THE 1D SCHRÖDINGER EQUATION WITH A SPACETIME WHITE NOISE: THE AVERAGE WAVE FUNCTION

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Abstract. For the 1D Schrödinger equation with a mollified spacetime white noise, we show that the average wave function converges to the Schrödinger equation with an effective potential after an appropriate renormalization.

Keywords: random Schrödinger equation, renormalization, path integral.

1. Main result

Consider the Schrödinger equation driven by a weak stationary spacetime Gaussian potential $V(t,x)$:

$$i\partial_t\phi(t,x)+\frac{1}{2}\Delta\phi(t,x)-\sqrt{\varepsilon}V(t,x)\phi(t,x)=0,\quad t>0,x\in\mathbb{R}, \tag{1.1}$$

on the diffusive scale $(t,x)\mapsto(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon})$,

$$\phi_\varepsilon(t,x):=\phi(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}) \tag{1.2}$$

satisfies

$$i\partial_t\phi_\varepsilon(t,x)+\frac{1}{2}\Delta\phi_\varepsilon(t,x)-\frac{1}{\varepsilon^{3/2}}V(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon})\phi_\varepsilon(t,x)=0. \tag{1.3}$$

With appropriate decorrelating assumptions on V, the rescaled large highly oscillatory potential $\varepsilon^{-3/2}V(t/\varepsilon^2,x/\varepsilon)$ converges in distribution to a spacetime white noise, denoted by $\dot{W}(t,x)$. To the best of our knowledge, the asymptotics of ϕ_ε and making sense of the limit of (1.3), which formally reads

$$i\partial_t\Phi(t,x)+\frac{1}{2}\Delta\Phi(t,x)-\dot{W}(t,x)\Phi(t,x)=0,$$

is an open problem. The goal of this short note is to take a first step by analyzing $\mathbb{E}[\phi_\varepsilon]$ as $\varepsilon\to0$.

1.1. **Assumptions on the randomness.**

We assume the spacetime white noise $\dot{W}(t,x)$ is built on the probability space $(\Omega,\mathcal{F},\mathbb{P})$, and

$$V(t,x)=\int_{\mathbb{R}^2}\varrho(t-s,x-y)\dot{W}(s,y)dyds$$

for some mollifier $\varrho\in\mathcal{C}_c^\infty$ with $\int\varrho=1$. By the scaling property of $\dot{W}$, we have

$$\begin{aligned}\frac{1}{\varepsilon^{3/2}}V(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon})=&\frac{1}{\varepsilon^{3/2}}\int_{\mathbb{R}^2}\varrho(\frac{t}{\varepsilon^2}-s,\frac{x}{\varepsilon}-y)\dot{W}(s,y)dyds\\=&\frac{1}{\varepsilon^{3/2}}\int_{\mathbb{R}^2}\frac{1}{\varepsilon^3}\varrho(\frac{t-s}{\varepsilon^2},\frac{x-y}{\varepsilon})\dot{W}(\frac{s}{\varepsilon^2},\frac{y}{\varepsilon})dyds\\\overset{\text{law}}{=}&\int_{\mathbb{R}^2}\frac{1}{\varepsilon^3}\varrho(\frac{t-s}{\varepsilon^2},\frac{x-y}{\varepsilon})\dot{W}(s,y)dyds,\end{aligned}$$

which converges in distribution to $\dot{W}$ independent of the choice of ϱ. For simplicity, we choose

$$\varrho(t,x) = \frac{\eta(t)}{\sqrt{\pi}} e^{-x^2},$$

with $\eta \in \mathcal{C}_c^\infty(\mathbb{R})$ and $\int \eta = 1$. The covariance function of V is

$$R(t,x) = \mathbb{E}[V(t,x)V(0,0)] = \int_{\mathbb{R}^2} \varrho(t+s,x+y)\varrho(s,y)dyds = R_\eta(t)q(x), \tag{1.4}$$

with

$$R_\eta(t) := \int_{\mathbb{R}} \eta(t+s)\eta(s)ds, \quad q(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}. \tag{1.5}$$

We define $\widetilde{R}(\omega,\xi)$ as the Fourier transform of R in (t,x):

$$\widetilde{R}(\omega,\xi) = \int_{\mathbb{R}^2} R(t,x)e^{-i\omega t - i\xi x} dtdx.$$

We use $\widehat{f}$ to denote the Fourier transform of f in the x variable:

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-i\xi x}dx.$$

1.2. **Main result.** Assuming the initial data $\phi_\varepsilon(0,x) = \phi_0(x) \in \mathcal{C}_c^\infty(\mathbb{R})$, so we have a low frequency wave before rescaling: $\phi(0,x) = \phi_0(\varepsilon x)$. The following is the main result:

Theorem 1.1. *There exists $z_1, z_2 \in \mathbb{C}$ depending on the mollifier ϱ, given by (2.13) and (2.18) respectively, such that for any $t > 0, \xi \in \mathbb{R}$,*

$$\mathbb{E}[\widehat{\phi}_\varepsilon(t,\xi)]e^{\frac{z_1 t}{\varepsilon}} \to \widehat{\phi}_0(\xi)e^{-\frac{i}{2}|\xi|^2 t + z_2 t}, \qquad \text{as } \varepsilon \to 0. \tag{1.6}$$

We make a few remarks.

