The Yamabe Problem, Spring 2013

University of Maryland, Department of Mathematics course 748F

HW 3 (19 February, 2013):

Please do not submit this.

Eigenfunctions of S^n

The goal of this homework is to prove several basic facts on eigenvalues and eigenfunctions of the Laplacian Δ on S^n equipped with the standard round metric.
We briefly recall a few facts about eigenfunctions of Δ. Consider the embedding $S^n = \{(x_1, \ldots, x_{n+1}) : \sum x_i^2 = 1\} \subset \mathbb{R}^{n+1}$. In particular, we will show that the first non-trivial eigenspace Λ_1 is spanned by $\{\phi_k := x_k|_{S^n}\}_{k=1}^{n+1}$ and

$$\Delta \phi_k = -n\phi_k. \tag{0}$$

1. Verify that

$$\Delta_{\mathbb{R}^{n+1}} = \frac{1}{r^n\sqrt{\det g_{S^n}}}\partial_i \circ g^{ij} r^n \sqrt{\det g_{S^n}}\partial_j = r^{-2}\Delta_{S^n} + \frac{n}{r}\partial_r + \partial_r^2. \tag{1}$$

To this end, use that the Euclidean metric on $\mathbb{R}^{n+1}$ can be written as a warped product $\sum dx_i^2 = dr^2 + r^2 g_{S^n}$ and the formula for the Laplacian from a previous homework.
Check that (0) follows.

2. First, given a function f on S^n define a function $\tilde{f}(x) := f(x/r)$ (where $r = |x|$) on $\mathbb{R}^{n+1} \setminus \{0\}$. From (1) then $\Delta_{S^n} f = \Delta_{\mathbb{R}^{n+1}} \tilde{f}\big|_{S^n}$. More generally, if F is positively-homogeneous of degree p on $\mathbb{R}^{n+1} \setminus \{0\}$ (i.e., $F(tx) = t^p F(x)$ for all $t > 0$) then show that $\Delta_{\mathbb{R}^{n+1}}(r^{-p}F) = -p(p+n-1)r^{-p-2}F + r^{-p}\Delta_{\mathbb{R}^{n+1}}F$. Thus, conclude that if F is harmonic it follows that f is an eigenfunction of Δ_{S^n} with eigenvalue $-p(p+n-1)$.

3. We already knew this with $p = 1$, but next we will show that all eigenfunctions arise in this way. Therefore, it follows that $-n$ must be the first non-trivial eigenvalue as well as that $\{\phi_i\}_{i=1}^{n+1} = \Lambda_1$.
(i) Indeed, check that if ϕ is an eigenfunction with eigenvalue λ then $f(x) := r^p\phi(x/r)$ is harmonic on $\mathbb{R}^{n+1} \setminus \{0\}$, where $p > 0$ is such that $-p(p+n-1) = \lambda$.
(ii) Then, the Fourier transform of f is then a tempered distribution supported at the origin (since $|\xi|^2\hat{f}(\xi) = 0$), which must then be a partial derivative of the Dirac delta function: this requires some work, you might want to look it up. Therefore, its Fourier transform must be a polynomial.

4.
As an aside, the dimension of the vector space $H^0(\mathbb{C}^{n+1}, O(d))$ of homogeneous polynomials of degree d in $n+1$ variables is $\binom{n+d}{d}$ (Griffiths–Harris, Principles of Algebraic Geometry, p. 166). Prove that $\Delta_{\mathbb{R}^{n+1}} : H^0(\mathbb{C}^{n+1}, O(d)) \to H^0(\mathbb{C}^{n+1}, O(d-2))$ is surjective.
(Hint: 1. To see this, consider the Hermitian inner product $(f, g) := \bar{f}(D)(g)(0)$, where $D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdot \partial x_{n+1}^{\alpha_{n+1}}}$. and show it is a positive inner product (check on monomials, which form an orthogonal basis). 2. $(\Delta f, g) = (f, r^2 g)$.)
In conclusion, $\dim\big(\ker \Delta_{\mathbb{R}^{n+1}} \cap H^0(\mathbb{C}^{n+1}, O(d))\big) = \binom{n+d}{n} - \binom{n+d-2}{n}$, which is also the dimension of the d-th eigenspace. When $d = 1$ we indeed obtain $n+1$.

5. Denote by $\mathrm{conf}(S^n, [g_{S^n}])$ the Lie algebra of conformal vector fields associated to the standard conformal class on the sphere. Show that $\mathrm{conf}(S^n, [g_{S^n}])$ is isomorphic as a complex vector space to $\Lambda_1 \oplus \sqrt{-1}\Lambda_1$.

6. Theorem (Lelong-Ferrand, Obata): Denote by $\mathrm{Conf}_0(M, [g])$ the identity component of the conformal diffeomorphism group associated to $(M, [g])$. Then $\mathrm{Conf}(M, [g])$ is compact, unless $(M, [g]) = (S^n, [g_{S^n}])$.

References: M. Obata, The conjectures on conformal transformations of Riemannian manifolds. J. Differential Geometry 6 (1971/72), 247–258.
J. Lafontaine, The theorem of Lelong-Ferrand and Obata, in: Conformal geometry (Bonn, 1985/1986), 93–103, Aspects Math., E12, Vieweg, Braunschweig, 1988.