MOMENTS OF THE 2D SHE AT CRITICALITY

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Abstract. We study the stochastic heat equation in two spatial dimensions with a multiplicative white noise, as the limit of the equation driven by a noise that is mollified in space and white in time. As the mollification radius $\varepsilon \to 0$, we tune the coupling constant near the critical point, and show that the single time correlation functions converge to a limit written in terms of an explicit non-trivial semigroup. Our approach consists of two steps. First we show the convergence of the resolvent of the (tuned) two-dimensional delta Bose gas, by adapting the framework of [DR04] to our setup of spatial mollification. Then we match this to the Laplace transform of our semigroup.

1. Introduction and main result

In this paper, we study the Stochastic Heat Equation (SHE), which informally reads

$$\partial_t Z = \tfrac{1}{2}\nabla^2 Z + \sqrt{\beta}\xi Z, \quad Z = Z(t,x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d,$$

where ∇^2 denotes the Laplacian, $d \in \mathbb{Z}_+$ denotes the spatial dimension, ξ denotes the spacetime white noise, and $\beta > 0$ is a tunable parameter. In broad terms, the SHE arises from a host of physical phenomena including the particle density of diffusion in a random environment, the partition function for a directed polymer in a random environment, and, through the inverse Hopf–Cole transformation, the height function of a random growth surface; the two-dimensional Kardar–Parisi–Zhang (KPZ) equation. We refer to [Cor12, Kho14, Com17] and the references therein.

When $d = 1$, the SHE enjoys a well-developed solution theory: For any $Z(0,x) = Z_{\mathrm{ic}}(x)$ that is bounded and continuous, and for each $\beta > 0$, the SHE (in $d = 1$) admits a unique $\mathscr{C}([0,\infty)\times\mathbb{R})$-valued mild solution, where $\mathscr{C}$ denotes continuous functions, c.f., [Wal86, Kho14]. Such a solution theory breaks down in $d \geqslant 2$, due to the deteriorating regularity of the spacetime white noise ξ, as the dimension d increases. In the language of stochastic PDE [Hai14, GIP15], $d = 2$ corresponds to the critical, and $d = 3, 4, \ldots$ the supercritical regimes.

Here we focus on the critical dimension $d = 2$. To set up the problem, fix a mollifier $\varphi \in \mathscr{C}^\infty_{\mathrm{c}}(\mathbb{R}^2)$, where $\mathscr{C}^\infty_{\mathrm{c}}$ denotes smooth functions with compact support, with $\varphi \geqslant 0$ and $\int \varphi\,\mathrm{d}x = 1$, and mollify the noise as

$$\xi_\varepsilon(t,x) := \int_{\mathbb{R}^2} \varphi_\varepsilon(x-y)\xi(t,y)\mathrm{d}y, \quad \varphi_\varepsilon(x) := \tfrac{1}{\varepsilon^2}\varphi(\tfrac{x}{\varepsilon}).$$

Consider the corresponding SHE driven by ξ_ε,

$$\partial_t Z_\varepsilon = \tfrac{1}{2}\nabla^2 Z_\varepsilon + \sqrt{\beta_\varepsilon}\xi_\varepsilon Z_\varepsilon, \quad Z_\varepsilon = Z_\varepsilon(t,x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2, \tag{1.1}$$

with a parameter $\beta_\varepsilon > 0$ that has to be finely tuned as $\varepsilon \to 0$. The noise ξ_ε is white in time, and we interpret the product between ξ_ε and Z_ε in the Itô sense. Let $\mathsf{p}(t,x) := \frac{1}{2\pi t}\exp(-\frac{|x|^2}{2t})$, $x \in \mathbb{R}^2$, denote the standard heat kernel in two dimensions. For fixed $Z(0,x) = Z_{\mathrm{ic}} \in \mathscr{L}^2(\mathbb{R}^2)$ and $\varepsilon > 0$, it is standard, though tedious, to show that the unique $\mathscr{C}((0,\infty)\times\mathbb{R}^2)$-valued mild solution of (1.1) is given by the chaos expansion

$$Z_\varepsilon(t,x) = \int_{\mathbb{R}^2} \mathsf{p}(t,x-x')Z_{\mathrm{ic}}(x')\,\mathrm{d}x' + \sum_{k=1}^\infty I_{\varepsilon,k}(t,x), \tag{1.2}$$

$$I_{\varepsilon,k}(t,x) := \int \Big(\prod_{s=1}^k \mathsf{p}(\tau_{s+1}-\tau_s, x^{(s+1)}-x^{(s)})\sqrt{\beta_\varepsilon}\xi_\varepsilon(\tau_s,x^{(s)})\mathrm{d}\tau_s\mathrm{d}x^{(s)}\Big)\mathsf{p}(\tau_1, x^{(1)}-x')Z_{\mathrm{ic}}(x')\,\mathrm{d}x', \tag{1.3}$$

where the integral goes over all $0 < \tau_1 < \ldots < \tau_k < t$ and $x', x^{(1)}, \ldots, x^{(k)} \in \mathbb{R}^2$, with the convention $x^{(k+1)} := x$ and $\tau_{k+1} := t$.

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From the expression (1.3) of $I_{\varepsilon,k}$, it is straightforward to check that, for fixed $\beta_\varepsilon = \beta > 0$ as $\varepsilon \to 0$, the variance $\mathrm{Var}[I_{\varepsilon,k}]$ diverges, confirming the breakdown of the standard theory in $d = 2$. We hence seek to tune $\beta_\varepsilon \to 0$ in a way so that a meaningful limit of Z_ε can be observed. A close examination shows that the divergence of $\mathrm{Var}[I_{\varepsilon,k}]$ originates from the singularity of $\mathsf{p}(t,0) = (2\pi t)^{-1}$ near $t = 0$, so it is natural to propose $\beta_\varepsilon = \frac{\beta_0}{|\log\varepsilon|} \to 0$, $\beta_0 > 0$. The $\varepsilon \to 0$ behavior of Z_ε for small values of β_0 has attracted much attention recently. For fixed $\beta_0 \in (0, 2\pi)$, [CSZ17] showed that the fluctuations of $Z_\varepsilon(t, \bullet)$ converge (as a random measure) to a Gaussian field, more precisely, the solution of the two-dimensional Edwards–Wilkinson (EW) equation. For $\beta_0 = \beta_{0,\varepsilon} \to 0$, [Fen15] showed that the corresponding polymer measure exhibits diffusive behaviors. The logarithm $h_\varepsilon(t, x) := \beta_\varepsilon^{-1/2} \log Z_\varepsilon(t, x)$ is also a quantity of interest: it describes the free energy of random polymers and the height function in surface growth phenomena which solves the two dimensional KPZ equation. The tightness of the centered height function was obtained in [CD18] for small enough β_0. It was then shown in [CSZ18b] that the centered height function converges to the EW equation for all $\beta_0 \in (0, 2\pi)$, and in [Gu18] for small enough β_0, i.e., the limit is Gaussian.

However, the $\varepsilon \to 0$ behavior of Z_ε goes through a *transition* at $\beta_0 = 2\pi$. Recall that by Itô calculus, the n-th order correlation function of the solution of the mollified SHE (1.1) at a fixed time,

$$u_\varepsilon(t, x_1, \ldots, x_n) := \mathbb{E}\Big[\prod_{i=1}^{n} Z_\varepsilon(t, x_i)\Big], \tag{1.4}$$

satisfies the n particle (approximate) delta Bose gas

$$\partial_t\, u_\varepsilon(t, x_1, \ldots, x_n) = -\big(\mathcal{H}_\varepsilon u_\varepsilon\big)(t, x_1, \ldots, x_n), \quad x_i \in \mathbb{R}^2, \quad u_\varepsilon(0) = Z_{\mathrm{ic}}^{\otimes n}, \tag{1.5}$$

where $\mathcal{H}_\varepsilon$ is the Hamiltonian

$$\mathcal{H}_\varepsilon := -\frac{1}{2}\sum_{i=1}^{n} \nabla_i^2 - \beta_\varepsilon \sum_{1\leqslant i<j\leqslant n} \delta_\varepsilon(x_i - x_j), \qquad \delta_\varepsilon(x) := \varepsilon^{-2}\Phi(\varepsilon^{-1}x), \qquad \Phi(x) := \int_{\mathbb{R}^2} \varphi(x+y)\varphi(y)\,\mathrm{d}y. \tag{1.6}$$

It can be shown (e.g., from [AGHKH88, Equation (I.5.56)]) that, for $n = 2$, the Hamiltonian $\mathcal{H}_\varepsilon$ has a vanishing/diverging principal eigenvalue as $\varepsilon \to 0$, respectively for $\beta_0 < 2\pi$ and $\beta_0 > 2\pi$. This phenomenon in turn suggests a transition in behaviors of Z_ε at $\beta_0 = 2\pi$. This transition is also demonstrated at the level of pointwise limit (in distribution) of $Z_\varepsilon(t, x)$ as $\varepsilon \to 0$ by [CSZ17].

