{1,p}(\Omega)$.

DEFINITION 8 (Norm on $(W_0^{1,p}(\Omega))^*$).

If $f \in (W_0^{1,p}(\Omega))^*$, then the norm is defined by

$$\| f \|_{(W_0^{1,p}(\Omega))^*} = \sup \left\{ \langle f, u \rangle : u \in W_0^{1,p}(\Omega), \| u \|_{W_0^{1,p}(\Omega)} \leq 1 \right\}.$$

LEMMA 1.