Remark 1.2. The limit in (1.6) is the solution to

$$i\partial_t \bar{\phi} + \frac{1}{2}\Delta\bar{\phi} - iz_2\bar{\phi} = 0, \quad \bar{\phi}(0,x) = \phi_0(x),$$

written in the Fourier domain:

$$\bar{\phi}(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\phi}_0(\xi)e^{-\frac{i}{2}|\xi|^2 t + z_2 t}e^{i\xi x}d\xi.$$

Remark 1.3. In the parabolic setting, a Wong-Zakai theorem is proved [3, 11, 13, 14] for

$$\partial_t u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + \frac{1}{\varepsilon^{3/2}}V(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon})u_\varepsilon, \quad u(0,x) = u_0(x).$$

The result says that there exists $c_1, c_2 > 0$ depending on ϱ such that

$$u_\varepsilon(t,x)e^{-\frac{c_1 t}{\varepsilon} - c_2 t} \Rightarrow \mathcal{U}(t,x) \quad \text{ in distribution,} \tag{1.7}$$

where $\mathcal{U}$ solves the stochastic heat equation with a multiplicative spacetime white noise

$$\partial_t \mathcal{U}(t,x) = \frac{1}{2}\Delta\mathcal{U}(t,x) + \mathcal{U}(t,x)\dot{W}(t,x), \quad \mathcal{U}(0,x) = u_0(x),$$

with the product $\mathcal{U}(t,x)\dot{W}(t,x)$ interpreted in the Itô's sense. Writing the above equation in the mild formulation, it is easy to see that $\mathbb{E}[\mathcal{U}]$ solves the unperturbed heat equation

$$\mathbb{E}[\mathcal{U}(t,x)] = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{|x-y|^2}{2t}} u_0(y)dy,$$

thus, a consequence of (1.7) is

$$\mathbb{E}[\widehat{u}_\varepsilon(t,\xi)]e^{-\frac{c_1 t}{\varepsilon}-c_2 t} \to \widehat{u}_0(\xi)e^{-\frac{1}{2}|\xi|^2 t},$$

which should be compared to (1.6) in the Schrödinger setting, with $-c_1, c_2$ corresponding to z_1, z_2.

Remark 1.4. Starting from the microscopic dynamics (1.1), if we consider a time scale that is shorter than the one in (1.2), and a low frequency initial data

$$(t,x) \mapsto (\frac{t}{\varepsilon}, \frac{x}{\sqrt{\varepsilon}}), \quad \phi(0,x) = \phi_0(\sqrt{\varepsilon}x),$$

a homogenization result was proved in [10]: for any $t > 0, \xi \in \mathbb{R}$,

$$\varepsilon^{\frac{d}{2}}\widehat{\phi}(\frac{t}{\varepsilon}, \sqrt{\varepsilon}\xi) \to \widehat{\phi}_0(\xi)e^{-\frac{i}{2}|\xi|^2 t - z_1 t} \tag{1.8}$$

in probability. Here z_1 is the same constant as in Theorem 1.1. If we instead consider a high frequency initial data $\phi(0,x) = \phi_0(x)$, which varies on the same scale as the random media, a kinetic equation was derived on the time scale of $\frac{t}{\varepsilon}$ in [2]:

$$\mathbb{E}[|\widehat{\phi}(\frac{t}{\varepsilon}, \xi)|^2] \to \overline{W}(t,\xi), \tag{1.9}$$

where $\overline{W}(t,\xi) = \int_{\mathbb{R}} W(t,x,\xi)dx$ and W solves the radiative transfer equation

$$\partial_t W(t,x,\xi) + \xi\cdot\nabla_x W(t,x,\xi) = \int_{\mathbb{R}} \widetilde{R}(\frac{|p|^2-|\xi|^2}{2}, p-\xi)(W(t,x,p) - W(t,x,\xi))\frac{dp}{2\pi}. \tag{1.10}$$

For similar results in the case of a spatial randomness, see [4, 7, 18]. The equation (1.10) shows that, in the high frequency regime where the wave and the random media interact fully, the momentum variable follows a jump process with the kernel given by $\widetilde{R}(\frac{|p|^2-|\xi|^2}{2}, p-\xi)$. The real part of the constant z_1, in (1.6) and (1.8), describes the total scattering cross-section, i.e., the jumping rate at the zero frequency:

$$2\mathrm{Re}[z_1] = \int_{\mathbb{R}} \widetilde{R}(\frac{|p|^2}{2}, p)\frac{dp}{2\pi}.$$

Thus, the renormalization in (1.6) can be viewed as a compensation of the exponential attenuation of wave propagation on the time scale of $\frac{t}{\varepsilon^2}$. We emphasize that the average wave function (more precisely, the term $\mathbb{E}[\widehat{\phi}_\varepsilon(t,\xi)]\mathbb{E}[\widehat{\phi}^*_\varepsilon(t,\xi)]$) only captures the ballistic component of wave.