The preceding observations point to an intriguing question of understanding the behavior of Z_ε and u_ε at this critical value $\beta_0 = 2\pi$. For the case of two particles ($n = 2$), by separating the center-of-mass and the relative motions, the delta Bose gas can be reduced to a system of one particle with a delta potential at the origin. Based on this reduction and the analysis of the one-particle system in [AGHKH88, Chapter I.5], [BC98] gave an explicit $\varepsilon \to 0$ limit of the second order correlation functions (tested against $\mathscr{L}^2$ functions). Further, given the radial symmetry of the delta potential, the one particle system (in $d = 2$) can be reduced to an one-dimensional problem along the radial direction. Despite its seeming simplicity, this one-dimensional problem already requires sophisticated analysis. Although the final answer is non-trivial, it does not rule out a lognormal limit. For $n > 2$, these reductions no longer exist, and to obtain information on the correlation functions stands as a challenging problem; the only existing results are on third order ($n = 3$): [Fen15] showed that for Z_ε the limiting ratio of the cube root of the third pointwise moment to the square root of the second moment is not what one would expect from a lognormal distribution, indicating (but not proving) non-trivial fluctuations. Using techniques developed in [CSZ18a] to control the chaos series, [CSZ18c] obtained the convergence of the third order correlations of Z_ε to a limit given in terms of a sum of integrals.

In this paper, we proceed through a different, functional analytic route, and obtain a unified description of the $\varepsilon \to 0$ limit of all correlation functions of Z_ε. We now prepare some notation for stating our main result. Hereafter throughout the paper, we set

$$\beta_\varepsilon := \frac{2\pi}{|\log\varepsilon|} + \frac{2\pi\beta_{\mathrm{fine}}}{|\log\varepsilon|^2}, \tag{1.7}$$

where $\beta_{\mathrm{fine}} \in \mathbb{R}$ is a fixed, fine-tuning constant. This fine-tuning constant does not complicate our analysis, though the limiting expressions do depend on β_{fine}. Let $\gamma_{\mathrm{EM}} = 0.577\ldots$ denote the Euler–Mascheroni constant, and, with Φ as in (1.1) and (1.6), set

$$\beta_\star := 2\,(\log 2 + \beta_{\text{fine}} - \beta_\Phi - \gamma_{\text{EM}}), \qquad \beta_\Phi := \int_{\mathbb{R}^4} \Phi(x_1) \log|x_1 - x_1'| \Phi(x_1')\,\mathrm{d}x_1 \mathrm{d}x_1', \tag{1.8}$$

and

$$\mathsf{j}(t, \beta_\star) := \int_0^\infty \frac{t^{\alpha-1} e^{\beta_\star \alpha}}{\Gamma(\alpha)}\,\mathrm{d}\alpha. \tag{1.9}$$

We will often work with vectors $x = (x_1, \ldots, x_n) \in \mathbb{R}^{2n}$, where $x_i \in \mathbb{R}^2$, and similarly $y = (y_2, \ldots, y_n) \in \mathbb{R}^{2n-2}$, $y_i \in \mathbb{R}^2$. We say x_i is the ***i*-th component** of x. For $1 \leqslant i < j \leqslant n$, consider a linear transformation $S_{ij} : \mathbb{R}^{2n-2} \to \mathbb{R}^{2n}$,

$$S_{ij}(y_2, \ldots, y_n) := (y_3, \ldots, \underbrace{y_2}_{i\text{-th}}, \ldots, \underbrace{y_2}_{j\text{-th}}, \ldots, y_n), \tag{1.10}$$

and the induced operator $\mathcal{S}_{ij} : \mathscr{L}^2(\mathbb{R}^{2n}) \to \mathscr{L}^2(\mathbb{R}^{2n-2})$,

$$\big(\mathcal{S}_{ij} u\big)(y) := u(S_{ij} y). \tag{1.11}$$

Let $\mathscr{H}^\alpha(\mathbb{R}^{2n})$ denote the Sobolev space of degree $\alpha \in \mathbb{R}$. As we will show in Lemma 4.1, (1.11) defines an *unbounded* operator $\mathscr{L}^2(\mathbb{R}^{2n}) \to \mathscr{L}^2(\mathbb{R}^{2n-2})$, and there exists an adjoint $\mathcal{S}_{ij}^* : \mathscr{L}^2(\mathbb{R}^{2n-2}) \to \cap_{a>1} \mathscr{H}^{-a}(\mathbb{R}^{2n})$. Adopting the shorthand notation $\nabla_i^2 := \nabla_{x_i}^2$, we let

$$\mathcal{P}_t := e^{\frac{t}{2}\sum_{i=1}^n \nabla_i^2}$$

denote the heat semigroup on $\mathscr{L}^2(\mathbb{R}^{2n})$; its integral kernel will be denoted $P(t, x) := \prod_{i=1}^n \frac{1}{2\pi t} \exp(-\frac{|x_i|^2}{2t})$. Define the operator $\mathcal{P}_t^{\mathcal{J}} : \mathscr{L}^2(\mathbb{R}^{2n-2}) \to \mathscr{L}^2(\mathbb{R}^{2n-2})$,

$$\mathcal{P}_t^{\mathcal{J}} := \mathsf{j}(t, \beta_\star) e^{\frac{t}{4}\nabla_2^2 + \frac{t}{2}\sum_{i=3}^n \nabla_i^2}. \tag{1.12}$$

As we will show in Lemma 8.5, $\mathcal{P}_t^{\mathcal{J}}$ is a bounded operator on $\mathscr{L}^2(\mathbb{R}^{2n-2})$.

We need to prepare some index sets. Hereafter we write $i < j$ for a pair of ordered indices in $\{1, \ldots, n\}$, i.e., $i < j \in \{1, \ldots, n\}$. For $n, m \in \mathbb{Z}_+$, we consider $\overrightarrow{(i,j)} = ((i_k, j_k))_{k=1}^m$ such that $(i_k < j_k) \neq (i_{k+1} < j_{k+1})$, i.e., *$m$ ordered pairs with consecutive pairs non-repeating.* Let

$$\mathrm{Dgm}(n, m) := \big\{\overrightarrow{(i,j)} \in (\{1, \ldots, n\}^2)^m : (i_k < j_k) \neq (i_{k+1} < j_{k+1})\big\}, \tag{1.13}$$

$$\mathrm{Dgm}(n) := \bigcup_{m=1}^\infty \mathrm{Dgm}(n, m) \tag{1.14}$$

denote the sets of all such indices, with the convention that $\mathrm{Dgm}(1, m) := \varnothing$, $m \in \mathbb{Z}_+$. The notation $\mathrm{Dgm}(n)$ refers to 'diagrams', as will be explained in Section 2. Let

$$\Sigma_m(t) := \big\{\vec{\tau} = (\tau_a)_{a \in \frac{1}{2}\mathbb{Z} \cap [0,m]} \in \mathbb{R}_+^{2m+1} : \tau_0 + \tau_{1/2} + \ldots + \tau_m = t\big\}, \tag{1.15}$$

so that for a fixed $t \in \mathbb{R}_+$, the integral $\int_{\Sigma_m(t)} (\,\bullet\,) \mathrm{d}\vec{\tau}$ denotes a $(2m+1)$-fold convolution over the set $\Sigma_m(t)$. For a bounded operator $\mathcal{Q} : \mathscr{K} \to \mathscr{K}'$ between Hilbert spaces $\mathscr{K}$ and $\mathscr{K}'$, let $\|\mathcal{Q}\|_{\text{op}} := \sup_{\|u\|_{\mathscr{K}} = 1} \|\mathcal{Q}u\|_{\mathscr{K}'}$ denote the inherited operator norm. We use the subscript 'op' (standing for 'operator') to distinguish the operator norm from the vector norm, and omit the dependence on $\mathscr{K}$ and $\mathscr{K}'$, since the spaces will always be specified along with a given operator. The $\mathscr{L}^2$ spaces in this paper are over $\mathbb{C}$, and we write $\langle f, g\rangle := \int_{\mathbb{R}^d} \overline{f(x)} g(x)\,\mathrm{d}x$ for the inner product. (Note our convention of taking complex conjugate in the first function.) Throughout this paper we use $C(a, b, \ldots)$ to denote a generic positive finite constant that may change from line to line, but depends only on the designated variables $a, b, \ldots$. We view the mollifier φ as fixed throughout this paper, so the dependence on φ will not be specified.

We can now state our main result.

Theorem 1.1.

(a) The operators

$$\mathcal{P}_t + \mathcal{D}_t^{\mathrm{Dgm}(n)}, \qquad \mathcal{D}_t^{\mathrm{Dgm}(n)} := \sum\nolimits_{\overrightarrow{(i,j)} \in \mathrm{Dgm}(n)} \mathcal{D}_t^{\overrightarrow{(i,j)}}, \quad t \geqslant 0, \tag{1.16}$$

define a norm-continuous semigroup on $\mathscr{L}^2(\mathbb{R}^{2n})$*, where, for* $\overrightarrow{(i,j)} = ((i_k, j_k))_{k=1}^m$*,*

$$\mathcal{D}_t^{\overrightarrow{(i,j)}} := \int_{\Sigma_m(t)} \mathcal{P}_{\tau_0} \mathcal{S}^*_{i_1 j_1} \big(4\pi \mathcal{P}^{\mathcal{J}}_{\tau_{1/2}}\big) \Big(\prod_{k=1}^{m-1} \mathcal{S}_{i_k j_k} \mathcal{P}_{\tau_k} \mathcal{S}^*_{i_{k+1} j_{k+1}} \big(4\pi \mathcal{P}^{\mathcal{J}}_{\tau_{k+1/2}}\big) \Big) \mathcal{S}_{i_m j_m} \mathcal{P}_{\tau_m} \, \mathrm{d}\vec{\tau}. \tag{1.17}$$

The sum in (1.16) *converges absolutely in operator norm, uniformly in* t *over compact subsets of* $[0,\infty)$*.*

(b) Start the mollified SHE (1.1) *from* $Z_\varepsilon(0,\bullet) = Z_{\mathrm{ic}}(\bullet) \in \mathscr{L}^2(\mathbb{R}^2)$*. For any* $f(x) = f(x_1,\ldots,x_n) \in \mathscr{L}^2(\mathbb{R}^{2n})$*,* $n \in \mathbb{Z}_+$*, we have*

$$\mathbb{E}\big[\langle f, Z^{\otimes n}_{\varepsilon,t}\rangle\big] := \mathbb{E}\Big[\int_{\mathbb{R}^{2n}} \overline{f(x)} \prod_{i=1}^n Z_\varepsilon(t,x_i)\,\mathrm{d}x\Big] \longrightarrow \Big\langle f, \big(\mathcal{P}_t + \mathcal{D}_t^{\mathrm{Dgm}(n)}\big)\, Z_{\mathrm{ic}}^{\otimes n} \Big\rangle \quad \text{as } \varepsilon \to 0, \tag{1.18}$$

uniformly in t *over compact subsets of* $[0,\infty)$*.*

Remark 1.2. Our result does not apply to the flat initial condition $Z_{\mathrm{ic}}(x) \equiv 1$. We conjecture that Theorem 1.1 extends to such initial data, and leave this to future work.

