Kähler manifolds, Autumn 2014

University of Maryland, Department of Mathematics course 868K

HW 1: (due by September 11.)

1. (Octonion fun)
(a) Show octonion multiplication is not associative.
(b) Show that, nevertheless, the following weaker form of associativity holds: The *associator of* $a, b, c \in \mathbb{O}$

$$[a, b, c] := (ab)c - a(bc),$$

vanishes when two of a, b, c are equal. Equivalently, the associator is alternating.
(c) Show that octonionic mutliplication induces an almost complex structure on the unit imaginary octonions (hint: use (b)).

2. (S^4 is not almost complex)
Hirzebruch signature theorem implies that for a 4-manifold the signature is given by the first Pontryagin class: $\sigma(M) = p_1(M).[M]/3$ (this can also be proved used index theory).
This can be used to show S^4 is not almost complex (reference: D. Auroux, notes to MIT course 966, 2007, lecture 12).
Fill in the details carefully.

3. (Riemann sphere)
Check the claim made in class that the cross product coincides with the standard complex structure on $\hat{\mathbb{C}}$ (the sphere S^2 endowed with the charts $(\mathbb{C}, z)$ and $(\mathbb{C}, w)$ with transition $w = 1/z$ on $\mathbb{C}^\star$).

4. (Eigenvalues of J)
Show that an almost complex manifold is even dimensional.

5. (symplectic and J)
Let (M, ω) be symplectic. Show there exists an almost complex structure J satisfying $\omega(JX, JY) = \omega(X, Y)$ for all X, Y.

6. (symplectic vs. J)
Let M be a $2n$-dimensional manifold. Show M admits an almost complex structure iff it admits a nondegenerate 2-form (i.e., a form α such that α^n is nowhere zero).