Remark 1.5. It is unclear at this stage what explicit information the convergence in (1.6) implies. On one hand, if we expect the family of random variables $\{\widehat{\phi}_\varepsilon(t,\xi)\}_{\varepsilon\in(0,1)}$ to converge in distribution to some random limit after any possible renormalization and assume the *uniform integrability* of this family of random variables, then our result shows that $e^{z_1 t/\varepsilon}$ is the only possible renormalization factor since the uniform integrability ensures the mean $\mathbb{E}[\widehat{\phi}_\varepsilon(t,\xi)]$ also converges after the same rescaling. On the other hand, without the uniform integrability it is a priori unclear whether the convergence of $\widehat{\phi}_\varepsilon(t,\xi)$ is related to the convergence of its first moment. In addition, based on the discussion in Remark 1.4, we know that the wave field $\widehat{\phi}(t,\xi)$ decays exponentially on the time scale t/ε^2 because $\mathrm{Re}[z_1] > 0$, and the lost energy escapes to high frequency regime through multiple scatterings. From this perspective, the physical meaning is unclear when we multiply the exponentially small solution by $e^{z_1 t/\varepsilon}$ in (1.6) so that something "nontrivial" can still be observed. In the parabolic setting, the renormalization in (1.7) can be naturally viewed as a shift of the height function by its average growing speed, which is described by the KPZ equation through a Hopf-Cole transformation

$$\log[u_\varepsilon(t,x)e^{-\frac{c_1 t}{\varepsilon}-c_2 t}] = \log u_\varepsilon(t,x) - \frac{c_1 t}{\varepsilon} - c_2 t.$$

For the Schrödinger equation, it is unclear to us what should be the right physical quantity to look at. Another choice is to consider $\widehat{\phi}(t,\xi)$ for $\xi \sim O(1)$ and $t \sim O(\varepsilon^{-\alpha})$ with some $\alpha > 1$. In light of (1.9) and the long time behavior of (1.10) analyzed in [15], we expect some diffusion equation to show up in the limit.

Remark 1.6. The convergences in (1.6), (1.8) and (1.9) hold in all dimensions $d \geqslant 1$. In other words, if we start from the microscopic dynamics (1.1), with a random potential of size $\sqrt{\varepsilon}$, then in all dimensions: (i) on the time scale t/ε, depending on the initial data, we have either (1.8) or (1.9); (ii) on the time scale t/ε^2, if we have a low frequency initial data, then (1.6) holds. The proof and the result does NOT depend on the dimensions. Nevertheless, with a random potential of size $\sqrt{\varepsilon}$, the change of variables $(t,x) \mapsto (\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})$ chosen in (1.3) only leads to a spacetime white noise in $d = 1$.

Remark 1.7. When the spacetime potential $V(t,x)$ is replaced by a spatial potential $V(x)$, similar problems (including nonlinear ones) have been analyzed in [1, 5, 6, 9, 12, 16, 19] in $d = 1, 2, 3$.

2. Proofs

The proof contains two steps. First, we derive a probabilistic representation of the average wave function $\mathbb{E}[\widehat{\phi}(t,\xi)]$ with some auxiliary Brownian motion $\{B_t\}_{t\geqslant 0}$ built on another probability space $(\Sigma, \mathcal{A}, \mathbb{P}_{\mathbf{B}})$. Using this probabilistic representation, we pass to the limit using tools from stochastic analysis. Similar proofs have already appeared in [9, 11].

2.1. Probabilistic representation.

Assuming $\{B_t\}_{t\geqslant 0}$ is a standard Brownian motion starting from the origin, defined on $(\Sigma, \mathcal{A}, \mathbb{P}_{\mathbf{B}})$. We denote the expectation with respect to $\{B_t\}_{t\geqslant 0}$ by $\mathbb{E}_{\mathbf{B}}$.

Lemma 2.1. *For the equation*

$$i\partial_t\psi + \frac{1}{2}\Delta\psi - V(t,x)\psi = 0, \quad t > 0, x \in \mathbb{R}, \tag{2.1}$$

with $\psi(0,x) = \psi_0(x)$*, we have*

$$\mathbb{E}[\widehat{\psi}(t,\xi)] = \widehat{\psi}_0(\xi)\mathbb{E}_{\mathbf{B}}[e^{i\sqrt{i}\xi B_t}e^{-\frac{1}{2}\int_0^t\int_0^t R(s-u,\sqrt{i}(B_s-B_u))dsdu}]. \tag{2.2}$$

On the formal level, (2.2) comes from an application of the Feynman-Kac formula to (2.1) then averaging with respect to V. We write (2.1) as

$$\partial_t\psi = \frac{i}{2}\Delta\psi - iV(t,x)\psi = 0,$$

and assume the following expression:

$$\psi(t,x) = \mathbb{E}_{\mathbf{B}}[\psi_0(x+\sqrt{i}B_t)e^{-i\int_0^t V(t-s,x+\sqrt{i}B_s)ds}].$$

Averaging with respect to V and using the Gaussianity yields

$$\mathbb{E}[\psi(t,x)] = \mathbb{E}_{\mathbf{B}}[\psi_0(x+\sqrt{i}B_t)e^{-\frac{1}{2}\int_0^t\int_0^t R(s-u,\sqrt{i}(B_s-B_u))dsdu}],$$

which, after taking the Fourier transform, gives (2.2).

