Kähler manifolds, Autumn 2014

University of Maryland, Department of Mathematics course 868K

HW 2: (due by September 23.)

1. (Pluriharmonic)
(a) $\partial\bar\partial f = 0$ iff $f = h + a$, where h is a holomorphic function, and a is an anti-holomorphic function.
(b) If X is compact and f is holomorphic on X, then f is constant.
(c) Prove or disprove: Let $f \in C^\infty(\mathbb{P}^1 \times \mathbb{P}^1)$. Then $(\sqrt{-1}\partial\bar\partial f)^2 = 0$ iff f is constant.

2. Prove the following facts used in the proof of the $\partial\bar\partial$-lemma:
(a) $G_\partial = G_{\bar\partial}$ commutes with $\partial, \bar\partial, \Delta_\partial = \Delta_{\bar\partial}$ as well as with $\partial^\star$ and $\bar\partial^\star$.
(b) Review the proof that $[\Lambda, \partial] = \sqrt{-1}\bar\partial^\star$ (see Griffiths–Harris page 111 and notation therein).
(c) Show that ∂ and $\bar\partial^\star$ anti-commute.

3. Show that $S^1 \times S^3$ admits a complex structure. Why does it not admit a Kähler structure?

4. (Holonomy and submanifolds) (a) Let (X, g) be Kahler. Let D be a submanifold. Then $Hol(D, g|_D)$ being induced by $Hol(X, g)$ consists of unitary matrices. Thus $(D, g|_D)$ is Kahler, too. What's false in this argument?
(b) Give a holonomy proof of: if D is complex submanifold then $(D, g|_D)$ is Kahler.
(c) Give at least one other proof of (b).

5. (Tangent bundle)
Let X be Kahler. Is TX Kahler?

6. (symplectic forms form a cone?)
Prove or disprove: Let ω_1, ω_2 be two symplectic forms. Then $a\omega_1 + b\omega_2$ is symplectic for any $a, b \in \mathbb{R}_+$.

7. Some prefer to use the real operators d and $d^c := \frac{\sqrt{-1}}{2}(\bar\partial - \partial)$. Show that these are indeed real operators (i.e., $\overline{Af} = A\overline{f}$). Show that $d \circ d^c = \sqrt{-1}\partial \circ \bar\partial$.