Theorem 1.1 gives a complete characterization of the $\varepsilon \to 0$ limit of fixed time, correlation functions of the SHE with an $\mathscr{L}^2$ initial condition. We will show in Section 2 that $\mathcal{D}^{\overrightarrow{(i,j)}}$ permits an explicit integral kernel, for each $\overrightarrow{(i,j)} \in \mathrm{Dgm}(n)$. Hence the limiting correlation functions (i.e., r.h.s. of (1.18)) can be expressed as a sum of integrals. From this expression, we check (in Remark 2.1) that for $n = 2$ our result matches that of [BC98], and for $n = 3$, we derive (in Proposition 2.2) an analogous expression of [CSZ18b, Equations (1.24)–(1.26)].

A question of interest arises as to whether one can uniquely characterize the limiting process of Z_ε. This does not follow directly from correlation functions, or moments, since we expect a very fast moment growth in n (see Remark 1.6). Still, as a simple corollary of Theorem 1.1, we are able to infer that every limit point of Z_ε must have corrlation functions given by the r.h.s. of (1.18):

Corollary 1.3. *Let* Z_{ic} *and* $Z_\varepsilon(t,x)$ *be as in Theorem 1.1, and, for each fixed* t*, view* $Z_\varepsilon(t,x)\mathrm{d}x := \mu_{\varepsilon,t}(\mathrm{d}x)$ *as a random measure. Then, for any fixed* $t \in \mathbb{R}_+$*, the law of* $\{\mu_{\varepsilon,t}(\mathrm{d}x)\}_{\varepsilon\in(0,1)}$ *is tight in the vague topology, and, for any limit point* $\mu_{*,t}(\mathrm{d}x)$ *of* $\{\mu_{\varepsilon,t,Z}(\mathrm{d}x)\}_{\varepsilon\in(0,1)}$*, and for any compactly supported, continuous* $f_1,\ldots,f_n \in \mathscr{C}_{\mathrm{c}}(\mathbb{R}^2)$*,* $n \in \mathbb{Z}_+$*,*

$$\mathbb{E}\Big[\prod_{i=1}^n \int_{\mathbb{R}^2} \overline{f_i(x_i)} \mu_{*,t}(\mathrm{d}x_i)\Big] = \Big\langle f_1 \otimes \cdots \otimes f_n, \big(\mathcal{P}_t + \mathcal{D}_t^{\mathrm{Dgm}(n)}\big)\, Z_{\mathrm{ic}}^{\otimes n} \Big\rangle. \tag{1.19}$$

Furthermore, if $Z_{\mathrm{ic}}(x), f(x) \geqslant 0$ *are nonnegative and not identically zero, then*

$$\mathbb{E}\Big[\Big(\int_{\mathbb{R}^2} f(x)\mu_{*,t}(\mathrm{d}x) - \mathbb{E}\Big[\int_{\mathbb{R}^2} f(x)\mu_{*,t}(\mathrm{d}x)\Big]\Big)^3\Big] > 0. \tag{1.20}$$

Due to the critical nature of our problem ($\beta_0 = 2\pi$ and $d = 2$), the moments go through a non-trivial transition as $\varepsilon \to 0$. To see this, in (1.2), use the orthogonality $\mathbb{E}[I_{\varepsilon,k}(t,x_1) I_{\varepsilon,k'}(t,x_2)] = 0$, $k \neq k'$, to express the second ($n = 2$) moment as

$$\mathbb{E}\Big[\Big(\int_{\mathbb{R}^2} Z_\varepsilon(t,x) f(x)\mathrm{d}x\Big)^2\Big] = \int_{\mathbb{R}^8} \prod_{i=1}^2 \mathsf{p}(t, x_i - x_i') f(x_i) Z_{\mathrm{ic}}(x_i')\,\mathrm{d}x_i'\mathrm{d}x_i + \sum_{k=1}^\infty \int_{\mathbb{R}^4} \mathbb{E}\Big[\prod_{i=1}^2 I_{\varepsilon,k}(t,x_i) f(x_i)\Big]\mathrm{d}x_1\mathrm{d}x_2.$$

As seen in [CSZ18b], the major contribution of the sum spans across a *divergent* number of terms — across all k's of order $|\log\varepsilon| \to \infty$. We are probing a regime where the limiting process 'escapes' to indefinitely high order chaos as $\varepsilon \to 0$, reminiscent of the large time behavior of the SHE/KPZ equation in $d = 1$.

Because of this, obtaining the $\varepsilon \to 0$ limit from chaos expansion requires elaborate and delicate analysis. In fact, just to obtain an ε-independent bound (for fixed Z_{ic} and test functions f_i's) from the chaos expansion is a challenging task. Such analysis is carried out for $n = 2, 3$ in [CSZ18b] (in a discrete setting and in the current continuum setting, both with $Z_{\mathrm{ic}} \equiv 1$). Here, we progress through a different route. From (1.4), (1.5), and (1.6) obtaining the limit of the correlation functions is equivalent to obtaining the limit of the semigroup $e^{-t\mathcal{H}_\varepsilon}$, which reduces to the study of $\mathcal{H}_\varepsilon$ itself, or its resolvent.

The delta Bose gas enjoys a long history of study, motivated in part by phenomena such as unbounded ground-state energy and infinite discrete spectrum observed in $d = 3$. We do not survey the literature here, and refer to the references in [AGHKH88]. Of most relevance to this paper is the work [DR04], which studied $d = 2$ with a momentum cutoff, and established the convergence of the resolvent of the Hamiltonian to an explicit limit [DR04, Equation (90)] (also (1.24)). Here, we follow the framework of [DR04], but instead of the momentum cutoff, we work with the space-mollification scheme as in (1.6), in order to connect the delta Bose gas to the SHE.

Hereafter we always assume $n \geqslant 2$, since the $n = 1$ case of Theorem 1.1 is trivial. We write $\mathbf{I}$ for the identity operator in Hilbert spaces. For $z \in \mathbb{C} \setminus [0, \infty)$, let

$$\mathcal{G}_z := \Big(-\frac{1}{2} \sum_{i=1}^{n} \nabla_i^2 - z\mathbf{I} \Big)^{-1} \tag{1.21}$$

denote the resolvent of the free Laplacian in $\mathbb{R}^{2n}$. Let $\mathcal{J}_z$ be the unbounded operator $\mathscr{L}^2(\mathbb{R}^{2n-2}) \to \mathscr{L}^2(\mathbb{R}^{2n-2})$ defined via its Fourier transform

$$\widehat{\mathcal{J}_z v}(p_{2-n}) := \log(\tfrac{1}{2}|p|^2_{2-n} - z)\widehat{v}(p_{2-n}), \tag{1.22}$$

where $p_{2-n} = (p_2, \ldots, p_n) \in \mathbb{R}^{2n-2}$ and

$$|p|^2_{2-n} := \tfrac{1}{2}|p_2|^2 + |p_3|^2 + \ldots + |p_n|^2,$$

with domain $\mathrm{Dom}(\mathcal{J}_z) := \{v \in \mathscr{L}^2(\mathbb{R}^{2n-2}) : \int_{\mathbb{R}^{2n}} \big|\widehat{v}(p_{2-n}) \log(|p|^2_{2-n} + 1)\big|^2 \mathrm{d}p_{2-n} < \infty\}$.