Proof. We follow the proof of [9, Proposition 2.1], where a similar formula is derived for spatial random potentials. For the convenience of readers, we provide all the details here.

Fix (t,ξ), we define the function

$$F_1(z) := \mathbb{E}_{\mathbf{B}}\big[e^{iz\xi B_t - \frac{1}{2}\int_0^t\int_0^t R(s-u, z(B_s-B_u))dsdu}\big], \quad z \in \bar{D}_0,$$

with $D_0 := \{z \in \mathbb{C} : \mathrm{Re}[z^2] > 0\}$. We also define the corresponding Taylor expansion

$$F_2(z) = \sum_{n=0}^{\infty} F_{2,n}(z), \quad z \in \bar{D}_0,$$

with

$$F_{2,n}(z) := \frac{(-1)^n}{2^n(2\pi)^n n!}\int_{[0,t]^{2n}}\int_{\mathbb{R}^n}\prod_{j=1}^n \widehat{R}(s_j - u_j, p_j)\mathbb{E}_{\mathbf{B}}\left[e^{iz\xi B_t}\prod_{j=1}^n e^{izp_j(B_{s_j}-B_{u_j})}\right]dpdsdu.$$

Recall that $R(t,x) = \frac{R_\eta(t)}{\sqrt{2\pi}}e^{-x^2/2}$. In the definition of F_1, we have extended the definition so that $R(t,z) = \frac{R_\eta(t)}{\sqrt{2\pi}}e^{-z^2/2}$ for all $z \in \mathbb{C}$. We also emphasize that $\widehat{R}(t,p)$ is the Fourier transform of $R(t,x)$ in the x−variable:

$$\widehat{R}(t,p) = R_\eta(t)e^{-\frac{1}{2}p^2}.$$

It is straightforward to check that both F_1 and F_2 are analytic on D_0 and continuous on $\bar{D}_0$. Note that $\sqrt{i} \in \partial D_0$. The goal is to show that

$$\mathbb{E}[\widehat{\psi}(t,\xi)] = \widehat{\psi}_0(\xi)F_1(\sqrt{i}). \tag{2.3}$$

Since $(z,s,u) \mapsto R(s-u, z(B_s - B_u))$ is bounded on $\bar{D}_0 \times \mathbb{R}_+^2$, we have

(2.4)

$$\begin{aligned}
F_1(z) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}\mathbb{E}_{\mathbf{B}}\left[e^{iz\xi B_t}\left(\int_{[0,t]^2} R(s-u, z(B_s-B_u))dsdu\right)^n\right] \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}\mathbb{E}_{\mathbf{B}}\left[e^{iz\xi B_t}\int_{[0,t]^{2n}}\prod_{j=1}^n R(s_j-u_j, z(B_{s_j}-B_{u_j}))dsdu\right] \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n(2\pi)^n n!}\mathbb{E}_{\mathbf{B}}\left[e^{iz\xi B_t}\int_{[0,t]^{2n}}\int_{\mathbb{R}^n}\prod_{j=1}^n \widehat{R}(s_j-u_j, p_j)e^{izp_j(B_{s_j}-B_{u_j})}dpdsdu\right].
\end{aligned}$$

For $z = x \in \mathbb{R}$, we can apply the Fubini theorem to see that $F_1(x) = F_2(x)$. Due to the analyticity and continuity of F_1 and F_2, we therefore have $F_1(z) = F_2(z)$ for all $z \in \bar{D}_0$. Hence, (2.3) is equivalent to

$$\mathbb{E}[\widehat{\psi}(t,\xi)] = \widehat{\psi}_0(\xi)\sum_{n=0}^{\infty}F_{2,n}(\sqrt{i}). \tag{2.5}$$

For a fixed n, we rewrite

$$\begin{aligned}
F_{2,n}(\sqrt{i}) = \frac{(-1)^n}{2^n(2\pi)^n n!}\int_{[0,t]^{2n}}\int_{\mathbb{R}^{2n}}\prod_{j=1}^n &\widehat{R}(s_{2j-1}-s_{2j}, p_{2j-1})\delta(p_{2j-1}+p_{2j}) \\
&\times \mathbb{E}_{\mathbf{B}}\left[e^{i\sqrt{i}\xi B_t}e^{-\sum_{j=1}^{2n} i\sqrt{i}p_j B_{s_j}}\right]dsdp.
\end{aligned}$$

Let σ denote the permutations of $\{1,\ldots,2n\}$. After a relabeling of the p-variables we can write

(2.6)

$$\begin{aligned}
F_{2,n}(\sqrt{i}) = \frac{(-1)^n}{2^n(2\pi)^n n!}\sum_{\sigma}\int_{[0,t]^{2n}_<}\int_{\mathbb{R}^{2n}}\prod_{j=1}^n &\widehat{R}(s_{\sigma(2j-1)}-s_{\sigma(2j)}, p_{\sigma(2j-1)})\delta(p_{\sigma(2j-1)}+p_{\sigma(2j)}) \\
&\times \mathbb{E}_{\mathbf{B}}\left[e^{i\sqrt{i}\xi B_t}e^{-\sum_{j=1}^{2n} i\sqrt{i}p_j B_{s_j}}\right]dsdp,
\end{aligned}$$

