HW 3: (due by October 2.)

1. (The classical Hopf fibration)
   (a) Prove Propositions 1 and 2 on page 13 of the second edition of Petersen’s “Riemannian geometry” textbook.
   (b) Prove that the standard metric on $S^m$ can be written as a rotationally symmetric metric $dr^2 + \sin r g_{S^{m-1}}(x)$ where $(r, x) \in (0, \pi) \times S^{m-1}$. Here, $g_{S^{m-1}}$ is the standard metric induced on the unit ball in $\mathbb{R}^n$ from the Euclidean metric.
   (c) Let $p \in \{1, \ldots, n - 2\}$. Similarly, show that the standard metric on $S^m$ can be written as a doubly warped product metric $dr^2 + \sin r g_{S^{m-1-p}}(x) + \cos r g_{S^p}(y)$ where $(r, x, y) \in (0, \pi/2) \times S^{m-1-p} \times S^p$.
   (d) Consider the case $m = 2n + 1, p = 1$. Let $S^1$ act diagonally on $\mathbb{C}^n \times \mathbb{C}$ (and trivially on the $(0, \pi/2)$ factor), and consider the induced action on $(0, \pi/2) \times S^{2n-1} \times S^1$ given by $\lambda.(r, z, w) = (r, \lambda z, \lambda w)$. Show that this action is an isometry w.r.t. to the standard metrics, and induces a smooth quotient space, in fact isometric to $(0, \pi/2) \times S^{2n-1}(\pi)$ (with what radius on the second factor?). The fibers of the projection map $\pi$ (restricted to the product of the spheres upstairs) are called Hopf fibers.
   (e) Denote by $g$ the metric on $S^{2n-1}$ for which the quotient map $\pi$ becomes a Riemannian submersion. Let $V = \ker d\pi$ be the distribution tangent to the Hopf fibers (upstairs). Decompose $g$ into a sum of $v$ and $h$, where $g = v + h$ and $v := \pi^* g_{S^{2n-1}}$. I.e., $v$ is a metric on $V$, and $h$ is a metric on its orthogonal complement w.r.t. $g_{S^{2n-1}}$. Show that the metric on the submersion $(0, \pi/2) \times S^{2n-1}$ (making this a Riemannian submersion) is given by $dr^2 + \sin^2(t)h + \sin^2(t)\cos^2(t)v$.
   (f) Prove that the quotient space with this doubly warped product metric is actually complex projective space equipped with a factor times the Fubini–Study metric (what factor?). Here, we take as a definition the Fubini–Study metric to be given as the unique metric determined by the Kähler form defined in class and the complex structure induced on our quotient by considering $S^{2n-1}$ as sitting in $\mathbb{C}^n$ (and pushing forward this complex structure under $\pi$).