Let $\mathscr{L}^2_{\mathrm{sym}}(\mathbb{R}^{2n})$ denote the subspace of $\mathscr{L}^2(\mathbb{R}^{2n})$ consisting of functions symmetric in the n-components, i.e., $u(x_1, \ldots, x_n) = u(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$, for all permutation $\sigma \in \mathbb{S}_n$. Recall $\beta_\star$ and β_{fine} from (1.8). As the major step toward proving Theorem 1.1, in Sections 3–7, we show

Proposition 1.4 (Limiting resolvent)**.** *There exists $C < \infty$ such that, for $z \in \mathbb{C}$ with $\mathrm{Re}(z) < -e^{Cn^2+\beta_\star}$,*

(a) the following defines a bounded operator on $\mathscr{L}^2(\mathbb{R}^{2n}) \to \mathscr{L}^2(\mathbb{R}^{2n})$:

$$\mathcal{R}_z = \mathcal{G}_z + \sum_{m=1}^{\infty} \sum_{\overrightarrow{(i,j)} \in \mathrm{Dgm}(n,m)} \mathcal{G}_z \mathcal{S}^*_{i_1 j_1} \left(4\pi(\mathcal{J}_z - \beta_\star \mathbf{I})^{-1}\right) \prod_{s=2}^{m} \Big(\mathcal{S}_{i_{s-1} j_{s-1}} \mathcal{G}_z \mathcal{S}^*_{i_s j_s} \left(4\pi(\mathcal{J}_z - \beta_\star \mathbf{I})^{-1}\right)\Big) \mathcal{S}_{i_m j_m} \mathcal{G}_z, \tag{1.23}$$

where the sum converges absolutely in operator norm;

(b) when restricted to $\mathscr{L}^2_{\mathrm{sym}}(\mathbb{R}^{2n})$, the operator takes a simpler form,

$$\mathcal{R}^{\mathrm{sym}}_z := \mathcal{G}_z + \frac{2}{n(n-1)} \Big(\sum_{i<j} \mathcal{G}_z \mathcal{S}^*_{ij} \Big) \Big(\frac{1}{4\pi} (\mathcal{J}_z - \beta_\star \mathbf{I}) - \frac{2}{n(n-1)} {\sum}^{\mathrm{d}} \mathcal{S}_{ij} \mathcal{G}_z \mathcal{S}^*_{k\ell} \Big)^{-1} \Big(\sum_{i<j} \mathcal{S}_{ij} \mathcal{G}_z \Big). \tag{1.24}$$

The sum ${\sum}^{\mathrm{d}}$ is over distinct pairs $(i < j) \neq (k < \ell)$.

Theorem 1.5 (Convergence of the resolvent)**.** *There exist constants $C_1 < \infty, C_2(\beta_{\mathrm{fine}}) > 0$ such that for all $\varepsilon \in (0, 1/C_2(\beta_{\mathrm{fine}}))$, and $z \in \mathbb{C}$ with $\mathrm{Re}(z) < -e^{C_1 n^2 + \beta_\star}$,*

(a) $(\mathcal{H}_\varepsilon - z)$ has a bounded inverse $\mathscr{L}^2(\mathbb{R}^{2n}) \to \mathscr{L}^2(\mathbb{R}^{2n})$;

(b) $\mathcal{R}_{\varepsilon,z} := (\mathcal{H}_\varepsilon - z)^{-1} \longrightarrow \mathcal{R}_z$ strongly on $\mathscr{L}^2(\mathbb{R}^{2n})$, as $\varepsilon \to 0$.

Remark 1.6. Given Theorem 1.5, by the Trotter–Kato Theorem, c.f., [RS72, Theorem VIII.22], there exists an (unbounded) self-adjoint operator $\mathcal{H}$ on $\mathscr{L}^2(\mathbb{R}^{2n})$, *the limiting Hamiltonian*, such that $\mathcal{R}_z = (\mathcal{H} - z\mathbf{I})^{-1}$, $\mathrm{Im}(z) \neq 0$. As implied by Theorem 1.5, the spectra of $\mathcal{H}_\varepsilon$ and $\mathcal{H}$ are bounded below by $-e^{Cn^2+\beta_\star}$. Such a bound is first obtained under the momentum cutoff in [DFT94]. Using a non-rigorous mean-field analysis, [Raj99] predicted that the lower end of the spectrum of $\mathcal{H}$ should approximate $-e^{c_\star n}$, for some $c_\star \in (0, \infty)$.

Remark 1.7 (SHE in $d \geqslant 3$)**.** In higher dimensions $d \geqslant 3$, the appropriate tuning parameter is $\beta_\varepsilon = \beta_0 \varepsilon^{d-2}$. For small β_0, the studies on the EW-equation limit of the SHE/KPZ equation include [MU18, GRZ18, DGRZ18], and results on the pointwise fluctuations of Z_ε and the phase transition in β_0 can be found in [MSZ16, CL17, CCM18, CCM19, CN19]. For discussions on directed polymers in a random environment, we refer to [Com17] and the references therein.

Outline. In Section 2 we give an explicit expression for the limiting semigroup in terms of diagrams and use this to derive Corollary 1.3 from Theorem 1.1. In Section 3, we derive the key expression (3.6) for the resolvent $\mathcal{R}_{\varepsilon,z}$, which allows the limit to be taken term by term: The limits are obtained in Sections 4 through 6, and these are used in Section 7 to prove Proposition 1.4(a)–(b), Theorem 1.5(a)–(b) and the convergence part of Theorem 1.1(b). In Section 8, we complete the proof of Theorem 1.1 by constructing the semigroup and matching its Laplace transform to the limiting resolvent $\mathcal{R}_z$.

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2. Diagram expansion

In this section, we give an explicit integral kernel $D^{\overrightarrow{(i,j)}}(t,x,x')$ of the operator $\mathcal{D}_t^{\overrightarrow{(i,j)}}$ in Theorem 1.1. and show how the kernel $D^{\overrightarrow{(i,j)}}(t,x,x')$ can be encoded in terms of diagrams. This is then used to show how Corollary 1.3 follows from Theorem 1.1. The operators $\mathcal{S}_{ij}\mathcal{P}_t$, $\mathcal{P}_t\mathcal{S}_{ij}^*$ and $\mathcal{S}_{ij}\mathcal{P}_t\mathcal{S}_{k\ell}^*$ have integral kernels

$$\big(\mathcal{S}_{ij}\mathcal{P}_t u\big)(y) = \int_{\mathbb{R}^{2n}} P(t, S_{ij}y - x)u(x)\,\mathrm{d}x, \qquad y = (y_2,\ldots,y_n) \in \mathbb{R}^{2n-2}, \tag{2.1}$$

$$\big(\mathcal{P}_t\mathcal{S}_{ij}^* v\big)(x) = \int_{\mathbb{R}^{2n-2}} P(t, x - S_{ij}y)v(y)\,\mathrm{d}y, \qquad x = (x_1,\ldots,x_n) \in \mathbb{R}^{2n}, \tag{2.2}$$

$$\big(\mathcal{S}_{ij}\mathcal{P}_t\mathcal{S}_{k\ell}^* v\big)(y) = \int_{\mathbb{R}^{2n-2}} P(t, S_{ij}y - S_{k\ell}y')v(y')\,\mathrm{d}y', \qquad y = (y_2,\ldots,y_n) \in \mathbb{R}^{2n-2}. \tag{2.3}$$

From this we see that $\mathcal{D}_t^{\overrightarrow{(i,j)}}$ has integral kernel

$$\begin{aligned} D^{\overrightarrow{(i,j)}}(t,x,x') = &\int_{\Sigma_m(t)} \mathrm{d}\vec{\tau} \int P\big(\tau_0, x - S_{i_1j_1}y^{(1/2)}\big)\,\mathrm{d}y^{(1/2)} \cdot 4\pi P^{\mathcal{J}}\big(\tau_{1/2}, y^{(1/2)} - y^{(1)}\big)\,\mathrm{d}y^{(1)} \\ &\cdot \prod_{k=1}^{m-1} \Big(P\big(\tau_k, S_{i_kj_k}y^{(k)} - S_{i_{k+1}j_{k+1}}y^{(k+1/2)}\big)\,\mathrm{d}y^{(k+1/2)} \cdot 4\pi P^{\mathcal{J}}\big(\tau_{k+1/2}, y^{(k+1/2)} - y^{(k+1)}\big)\,\mathrm{d}y^{(k+1)}\Big) \\ &\cdot P\big(\tau_m, S_{i_mj_m}y^{(m)} - x'\big), \end{aligned} \tag{2.4}$$

where $\Sigma_m(t)$ is defined in (1.15), $x, x' \in \mathbb{R}^{2n}$, and $y^{(a)} \in \mathbb{R}^{2n-2}$ with $a \in (\frac{1}{2}\mathbb{Z}) \cap (0,m]$.

We wish to further reduce (2.4) to an expression that involves only the two-dimensional heat kernel $\mathsf{p}(\tau, x_i)$ and $\mathsf{j}(\tau,\beta_\star)$. Recall from (1.10) that $(S_{ij}y) := x$ is a vector in $\mathbb{R}^{2n}$ such that $x_i = x_j$. In (2.4), we write

$$S_{i_kj_k}y^{(a)} = (y_3^{(a)}, \ldots, \underbrace{y_2^{(a)}}_{i_k\text{-th}}, \ldots, \underbrace{y_2^{(a)}}_{j_k\text{-th}}, \ldots, y_n^{(a)}) := (x_1^{(a)}, \ldots, x_n^{(a)})\mathbf{1}\{x_{i_k}^{(a)} = x_{j_k}^{(a)}\},$$

and accordingly, $\mathrm{d}y^{(a)} = \mathrm{d}'x^{(a)}$, where $a = k - \frac{1}{2}, k$. The vector $x^{(a)}$ is in $\mathbb{R}^{2n}$, but the integrator $\mathrm{d}'x^{(a)}$ is $(2n-2)$-dimensional due to the contraction $x_{i_k}^{(a)} = x_{i_k}^{(a)}$. More explicitly,

$$\mathrm{d}'x^{(a)} := \Big(\mathrm{d}x_{i_k}^{(a)} \prod_{\ell \neq i_k, j_k} \mathrm{d}x_\ell^{(a)}\Big) = \Big(\mathrm{d}x_{j_k}^{(a)} \prod_{\ell \neq i_k, j_k} \mathrm{d}x_\ell^{(a)}\Big), \qquad a = k - \tfrac{1}{2}, k.$$