where $[0,t]^{2n}_{<} := \{(s_1,\ldots,s_{2n}) : 0 \leqslant s_{2n} \leqslant \ldots \leqslant s_1 \leqslant t\}$. Let $\mathcal{F}$ denote the pairings formed over $\{1,\ldots,2n\}$. It is straightforward to check that

$$F_{2,n}(\sqrt{i}) = \frac{1}{i^{2n}(2\pi)^n}\sum_{\mathcal{F}}\int_{[0,t]^{2n}_{<}}\int_{\mathbb{R}^{2n}}\prod_{(k,l)\in\mathcal{F}}\widehat{R}(s_k-s_l,p_k)\delta(p_k+p_l) \times \mathbb{E}_{\mathbf{B}}\left[e^{i\sqrt{i}\xi B_t}e^{-\sum_{j=1}^{2n} i\sqrt{i}p_jB_{s_j}}\right]dsdp. \tag{2.7}$$

The pre-factors in (2.6) and (2.7) differ by a factor of $2^n n!$ since $i^{-2n}=(-1)^n$, and this comes from the mapping between the sets of permutations and pairings: for a given pairing with n pairs, we have $n!$ ways of permutating the pairs, and inside each pair, we have 2 options which leads to the additional factor of 2^n.

The phase factor inside the integral in (2.7) can be computed explicitly:

$$\mathbb{E}_{\mathbf{B}}\left[e^{i\sqrt{i}\xi B_t}e^{-\sum_{j=1}^{2n} i\sqrt{i}p_jB_{s_j}}\right] = e^{-\frac{i}{2}|\xi|^2(t-s_1)-\frac{i}{2}|\xi-p_1|^2(s_1-s_2)-\ldots-\frac{i}{2}|\xi-\ldots-p_{2n}|^2s_{2n}}. \tag{2.8}$$

On the other hand, the equation (2.1) is written in the Fourier domain as

$$\partial_t\widehat{\psi} = -\frac{i}{2}|\xi|^2\widehat{\psi} + \int_{\mathbb{R}}\frac{\widehat{V}(t,dp)}{2\pi i}\widehat{\psi}(t,\xi-p), \quad \widehat{\psi}(0,\xi)=\widehat{\psi}_0(\xi),$$

where $V(t,x)$ admits the spectral representation $V(t,x)=\int_{\mathbb{R}}\frac{\widehat{V}(t,dp)}{2\pi}e^{ipx}$. Using the above formula, we can write the solution $\widehat{\psi}(t,\xi)$ as an infinite series

$$\widehat{\psi}(t,\xi) = \sum_{n=0}^{\infty}\int_{[0,t]^n_{<}}\int_{\mathbb{R}^n}\prod_{j=1}^n\frac{\widehat{V}(s_j,dp_j)}{2\pi i}e^{-\frac{i}{2}|\xi|^2(t-s_1)-\frac{i}{2}|\xi-p_1|^2(s_1-s_2)-\ldots-\frac{i}{2}|\xi-\ldots-p_n|^2s_n} \times \widehat{\psi}_0(\xi-p_1-\ldots-p_n)ds. \tag{2.9}$$

Evaluating the expectation $\mathbb{E}[\widehat{\psi}(t,\xi)]$ in (2.9), using the Wick formula for computing the Gaussian moment

$$\mathbb{E}[\widehat{V}(s_1,dp_1)\ldots\widehat{V}(s_n,dp_n)],$$

and the fact that

$$\mathbb{E}[\widehat{V}(s_i,dp_i)\widehat{V}(s_j,dp_j)] = 2\pi\widehat{R}(s_i-s_j,p_i)\delta(p_i+p_j)dp_idp_j,$$

and comparing the result to (2.7)-(2.8), we conclude that (2.5) holds, which completes the proof. □

2.2. **Convergence of Brownian functionals.** By Lemma 2.1, the interested quantity is written as

$$\mathbb{E}[\widehat{\phi}_\varepsilon(t,\xi)] = \widehat{\phi}_0(\xi)\mathbb{E}_{\mathbf{B}}[e^{i\sqrt{i}\xi B_t}e^{-\frac{1}{2}\int_0^t\int_0^t R_\varepsilon(s-u,\sqrt{i}(B_s-B_u))dsdu}],$$

with R_ε defined as the covariance function of $\varepsilon^{-3/2}V(t/\varepsilon^2,x/\varepsilon)$:

$$R_\varepsilon(t,x) = \frac{1}{\varepsilon^3}R(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}).$$

After a change of variable and using the scaling property of the Brownian motion, we have

$$\begin{aligned}&\int_0^t\int_0^t R_\varepsilon(s-u,\sqrt{i}(B_s-B_u))dsdu\\ &= \varepsilon\int_0^{t/\varepsilon^2}\int_0^{t/\varepsilon^2} R_\eta(s-u)q(\sqrt{i}(B_{\varepsilon^2 s}-B_{\varepsilon^2 u})/\varepsilon)dsdu\\ &\overset{\text{law}}{=} \varepsilon\int_0^{t/\varepsilon^2}\int_0^{t/\varepsilon^2} R_\eta(s-u)q(\sqrt{i}(B_s-B_u))dsdu,\end{aligned}$$

where R_η and q were defined in (1.5). Thus, by defining

$$X_t^\varepsilon := \frac{\varepsilon}{2}\int_0^{t/\varepsilon^2}\int_0^{t/\varepsilon^2} R_\eta(s-u)q(\sqrt{i}(B_s-B_u))dsdu, \tag{2.10}$$

we have

$$\mathbb{E}[\widehat{\phi}_\varepsilon(t,\xi)] = \widehat{\phi}_0(\xi)\mathbb{E}_{\mathbf{B}}[e^{i\sqrt{i}\xi\varepsilon B_{t/\varepsilon^2}-X_t^\varepsilon}]. \tag{2.11}$$

