Kähler manifolds, Autumn 2014

University of Maryland, Department of Mathematics course 868K

HW 2: (due by September 23.)

1. (Pluriharmonic)
   (a) \( \partial \bar{\partial} f = 0 \) iff \( f = h + a \), where \( h \) is a holomorphic function, and \( a \) is an anti-holomorphic function.
   (b) If \( X \) is compact and \( f \) is holomorphic on \( X \), then \( f \) is constant.
   (c) Prove or disprove: Let \( f \in C^\infty(\mathbb{P}^1 \times \mathbb{P}^1) \). Then \( (\sqrt{-1} \partial \bar{\partial} f)^2 = 0 \) iff \( f \) is constant.

2. Prove the following facts used in the proof of the \( \partial \bar{\partial} \)-lemma:
   (a) \( G_\theta = G_{\bar{\theta}} \) commutes with \( \partial, \bar{\partial}, \Delta = \Delta_{\bar{\theta}} \) as well as with \( \partial^* \) and \( \bar{\partial}^* \).
   (b) Review the proof that \( [\Lambda, \partial] = \sqrt{-1} \bar{\partial}^* \) (see Griffiths–Harris page 111 and notation therein).
   (c) Show that \( \partial \) and \( \bar{\partial}^* \) anti-commute.

3. Show that \( S^1 \times S^3 \) admits a complex structure. Why does it not admit a Kähler structure?

4. (Holonomy and submanifolds) (a) Let \( (X, g) \) be Kahler. Let \( D \) be a submanifold. Then \( Hol(D, g|_D) \) being induced by \( Hol(X, g) \) consists of unitary matrices. Thus \( (D, g|_D) \) is Kahler, too. What’s false in this argument?
   (b) Give a holonomy proof of: if \( D \) is complex submanifold then \( (D, g|_D) \) is Kahler.
   (c) Give at least one other proof of (b).

5. (Tangent bundle)
   Let \( X \) be Kahler. Is \( TX \) Kahler?

6. (symplectic forms form a cone?)
   Prove or disprove: Let \( \omega_1, \omega_2 \) be two symplectic forms. Then \( a\omega_1 + b\omega_2 \) is symplectic for any \( a, b \in \mathbb{R}^+ \).

7. Some prefer to use the real operators \( d \) and \( d^c := \sqrt{-1} (\bar{\partial} - \partial) \). Show that these are indeed real operators (i.e., \( \overline{Af} = Af \)). Show that \( d \circ d^c = \sqrt{-1} \partial \circ \bar{\partial} \).