We express P as the product of two dimensional heat kernels, i.e., $P(\tau, x) = \prod_{\ell=1}^n \mathsf{p}(\tau, x_\ell)$ with $x = (x_1, \ldots, x_n)$, and similarly for $P^{\mathcal{J}}(\tau, \bullet)$ (as in (8.6)). This gives

$$\begin{aligned} D^{\overrightarrow{(i,j)}}(t,x,x') := &\int_{\Sigma_m(t)} \mathrm{d}\vec{\tau} \int \prod_{\ell=1}^n \mathsf{p}\big(\tau_0, x_\ell - x_\ell^{(1/2)}\big)\mathbf{1}\big\{x_{i_1}^{(1/2)} = x_{j_1}^{(1/2)}\big\}\,\mathrm{d}'x^{(1/2)} \\ &\cdot 4\pi\mathsf{j}(\tau_{1/2}, \beta_\star)\mathsf{p}\big(\tfrac{1}{2}\tau_{1/2}, x_{i_1}^{(1/2)} - x_{i_1}^{(1)}\big) \prod_{\ell \neq i_1, j_1} \mathsf{p}(\tau_{1/2}, x_\ell^{(1/2)} - x_\ell^{(1)})\,\mathrm{d}'x^{(1)} \\ &\cdot \prod_{k=1}^{m-1} \Bigg(\prod_{\ell=1}^n \mathbf{1}\big\{x_{i_k}^{(k)} = x_{j_k}^{(k)}\big\}\mathsf{p}\big(\tau_k, x_\ell^{(k)} - x_\ell^{(k+1/2)}\big)\mathbf{1}\big\{x_{i_{k+1}}^{(k+1/2)} = x_{j_{k+1}}^{(k+1/2)}\big\}\mathrm{d}'x^{(k)} \end{aligned} \tag{2.5}$$

$$\cdot\, 4\pi \mathsf{j}\big(\tau_{k+1/2},\beta_\star\big)\mathsf{p}\big(\tfrac12\tau_{k+1/2}, x^{(k+1/2)}_{i_{k+1}} - x^{(k+1)}_{i_{k+1}}\big)\prod_{\ell\neq i_k,j_k}\mathsf{p}\big(\tau_{k+1/2}, x^{(k+1/2)}_\ell - x^{(k+1)}_\ell\big)\,\mathrm{d}'x^{(k+1)}\Bigg)$$

$$\cdot\prod_{\ell=1}^{n}\mathsf{p}\big(\tau_m, x^{(m)}_\ell - x'_\ell\big).$$

This complicated looking formula can be conveniently recorded in terms of diagrams. Set $A := (\frac12\mathbb{Z})\cap[0,m+\frac12]$, and adopt the convention $x^{(0)} := x$ and $x^{(m+1/2)} := x'$. We schematically represent spacetime $\mathbb{R}_+\times\mathbb{R}^2$ by the plane, with the horizontal direction being the time axis $\mathbb{R}_+$, and the vertical direction representing space $\mathbb{R}^2$. We put dots on the plan representing $x^{(a)}_\ell$, $a\in A$. Dots with smaller a sit to the left of those with bigger a, and those with the same a lie on the same vertical line. The horizontal distance between $x^{(a-1/2)}_\ell$ and $x^{(a)}_\ell$, $a\in A$, represents a time lapse $\tau_a>0$. We fix the time horizon between $x_\ell = x^{(0)}_\ell$ and $x'_\ell = x^{(m+1/2)}_\ell$ to be t, which forces $\tau_0+\tau_{1/2}+\ldots+\tau_m = t$. The points $x^{(a)}_\ell$, are generically represented by distinct dots, expect that $x^{(a)}_{i_k}$ and $x^{(a)}_{j_k}$ are joined for $k=a-1/2, a$. In these cases we call the dot double, otherwise single. See Figure 1 for an example with $n=4$ and $\overrightarrow{(i,j)} = ((1<2),(2<3),(3<4))$.

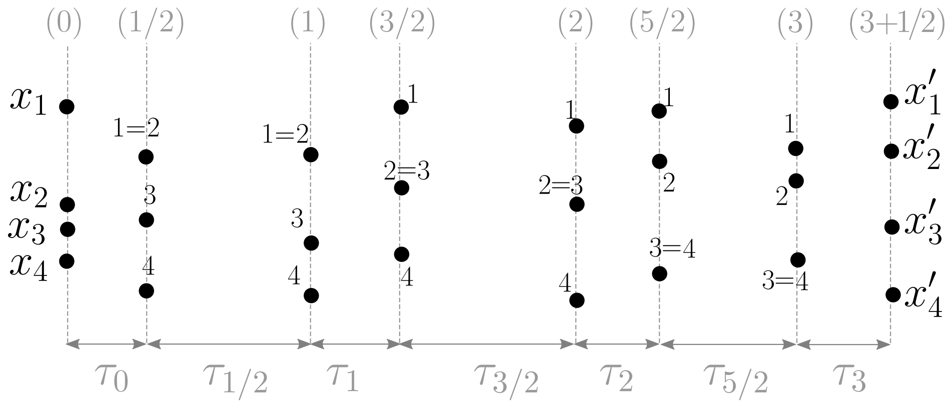

FIGURE 1. Schematic representation of $x^{(a)}_\ell$, with $n=4$ and $\vec{(i,j)} = ((1<2),(2<3),(3<4))$. Each dot represents a point $x^{(a)}_\ell$, $a\in(\frac12\mathbb{Z})\cap[0,3+\frac12]$, with the convention $x_\ell := x^{(0)}_\ell$ and $x'_\ell := x^{(3+1/2)}_\ell$. In the figure, the ℓ indices are printed in back next to the dot, while the a superscripts are put over the vertical, dashed line. The horizontal distances between dash lines represent time lapses τ_a.

Next, connect dots that represent $x^{(a-1/2)}_\ell$ and $x^{(a)}_\ell$ together, by a 'single' line except for the case when both ends are double points, by a 'double' line otherwise. To each regular line we assign a two-dimensional heat kernel $\mathsf{p}(\tau_a, x^{(a-1/2)}_\ell - x^{(a)}_\ell)$, and to each double line assign the quantity $4\pi\mathsf{j}(\tau_a,\beta_\star)\mathsf{p}(\frac12\tau_a, x^{(a-1/2)}_\ell - x^{(a)}_\ell)$. The kernel $D^{\overrightarrow{(i,j)}}(t,x,x')$ is then obtained by multiplying together the quantities assigned to the (regular and double) lines, and integrate the $x^{(a)}$'s and τ_a's, with the points $x_\ell := x^{(0)}_\ell$ and $x'_\ell = x^{(m+1/2)}_\ell$ being fixed. See Figure 2 for an example with $n=4$ and $\overrightarrow{(i,j)} = ((1<2),(2<3),(3<4))$.

In the follow two subsections, we examine the $n=2,3$ cases, and derive some useful formulas.

2.1. **The $n=2$ case.** In this case, the only index is the singleton $\overrightarrow{(i,j)} = ((1<2))$, whereby

$$\big(P+D^{\mathrm{Dgm}(2)}\big)(t,x_1,x_2,x'_1,x'_2) = \prod_{\ell=1}^{2}\mathsf{p}(t,x_\ell-x'_\ell) + \int_{\tau_0+\tau_{1/2}+\tau_1=t}\mathrm{d}\vec{\tau}\int\prod_{\ell=1}^{2}\mathsf{p}\big(\tau_0, x_\ell - x^{(1/2)}_1\big)\mathrm{d}x^{(1/2)}_1 \tag{2.6a}$$

$$\cdot\, 4\pi\mathsf{j}\big(\tau_{1/2},\beta_\star\big)\mathsf{p}\big(\tfrac12\tau_{1/2}, x^{(1/2)}_1 - x^{(1)}_1\big)\mathrm{d}x^{(1)}_1 \tag{2.6b}$$

$$\cdot\prod_{\ell=1}^{2}\mathsf{p}\big(\tau_1, x^{(1)}_1 - x'_\ell\big), \tag{2.6c}$$

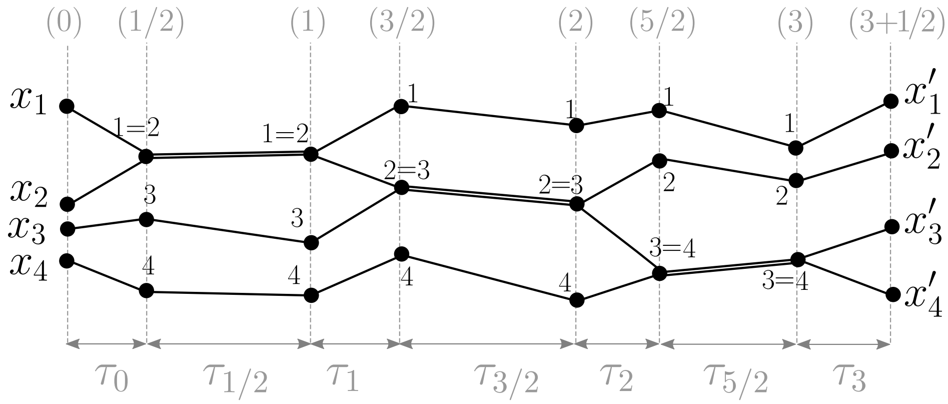

FIGURE 2. The diagram representation for $D^{(\vec{i,j})}(t,x,x')$, with $n=4$ and $(\vec{i,j}) = ((1<2),(2<3),(3<4))$. Each regular (single) line between dots is assigned $\mathsf{p}(\tau, x_\ell^{(a-1/2)} - x_\ell^{(a)})$, while each double line is assigned $4\pi\mathsf{j}(\tau,\beta_\star)\mathsf{p}(\frac{1}{2}\tau, x_\ell^{(a-1/2)} - x_\ell^{(a)})$, where $x_\ell^{(a-1/2)}$ and $x_\ell^{(a)}$ are represented by the dots at the two ends, and τ is the horizontal distance between these dots.

and the diagram of $D^{((12))}(t,x,x')$ is given in Figure 3.