To pass to the limit of $\mathbb{E}[\widehat{\phi}_\varepsilon(t,\xi)]$, it suffices to prove the weak convergence of the random vector $(\varepsilon B_{t/\varepsilon^2}, X_t^\varepsilon)$ (for fixed $t>0$) and a uniform integrability condition. The proof of Theorem 1.1 reduces to the following three lemmas.

Lemma 2.2. $\mathbb{E}_{\mathbf{B}}[X_t^\varepsilon] = \frac{z_1 t}{\varepsilon} + O(\varepsilon)$ *with* z_1 *defined in* (2.13).

Lemma 2.3. *For fixed* $t>0$, *as* $\varepsilon\to 0$,

$$(\varepsilon B_{t/\varepsilon^2}, X_t^\varepsilon - \mathbb{E}_{\mathbf{B}}[X_t^\varepsilon]) \Rightarrow (N_1, N_2+iN_3) \tag{2.12}$$

in distribution, where $N_1\sim N(0,t)$ *and is independent of* $(N_2,N_3)\sim N(0,t\mathsf{A})$, *with the* 2×2 *covariance matrix* A *defined in* (2.15).

Lemma 2.4. *For any* $\lambda\in\mathbb{R}$, *there exists a constant* $C>0$ *such that*

$$\mathbb{E}_{\mathbf{B}}[|e^{\lambda(X_t^\varepsilon-\mathbb{E}_{\mathbf{B}}[X_t^\varepsilon])}|] \leqslant C$$

uniformly in $\varepsilon>0$.

Remark 2.5. With some extra work as in [11, Proposition 2.3], the convergence in (2.12) can be upgraded to the process level. To keep the argument short, we only consider the marginal distributions, which is enough for the proof of Theorem 1.1.

Proof of Lemma 2.2. A straightforward calculation gives

$$\begin{aligned}\mathbb{E}_{\mathbf{B}}[X_t^\varepsilon] =&\varepsilon\int_0^{t/\varepsilon^2} ds\int_0^s \frac{R_\eta(s-u)}{\sqrt{2\pi}}\mathbb{E}_{\mathbf{B}}[e^{-\frac{i}{2}|B_s-B_u|^2}]du\\ =&\varepsilon\int_0^{t/\varepsilon^2} ds\int_0^s \frac{R_\eta(u)}{\sqrt{2\pi}}\mathbb{E}_{\mathbf{B}}[e^{-\frac{i}{2}|B_s-B_{s-u}|^2}]du.\end{aligned}$$

Since R_η is compactly supported, it is clear that

$$\mathbb{E}_{\mathbf{B}}[X_t^\varepsilon] = \frac{z_1 t}{\varepsilon} + O(\varepsilon),$$

where

$$z_1 = \int_0^\infty \frac{R_\eta(u)}{\sqrt{2\pi}}\mathbb{E}_{\mathbf{B}}[e^{-\frac{i}{2}|B_u|^2}]du = \int_0^\infty \frac{R_\eta(u)}{\sqrt{2\pi(1+iu)}}du. \tag{2.13}$$

The proof is complete. □

Proof of Lemma 2.3. The proof is based on a martingale decomposition. Denote the Brownian filtration by $\mathscr{F}_r$ and the Malliavin derivative with respect to dB_r by D_r. An application of the Clark-Ocone formula [17, Proposition 1.3.14] leads to

$$X_t^\varepsilon - \mathbb{E}_{\mathbf{B}}[X_t^\varepsilon] = \int_0^{t/\varepsilon^2}\mathbb{E}_{\mathbf{B}}[D_r X_t^\varepsilon|\mathscr{F}_r]dB_r.$$

Recall that X_t^ε is defined in (2.10), by chain rule and the fact that

$$D_r(B_s-B_u) = D_r\int_u^s dB_r = 1_{[u,s]}(r), \tag{2.14}$$

we have

$$D_r X_t^\varepsilon = -i\varepsilon\int_0^{t/\varepsilon^2}\int_0^s \frac{R_\eta(s-u)}{\sqrt{2\pi}}e^{-\frac{i}{2}|B_s-B_u|^2}(B_s-B_u)1_{[u,s]}(r)duds, \quad r\in[0,t/\varepsilon^2].$$

