Kähler manifolds, Autumn 2014

University of Maryland, Department of Mathematics course 868K

HW 3: (due by October 2.)

1. (The classical Hopf fibration)

(a) Prove Propositions 1 and 2 on page 13 of the second edition of Petersen's "Riemannian geometry" textbook.

(b) Prove that the standard metric on S^m can be written as a rotationally symmetric metric $dr^2 + \sin r g_{S^{m-1}}(x)$ where $(r, x) \in (0, \pi) \times S^{m-1}$. Here, $g_{S^{m-1}}$ is the standard metric induced on the unit ball in $\mathbb{R}^n$ from the Euclidean metric.

(c) Let $p \in \{1, \dots, n-2\}$. Similarly, show that the standard metric on S^m can be written as a doubly warped product metric $dr^2 + \sin r g_{S^{m-1-p}}(x) + \cos r g_{S^p}(y)$ where $(r, x, y) \in (0, \pi/2) \times S^{m-1-p} \times S^p$.

(d) Consider the case $m = 2n+1, p = 1$. Let S^1 act diagonally on $\mathbb{C}^n \times \mathbb{C}$ (and trivially on the $(0, \pi/2)$ factor), and consider the induced action on $(0, \pi/2) \times S^{2n-1} \times S^1$ given by $\lambda.(r, z, w) = (r, \lambda z, \lambda w)$. Show that this action is an isometry w.r.t. to the standard metrics, and induces a smooth quotient space, in fact isometric to $(0, \pi/2) \times S^{2n-1}$ (with what radius on the second factor?). The fibers of the projection map π (restricted to the product of the spheres upstairs) are called Hopf fibers.

(e) Denote by g the metric on S^{2n-1} for which the quotient map π becomes a Riemannian submersion. Let $V = \ker d\pi$ be the distribution tangent to the Hopf fibers (upstairs). Decompose g into a sum of v and h, where $g = v + h$ and $v := \pi^\star g_{S^{2n-1}}$. I.e., v is a metric on V, and h is a metric on its orthogonal complement w.r.t. $g_{S^{2n-1}}$.

Show that the metric on the submersion $(0, \pi/2) \times S^{2n-1}$ (making this a Riemannian submersion) is given by

$$dr^2 + \sin^2(t)h + \sin^2(t)\cos^2(t)v.$$

(f) Prove that the quotient space with this doubly warped product metric is actually complex projective space equipped with a factor times the Fubini–Study metric (what factor?). Here, we take as a definition the Fubini–Study metric to be given as the unique metric determined by the Kähler form defined in class and the complex structure induced on our quotient by considering S^{2n-1} as sitting in $\mathbb{C}^n$ (and pushing forward this complex structure under π).