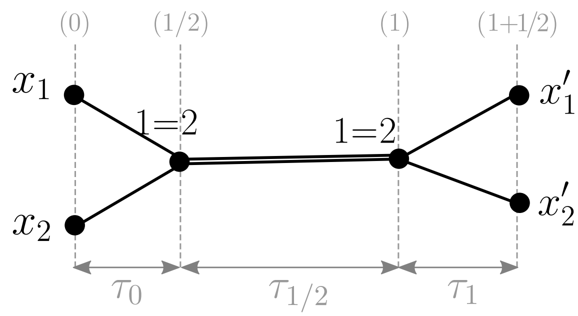

FIGURE 3. The diagram of $D^{((12))}(t,x,x')$.

In (2.6a), rewrite the products in the center-of-mass and relative coordinates,

$$\prod_{\ell=1}^{2} \mathsf{p}(\tau, x_\ell) = \mathsf{p}\big(\tfrac{1}{2}\tau, \tfrac{x_1+x_2}{2}\big)\mathsf{p}(2\tau, x_1 - x_2),$$

and then integrate over $x_1^{(1/2)}, x_1^{(1)} \in \mathbb{R}^2$, using the semigroup property of $\mathsf{p}(\bullet,\bullet)$. We then obtain

$$\begin{aligned}&\big(P + D^{\mathrm{Dgm}(2)}\big)(t, x_1, x_2, x'_1, x'_2)\\ &= \mathsf{p}\big(\tfrac{1}{2}t, x_{\mathrm{c}} - x'_{\mathrm{c}}\big)\Big(\mathsf{p}\big(2t, x_{\mathrm{d}} - x'_{\mathrm{d}}\big) + \int_{\tau_0+\tau_{1/2}+\tau_1=t} \mathrm{d}\vec{\tau}\, \mathsf{p}\big(2\tau_0, x_{\mathrm{d}}\big)\, 4\pi\mathsf{j}\big(\tau_{1/2},\beta_\star\big)\, \mathsf{p}\big(2\tau_1, x'_{\mathrm{d}}\big)\Big),\end{aligned} \tag{2.7}$$

where $x_{\mathrm{c}} := \frac{x_1+x_2}{2}$, $x_{\mathrm{d}} := x_1 - x_2$, and similarly for $x'_\bullet$.

Remark 2.1. The formula (2.7) matches [BC98, Equation (3.11)–(3.12)] after a reparametrization. Recall $\beta_\star$ from (1.8). Comparing our parameterization (1.7) and with [BC98, Equation (2.6)], we see that $\beta_\star$ here corresponds to $\log\beta$ in [BC98]. The expression in (2.7) matches [BC98, Equation (3.11)–(3.12)] upon replacing $(x_{\mathrm{d}}, x'_{\mathrm{d}}) \mapsto (x,y)$, $\beta_\star \mapsto \log\beta$, and using the identity:

$$\int_0^\tau \mathsf{p}(2(\tau-s), x_{\mathrm{d}})\mathsf{p}(2s, x'_{\mathrm{d}})\,\mathrm{d}s = \frac{1}{8\pi^2\tau}\exp\big(-\tfrac{1}{4\tau}(|x_{\mathrm{d}}|^2 + |x'_{\mathrm{d}}|^2)\big)K_0\big(\tfrac{|x_{\mathrm{d}}||x'_{\mathrm{d}}|}{2\tau}\big), \tag{2.8}$$

where K_ν denotes the modified Bessel function of the second kind.

To prove (2.8), by scaling in τ, without lost of generality we assume $\tau = 1$. On the l.h.s. of (2.8), factor out $\exp(-\frac{1}{4}(|x_{\mathrm{d}}|^2+|x'_{\mathrm{d}}|^2))$, decompose the resulting integral into $s \in (0,1/2)$ and $s \in (1/2,1)$, for the former perform the change of variable $u = (1-s)/s$, and for the latter $u = s/(1-s)$. We have

$$(\text{l.h.s. of (2.8)}) = \exp\big(-\tfrac{1}{4}(|x_{\mathrm{d}}|^2+|x'_{\mathrm{d}}|^2)\big)I_\star, \qquad I_\star := \ 2\int_1^\infty \frac{1}{(4\pi)^2 u} e^{-\frac{1}{4}(u|x_{\mathrm{d}}|^2+\frac{1}{u}|x'_{\mathrm{d}}|^2)}\,\mathrm{d}u.$$

The integrand within the last integral stays unchanged upon the change of variable $u \mapsto 1/u$, while the range maps to $(0,1)$. We hence replace $2\int_1^\infty(\,\cdot\,)\,\mathrm{d}u$ with $\int_0^\infty(\,\cdot\,)\,\mathrm{d}u$. Within the result, perform a change of variable $v = 2u|x_{\mathrm{d}}|^2$, and from the result recognize $\frac{1}{2\pi v}e^{-\frac{1}{2v}(|x_{\mathrm{d}}|^2|x'_{\mathrm{d}}|^2)} = \mathsf{p}(v, |x_{\mathrm{d}}||x'_{\mathrm{d}}|)$. We get

$$I_\star = \int_0^\infty \frac{1}{(4\pi)^2 v} e^{-\frac{|x_{\mathrm{d}}|^2|x'_{\mathrm{d}}|^2}{2v}} e^{-\frac{v}{8}}\,\mathrm{d}v = \frac{1}{8\pi}\mathsf{G}_{-\frac{1}{8}}\big(|x_{\mathrm{d}}||x'_{\mathrm{d}}|\big),$$

where $\mathsf{G}_z(|x|) = \mathsf{G}_z(x)$ denote two-dimensional Green's function. As argued in the proof of Lemma 6.2, $\mathsf{G}_z(x) = \frac{1}{\pi}K_0(\sqrt{-2z}|x|)$. This gives (2.8).

2.2. **The $n = 3$ case.** Here we derive a formula for the limiting centered third moment. We say $\overrightarrow{(i,j)} = ((i_k < j_k))_{k=1}^m \in \mathrm{Dgm}(n)$ is **degenerate** if $\cup_{k=1}^m\{i_k, j_k\} \subsetneqq \{1,\ldots,n\}$, and otherwise nondegenerate. Let $\mathrm{Dgm}'(n)$ denote the set of all nondegenerate elements of $\mathrm{Dgm}(n)$, and, accordingly,

$$\mathcal{D}_t^{\mathrm{Dgm}'(n)} := \sum\nolimits_{\overrightarrow{(i,j)}\in\mathrm{Dgm}'(n)} \mathcal{D}_t^{\overrightarrow{(i,j)}}.$$

Proposition 2.2. *Start the SHE from $Z_\varepsilon(0,\cdot) = Z_{\mathrm{ic}}(\cdot) \in \mathscr{L}^2(\mathbb{R}^2)$. For any $f \in \mathscr{L}^2(\mathbb{R}^2)$,*

$$\mathbb{E}\Big[\Big(\langle f, Z_{\varepsilon,t}\rangle - \mathbb{E}[\langle f, Z_{\varepsilon,t}\rangle]\Big)^3\Big] \longrightarrow \big\langle f^{\otimes 3}, \mathcal{D}_t^{\mathrm{Dgm}'(3)} Z_{\mathrm{ic}}^{\otimes 3}\big\rangle \quad \text{as } \varepsilon \to 0, \tag{2.9}$$

uniformly in t over compact subsets of $[0,\infty)$.

Proof. Expand the l.h.s. of (2.9) into a sum of products of $n' = 1,2,3$ moments of $\langle f, Z_{\varepsilon,t}\rangle$ as

$$\mathbb{E}\Big[\Big(\langle f, Z_{\varepsilon,t}\rangle - \mathbb{E}[\langle f, Z_{\varepsilon,t}\rangle]\Big)^3\Big] = \mathbb{E}\big[\langle f, Z_{\varepsilon,t}\rangle^3\big] - 3\mathbb{E}\big[\langle f, Z_{\varepsilon,t}\rangle^2\big]\,\mathbb{E}\big[\langle f, Z_{\varepsilon,t}\rangle\big] + 2\big(\mathbb{E}\big[\langle f, Z_{\varepsilon,t}\rangle\big]\big)^3. \tag{2.10}$$