Taking the conditional expectation with respect to $\mathscr{F}_r$ and computing the expectation

$$\mathbb{E}[e^{-\frac{i}{2}X^2}X \,|\, B_r - B_u]$$

with $X \sim N(B_r - B_u, s-r)$ explicitly yields

$$\begin{aligned} Y_r^{\varepsilon,t} &:= \varepsilon^{-1}\mathbb{E}_{\mathbf{B}}[D_r X_t^\varepsilon|\mathscr{F}_r] \\ &= -i\int_0^{t/\varepsilon^2}\int_0^s \frac{R_\eta(s-u)}{\sqrt{2\pi}(1+i(s-r))^{3/2}}e^{-\frac{i|B_r-B_u|^2}{2(1+i(s-r))}}(B_r-B_u)1_{[u,s]}(r)duds. \end{aligned}$$

By the assumption, there exists $M>0$ such that $R_\eta(s-u)=0$ if $s-u\geqslant M$. Using the indicator function $1_{[u,s]}(r)$ in the above expression, we have for $M\leqslant r\leqslant t/\varepsilon^2-M$ that

$$\begin{aligned} Y_r^{\varepsilon,t} = Y_r :&= -i\int_r^{r+M}\int_{r-M}^r \frac{R_\eta(s-u)}{\sqrt{2\pi}(1+i(s-r))^{3/2}}e^{-\frac{i|B_r-B_u|^2}{2(1+i(s-r))}}(B_r-B_u)1_{[u,s]}(r)duds \\ &= -i\int_0^M\int_0^M \frac{R_\eta(s+u)}{\sqrt{2\pi}(1+is)^{3/2}}e^{-\frac{i|B_r-B_{r-u}|^2}{2(1+is)}}(B_r-B_{r-u})duds. \end{aligned}$$

The Y_r defined above is only for $r\in[M,t/\varepsilon^2-M]$, but we can extend the definition to $r\in\mathbb{R}$ by interpreting B as a two-sided Brownian motion. Thus, by the fact that the Brownian motion has stationary and independent increments, we know $\{Y_r\}_{r\in\mathbb{R}}$ is a stationary process with a finite range of dependence.

It is easy to check that

$$X_t^\varepsilon - \mathbb{E}_{\mathbf{B}}[X_t^\varepsilon] - \varepsilon\int_0^{t/\varepsilon^2} Y_r dB_r = \varepsilon\int_0^{t/\varepsilon^2}(Y_r^{\varepsilon,t}-Y_r)dB_r \to 0$$

in probability. Define $Y_{1,r}=\mathrm{Re}[Y_r]$ and $Y_{2,r}=\mathrm{Im}[Y_r]$, applying Ergodic theorem, we have

$$\varepsilon^2\int_0^{t/\varepsilon^2} Y_{j,r}Y_{l,r}dr \to t\,\mathbb{E}[Y_{j,r}Y_{l,r}],\quad j,l=1,2,$$

and

$$\varepsilon^2\int_0^{t/\varepsilon^2} Y_r ds \to t\,\mathbb{E}[Y_r]=0,$$

almost surely. We apply the martingale central limit theorem [8, pp. 339] to derive

$$(\varepsilon B_{t/\varepsilon^2}, \varepsilon\int_0^{t/\varepsilon^2} Y_r dB_r) \Rightarrow (B_t, W_t^1 + iW_t^2)$$

in $\mathcal{C}[0,\infty)$, where B_t is a standard Brownian motion, independent of the two-dimensional Brownian motion (W_t^1,W_t^2) with the covariance matrix $\mathsf{A}=(\mathsf{A}_{jl})_{j,l=1,2}$ given by

$$\mathsf{A}_{jl} = \mathbb{E}[Y_{j,r}Y_{l,r}]. \tag{2.15}$$

The proof is complete. □

Proof of Lemma 2.4. We write

$$X_t^\varepsilon - \mathbb{E}_{\mathbf{B}}[X_t^\varepsilon] = \varepsilon\int_0^{t/\varepsilon^2}\mathcal{Z}_s ds,$$

where

$$\mathcal{Z}_s := \int_0^s \frac{R_\eta(u)}{\sqrt{2\pi}}\left(e^{-\frac{i}{2}|B_s-B_{s-u}|^2} - \mathbb{E}_{\mathbf{B}}[e^{-\frac{i}{2}|B_s-B_{s-u}|^2}]\right)du.$$

Again, assuming that $R_\eta(u) = 0$ for $|u| \geqslant M$. Let $N_\varepsilon = [\frac{t}{M\varepsilon^2}]$, we have

$$\begin{aligned} X_t^\varepsilon - \mathbb{E}_{\mathbf{B}}[X_t^\varepsilon] =& \varepsilon \sum_{k=2}^{N_\varepsilon} \int_{(k-1)M}^{kM} \mathcal{Z}_s ds + \varepsilon \left(\int_0^M + \int_{N_\varepsilon M}^{t/\varepsilon^2} \right) \mathcal{Z}_s ds \\ =& \varepsilon \sum_{k=2}^{N_\varepsilon} Z_k + \varepsilon \left(\int_0^M + \int_{N_\varepsilon M}^{t/\varepsilon^2} \right) \mathcal{Z}_s ds \end{aligned}$$

where we defined $Z_k := \int_{(k-1)M}^{kM} \mathcal{Z}_s ds$ for $2 \leqslant k \leqslant N_\varepsilon$. Since $\mathcal{Z}_s$ is uniformly bounded, we have

$$\left| \varepsilon \left(\int_0^M + \int_{N_\varepsilon M}^{t/\varepsilon^2} \right) \mathcal{Z}_s ds \right| \lesssim \varepsilon.$$