For the $n' = 1$ moment, rewriting the SHE (1.1) in the mild (i.e., Duhamel) form and take expectation gives

$$\mathbb{E}[\langle f, Z_{\varepsilon,t}\rangle] = \langle f, \mathsf{p} * Z_{\mathrm{ic}}\rangle = \int_{\mathbb{R}^4} \overline{f(x')}\mathsf{p}(t, x'-x) Z_{\mathrm{ic}}(x)\,\mathrm{d}x\mathrm{d}x',$$

where $*$ denotes convolution in $x \in \mathbb{R}^2$. Note that for $n' = 2$ the only index $\mathrm{Dgm}(2) = \{((1<2))\}$ is the singleton and that $\langle f^{\otimes n'}, \mathcal{P}_t Z_{\mathrm{ic}}^{\otimes n'}\rangle = \langle f, \mathsf{p} * Z_{\mathrm{ic}}\rangle^{n'}$. We then have

$$\lim_{\varepsilon\to 0}\mathbb{E}\big[\langle f, Z_{\varepsilon,t}\rangle^3\big] = \big(\langle f, \mathsf{p} * Z_{\mathrm{ic}}\rangle\big)^3 + \big\langle f^{\otimes 3}, \mathcal{D}_t^{\mathrm{Dgm}(3)} Z_{\mathrm{ic}}^{\otimes 3}\big\rangle, \tag{2.11}$$

$$\lim_{\varepsilon\to 0}\mathbb{E}\big[\langle f, Z_{\varepsilon,t}\rangle^2\big] = \big(\langle f, \mathsf{p} * Z_{\mathrm{ic}}\rangle\big)^2 + \big\langle f^{\otimes 2}, \mathcal{D}_t^{((12))} Z_{\mathrm{ic}}^{\otimes 2}\big\rangle. \tag{2.12}$$

Inserting (2.11)–(2.12) into (2.10) gives

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[\Big(\langle f, Z_{\varepsilon,t}\rangle - \mathbb{E}[\langle f, Z_{\varepsilon,t}\rangle]\Big)^3\Big] = \big\langle f^{\otimes 3}, \mathcal{D}_t^{\mathrm{Dgm}(3)} Z_{\mathrm{ic}}^{\otimes 3}\big\rangle - 3\langle f, \mathsf{p} * Z_{\mathrm{ic}}\rangle\,\big\langle f^{\otimes 2}, \mathcal{D}_t^{((12))} Z_{\mathrm{ic}}^{\otimes 2}\big\rangle. \tag{2.13}$$

For $n' = 3$, degenerate indices in $\mathrm{Dgm}(3)$ are the singletons $((1<2)), ((1<3)), ((2<3))$. This being the case, we see that the last term in (2.13) exactly cancels the contribution of degenerate indices in $\big\langle f^{\otimes 3}, \mathcal{D}_t^{\mathrm{Dgm}(3)} Z_{\mathrm{ic}}^{\otimes 3}\big\rangle$. The desired result follows. ☐

2.3. **Proof of Corollary 1.3.** Here we prove Corollary 1.3 assuming Theorem 1.1 (which will be proven in Section 8). Our first goal is to show $Z_\varepsilon(t, x_1)\mathrm{d}x_1 := \mu_{\varepsilon,t,Z}(\mathrm{d}x_1)$, as a random measure on $\mathbb{R}^2$, is tight in ε, under the vague topology. By [Kal97, Lemma 14.15], this amounts to showing $\int_{\mathbb{R}^2} g(x)\mu_{\varepsilon,t,Z}(\mathrm{d}x) = \langle \overline{g}, Z_{\varepsilon,t}\rangle$ is tight (as a $\mathbb{C}$-valued random variable), for each $g \in \mathscr{C}_{\mathrm{c}}(\mathbb{R}^2)$. Apply Theorem 1.1 with $n = 2$, with $Z_{\mathrm{ic}}(x_1) \mapsto |Z_{\mathrm{ic}}(x_1)| \in \mathscr{L}^2(\mathbb{R}^2)$, and with $f(x_1,x_2) = |g(x_1)g(x_2)|$. We obtain that $\mathbb{E}[|\langle Z_{\varepsilon,t}, g\rangle|^2]$ is uniformly bounded in ε, so $\int_{\mathbb{R}^2} g(x)\mu_{\varepsilon,t,Z}(\mathrm{d}x)$ is tight.

Fixing a limit point $\mu_{*,t}$ of $\{\mu_{\varepsilon,t,Z}\}_\varepsilon$, we proceed to show (1.19). Fix a sequence $\varepsilon_k \to 0$ such that $\mu_{\varepsilon_k,t,Z} \to \mu_{*,t}$ vaguely, as $k \to \infty$. The desired result (1.19) follows from Theorem 1.1 if we can upgrade the preceding vague convergence of $\mu_{\varepsilon_k,t,Z}$ to convergence in moments. For fixed $f_1,\ldots,f_n \in \mathscr{C}_{\mathrm{c}}(\mathbb{R}^2)$, applying Theorem 1.1

with $n \mapsto 2n$, with $Z_{\mathrm{ic}}(x_1) \mapsto |Z_{\mathrm{ic}}(x_1)| \in \mathscr{L}^2(\mathbb{R}^2)$, and with $f(x_1,\ldots,x_{2n}) = \prod_{i=1}^n |f_i(x_i) f_i(x_{n+i})|$, we obtain that

$$\mathbb{E}\Big[\langle f, |Z_{\varepsilon,t}|^{\otimes 2n}\rangle\Big] = \mathbb{E}\Big[\Big|\langle f_1 \otimes \cdots \otimes f_n, Z_{\varepsilon,t}^{\otimes n}\rangle\Big|^2\Big] = \mathbb{E}\Big[\Big|\prod_{i=1}^n \int_{\mathbb{R}^2} \overline{f_i(x_i)} \mu_{\varepsilon,t,Z}(\mathrm{d}x_i)\Big|^2\Big]$$

is uniformly bounded in ε. Hence $(\prod_{i=1}^n \int_{\mathbb{R}^2} \overline{f_i(x_i)}\mu_{\varepsilon,t,Z}(\mathrm{d}x_i))$ is uniformly integrable in ε (as $\mathbb{C}$-valued random variables), which guarantees the desired convergence in moments.

We now move on to showing (1.20). For $Z_{\mathrm{ic}}(x_1), f_1(x_1) \geqslant 0$, both not identically zero, we apply Proposition 2.2 to obtain the $\varepsilon \to 0$ limit of the centered, third moment of $\int_{\mathbb{R}^2} f_1(x_1)\mu_{\varepsilon,t,Z}(\mathrm{d}x_1)$. As just argued, such a limit is also inherited by $\mu_{*,t}$, whereby

$$\mathbb{E}\Big[\Big(\int_{\mathbb{R}^2} f_1(x_1)\mu_{*,t}(\mathrm{d}x_1) - \mathbb{E}\Big[\int_{\mathbb{R}^2} f_1(x_1)\mu_{*,t}(\mathrm{d}x_1)\Big]\Big)^3\Big] = \langle f_1^{\otimes 3}, \mathcal{D}_t^{\mathrm{Dgm}'(3)} Z_{\mathrm{ic}}^{\otimes 3}\rangle. \tag{2.14}$$

As seem from (2.5), the operator $\mathcal{D}^{\overrightarrow{(i,j)}}$ has a *strictly* positive integral kernel. Under current assumption $Z_{\mathrm{ic}}(x_1), f_1(x_1) \geqslant 0$ and not identically zero, we see that the r.h.s. of (2.14) is strictly positive.

3. Resolvent identity

In this section we derive the identity (3.6) for the resolvent $\mathcal{R}_{\varepsilon,z} = (\mathcal{H}_\varepsilon - z)^{-1}$ which is the key to our analysis.

Let $\mathcal{H}_{\mathrm{fr}} := -\frac{1}{2}\sum_i \nabla_i^2$ denote the 'free Hamiltonian', and let $\mathcal{V}_\varepsilon : \mathscr{L}^2(\mathbb{R}^{2n}) \to \mathscr{L}^2(\mathbb{R}^{2n})$

$$\mathcal{V}_\varepsilon u(x) := \sum_{i<j} \delta_\varepsilon(x_i - x_j) u(x)$$

denote the operator of multiplication by the approximate delta potential, which is a bounded operator for each $\varepsilon > 0$. The Hamiltonian $\mathcal{H}_\varepsilon$ is then an unbounded operator on $\mathscr{L}^2(\mathbb{R}^{2n})$ with domain $\mathscr{H}^2(\mathbb{R}^{2n})$ (the Sobolev space), i.e.,

$$\mathcal{H}_\varepsilon := \mathcal{H}_{\mathrm{fr}} - \beta_\varepsilon \mathcal{V}_\varepsilon, \qquad \mathrm{Dom}(\mathcal{H}_\varepsilon) := \mathscr{H}^2(\mathbb{R}^{2n}) \subset \mathscr{L}^2(\mathbb{R}^{2n}). \tag{3.1}$$

The first step is to built a 'square root' of $\mathcal{V}_\varepsilon$. More precisely, we seek to construct an operator $\mathcal{S}_{\varepsilon ij}$, indexed by a pair $i < j$, and its adjoint $\mathcal{S}_{\varepsilon ij}^*$ such that $\mathcal{V}_\varepsilon = \sum_{i<j} \mathcal{S}_{\varepsilon ij}^* \mathcal{S}_{\varepsilon ij}$. To this end, for each $\varepsilon > 0$ and $1 \leqslant i < j \leqslant n$, consider the linear transformation $T_{\varepsilon ij} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$:

$$T_{\varepsilon ij}(x_1,\ldots,x_n) := \big(\tfrac{x_i - x_j}{\varepsilon}, \tfrac{x_i + x_j}{2}, x_{\overline{ij}}\big), \tag{3.2}$$

where $x_{\overline{ij}} \in \mathbb{R}^{2(n-2)}$ denotes the vector obtained by removing the i,j-th components from $x \in \mathbb{R}^{2n}$. In other words, the transformation $T_{\varepsilon ij}$ places the relative distance (on the scale of ε) and the center of mass corresponding to (x_i, x_j) in the first two components, while keeping all other components unchanged. The transformation $T_{\varepsilon ij}$ has inverse $S_{\varepsilon ij} = T_{\varepsilon ij}^{-1} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$:

$$S_{\varepsilon ij}(y_1,\ldots,y_n) := (y_3, \ldots, \underbrace{y_2 + \tfrac{\varepsilon y_1}{2}}_{i\text{-th}}, \ldots, \underbrace{y_2 - \tfrac{\varepsilon y_1}{2}}_{j\text{-th}}, \ldots, y_n). \tag{3.3}$$