For the first part, we write

$$\varepsilon \sum_{k=2}^{N_\varepsilon} Z_k = \left(\sum_{k \in A_{\varepsilon,1}} + \sum_{k \in A_{\varepsilon,2}} \right) \varepsilon Z_k,$$

with $A_{\varepsilon,1} = \{2 \leqslant k \leqslant N_\varepsilon : k \text{ even } \}$ and $A_{\varepsilon,2} = \{2 \leqslant k \leqslant N_\varepsilon : k \text{ odd } \}$. By the independence of the increments of the Brownian motion, we know that $\{Z_k\}_{k \in A_{\varepsilon,j}}$ are i.i.d. for $j = 1$ and 2. Therefore,

$$\begin{aligned} \mathbb{E}_{\mathbf{B}}[|e^{\lambda(X_t^\varepsilon - \mathbb{E}_{\mathbf{B}}[X_t^\varepsilon])}|] \lesssim& \mathbb{E}_{\mathbf{B}}[e^{\lambda\varepsilon \sum_{k=2}^{N_\varepsilon} \mathrm{Re}[Z_k]}] \\ \lesssim& \sqrt{\mathbb{E}_{\mathbf{B}}[e^{2\lambda\varepsilon \sum_{k \in A_{\varepsilon,1}} \mathrm{Re}[Z_k]}] \mathbb{E}_{\mathbf{B}}[e^{2\lambda\varepsilon \sum_{k \in A_{\varepsilon,2}} \mathrm{Re}[Z_k]}]}. \end{aligned}$$

By the fact that $\{Z_k\}$ are bounded random variables with zero mean, we have for $j = 1, 2$ that

$$\begin{aligned} \mathbb{E}_{\mathbf{B}}[e^{2\lambda\varepsilon \sum_{k \in A_{\varepsilon,j}} \mathrm{Re}[Z_k]}] =& \prod_{k \in A_{\varepsilon,j}} \mathbb{E}_{\mathbf{B}}[e^{2\lambda\varepsilon \mathrm{Re}[Z_k]}] \\ \leqslant& \prod_{k=1}^{N_\varepsilon} \left(1 + 2\lambda^2\varepsilon^2 \mathbb{E}_{\mathbf{B}}[|\mathrm{Re}[Z_k]|^2] + O(\varepsilon^3)\right) \lesssim 1. \end{aligned} \tag{2.16}$$

The proof is complete. ☐

2.3. **Proof of Theorem 1.1.** By (2.11), we have

$$\mathbb{E}[\widehat{\phi}_\varepsilon(t,\xi)] e^{\mathbb{E}_{\mathbf{B}}[X_t^\varepsilon]} = \widehat{\phi}_0(\xi) \mathbb{E}_{\mathbf{B}}[e^{i\sqrt{i}\xi\varepsilon B_{t/\varepsilon^2} - (X_t^\varepsilon - \mathbb{E}_{\mathbf{B}}[X_t^\varepsilon])}]. \tag{2.17}$$

By Lemma 2.3, we know that, for fixed $t > 0, \xi \in \mathbb{R}$, the random variable

$$i\sqrt{i}\xi\varepsilon B_{t/\varepsilon^2} - (X_t^\varepsilon - \mathbb{E}_{\mathbf{B}}[X_t^\varepsilon]) \Rightarrow i\sqrt{i}\xi N_1 - (N_2 + iN_3)$$

in distribution, where $N_1 \sim N(0,t)$ independent of $(N_2, N_3) \sim N(0, t\mathsf{A})$. Since Lemma 2.4 provides the uniform integrability:

$$\mathbb{E}_{\mathbf{B}}[|e^{i\sqrt{i}\xi\varepsilon B_{t/\varepsilon^2} - (X_t^\varepsilon - \mathbb{E}_{\mathbf{B}}[X_t^\varepsilon])}|^2] \leqslant \sqrt{\mathbb{E}_{\mathbf{B}}[|e^{i\sqrt{i}\xi\varepsilon B_{t/\varepsilon^2}}|^2] \mathbb{E}_{\mathbf{B}}[|e^{-(X_t^\varepsilon - \mathbb{E}_{\mathbf{B}}[X_t^\varepsilon])}|^2]} \lesssim 1,$$

sending $\varepsilon \to 0$ on both sides of (2.17) and applying Lemma 2.2, we have

$$\mathbb{E}[\widehat{\phi}_\varepsilon(t,\xi)] e^{\frac{z_1 t}{\varepsilon}} \to \widehat{\phi}_0(\xi) \mathbb{E}_{\mathbf{B}}[e^{i\sqrt{i}\xi N_1 - (N_2 + iN_3)}] = \widehat{\phi}_0(\xi) e^{-\frac{i}{2}|\xi|^2 t} e^{\frac{1}{2}(\mathsf{A}_{11} - \mathsf{A}_{22} + 2i\mathsf{A}_{12})t}.$$

Define

$$z_2 = \frac{1}{2}(\mathsf{A}_{11} - \mathsf{A}_{22} + 2i\mathsf{A}_{12}), \tag{2.18}$$

the proof of Theorem 1.1 is complete.

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