Accordingly, we let $\mathcal{S}_{\varepsilon ij}$ and $\mathcal{S}_{\varepsilon ij}^*$ be the induced operators $\mathscr{L}^2(\mathbb{R}^{2n}) \to \mathscr{L}^2(\mathbb{R}^{2n})$,

$$\big(\mathcal{S}_{\varepsilon ij} u\big)(y) := u(S_{\varepsilon ij} y), \quad \big(\mathcal{S}_{\varepsilon ij}^* v\big)(x) := \varepsilon^{-2} v(T_{\varepsilon ij} x). \tag{3.4}$$

It is straightforward to check that $\mathcal{S}_{\varepsilon ij}^*$ is the adjoint of $\mathcal{S}_{\varepsilon ij}$, i.e., the unique operator for which $\langle \mathcal{S}_{\varepsilon ij}^* v, u\rangle = \langle v, \mathcal{S}_{\varepsilon ij} u\rangle$, $\forall u, v \in \mathscr{L}^2(\mathbb{R}^{2n})$. Since $S_{\varepsilon ij}, T_{\varepsilon ij}$ are both invertiable, the operators $\mathcal{S}_{\varepsilon ij}, \mathcal{S}_{\varepsilon ij}^*$ are bounded for each $\varepsilon > 0$. Φ is even and non-negative, so we can set $\phi(x) := \sqrt{\Phi}(x)$ and view $(\phi v)(y) := \phi(y_1) v(y_1, \ldots, y_n)$ as a bounded multiplication operator on $\mathscr{L}^2(\mathbb{R}^{2n})$. From (3.4), it is straightforward to check

$$\mathcal{V}_\varepsilon = \sum_{i<j} \mathcal{S}_{\varepsilon ij}^* \phi\, \phi \mathcal{S}_{\varepsilon ij}. \tag{3.5}$$

Remark 3.1. We comment on how our setup compares to that of [DR04]. They work in $\mathscr{L}^2_{\text{sym}}(\mathbb{R}^{2n})$, corresponding to n Bosons in $\mathbb{R}^2$, the key idea being to decompose the action of the delta potential $\mathcal{V}_\varepsilon$ on $\mathscr{L}^2_{\text{sym}}(\mathbb{R}^{2n})$ into some intermediate actions from $\mathscr{L}^2_{\text{sym}}(\mathbb{R}^{2n})$ into an 'auxiliary space', consisting of $n-2$ Bosons and an 'angle particle'. In our current setting, the auxiliary space is $\mathscr{L}^2(\mathbb{R}^{2n}) \ni v = v(y_1, y_2, y_3, \ldots, y_n)$. The components $y_3, \ldots, y_n$ correspond to the $n-2$ particles, the component y_2 corresponds to the angle particle, while y_1 is a 'residual' component that arises from our space-mollification scheme, and is not presented under the momentum-cutoff scheme of [DR04].

Given (3.5), the next step is to develop an expression for the resolvent $\mathcal{R}_{\varepsilon,z} = (\mathcal{H}_\varepsilon - z)^{-1}$ that is amenable for the $\varepsilon \to 0$ asymptotic. In the case of momentum cutoff, such a resolvent expression is obtained in (Eq (68) of) [DR04] by comparing two different ways of inverting a two-by-two (operator-valued) matrix. Here, we derive the analogous expression (i.e., (3.6)) using a more straightforward procedure — power-series expansion of (operator-valued) geometric series. Recall $\text{Dgm}(n,m)$ from (1.13), recall that $\|\mathcal{Q}\|_{\text{op}}$ denotes the operator norm of $\mathcal{Q}$, and recall from (1.21) that $\mathcal{G}_z$ denotes the resolvent of the Laplacian.

Lemma 3.2. *For all* $\varepsilon \in (0,1)$ *and* $z \in \mathbb{C}$ *such that* $\text{Re}(z) < -\beta_\varepsilon(1 + \sum_{i<j} \|\mathcal{S}_{\varepsilon ij}\phi\|_{\text{op}})^2$, *we have*

$$\mathcal{R}_{\varepsilon,z} := (\mathcal{H}_\varepsilon - z\mathbf{I})^{-1} = \mathcal{G}_z + \sum_{m=1}^{\infty} \sum_{\overrightarrow{(i,j)} \in \text{Dgm}(n,m)} \big(\mathcal{G}_z \mathcal{S}^*_{\varepsilon i_1 j_1}\phi\big) \tag{3.6a}$$

$$\cdot \big(\beta_\varepsilon^{-1}\mathbf{I} - \phi\mathcal{S}_{\varepsilon 12}\mathcal{G}_z\mathcal{S}^*_{\varepsilon 12}\phi\big)^{-1} \prod_{k=2}^{m} \Big(\big(\phi\mathcal{S}_{\varepsilon i_{k-1}j_{k-1}}\mathcal{G}_z\mathcal{S}^*_{\varepsilon i_k j_k}\phi\big) \big(\beta_\varepsilon^{-1}\mathbf{I} - \phi\mathcal{S}_{\varepsilon 12}\mathcal{G}_z\mathcal{S}^*_{\varepsilon 12}\phi\big)^{-1} \Big) \tag{3.6b}$$

$$\cdot \big(\phi\mathcal{S}_{i_m j_m}\mathcal{G}_z\big). \tag{3.6c}$$

Remark 3.3. As stated, Lemma 3.2 holds for $\text{Re}(z) < -C_1(\varepsilon, n)$, with a threshold $C_1(\varepsilon, n)$ that depends on ε. This may not seem useful as $\varepsilon \to 0$, however, as we will show later in Section 7, the r.h.s. of (3.6) is actually analytic (in norm) in $\{z : \text{Re}(z) < -C_2(n)\}$, for some threshold $C_2(n) < \infty$ that is *independent* of ε. It then follows immediately (as argued in Section 7) that (3.6) extends to all $\text{Re}(z) < -C_2(n)$.

Proof. To simplify notation, set $\widetilde{\mathcal{S}}_{ij} := \beta_\varepsilon^{1/2}\phi\mathcal{S}_{\varepsilon ij}$, $\widetilde{\mathcal{S}}^{ij} := (\widetilde{\mathcal{S}}_{ij})^* = \beta_\varepsilon^{1/2}\mathcal{S}^*_{\varepsilon ij}\phi$, and $\widetilde{\mathcal{G}}^{k\ell}_{ij} := \widetilde{\mathcal{S}}_{ij}\mathcal{G}_z\widetilde{\mathcal{S}}^{k\ell}$. In (3.6b), factor β_ε^{-1} from the inverse. Under the preceding shorthand notation, we rewrite (3.6) as

$$\mathcal{R}_{\varepsilon,z} = \mathcal{G}_z + \sum_{m=1}^{\infty} \sum_{\overrightarrow{(i,j)} \in \text{Dgm}(n,m)} \mathcal{G}_z\widetilde{\mathcal{S}}^{i_1j_1} \cdot \big(\mathbf{I} - \widetilde{\mathcal{G}}^{12}_{12}\big)^{-1} \prod_{k=2}^{m} \widetilde{\mathcal{G}}^{i_kj_k}_{i_{k-1}j_{k-1}} \big(\mathbf{I} - \widetilde{\mathcal{G}}^{12}_{12}\big)^{-1} \cdot \widetilde{\mathcal{S}}_{i_mj_m}\mathcal{G}_z. \tag{3.7}$$

Our goal is to expand the inverse in (3.7), and then simplify the result to match $(\mathcal{H}_\varepsilon - z\mathbf{I})^{-1}$.

To expand the inverse in (3.7), we utilize the geometric series $(\mathbf{I} - \mathcal{Q})^{-1} = \mathbf{I} + \sum_{k=1}^\infty \mathcal{Q}^k$, valid for $\|\mathcal{Q}\|_{\text{op}} < 1$. Indeed, $\|\mathcal{G}_z\|_{\text{op}} \leqslant 1/(-\text{Re}(z))$, so under the assumption on the range of $\text{Re}(z)$ we have $\|\widetilde{\mathcal{S}}^{12}_{12}\|_{\text{op}} < 1$. Using the geometric series for $\mathcal{Q} = \widetilde{\mathcal{G}}^{12}_{12}$, and inserting the result into (3.7) gives

$$\mathcal{R}_{\varepsilon,z} = \mathcal{G}_z + \sum \mathcal{G}_z\widetilde{\mathcal{S}}^{i_1j_1} \underbrace{\widetilde{\mathcal{G}}^{12}_{12}\cdots\widetilde{\mathcal{G}}^{12}_{12}}_{\ell_1} \widetilde{\mathcal{G}}^{i_2j_2}_{i_1j_1} \underbrace{\widetilde{\mathcal{G}}^{12}_{12}\cdots\widetilde{\mathcal{G}}^{12}_{12}}_{\ell_2} \widetilde{\mathcal{G}}^{i_3j_3}_{i_2j_2} \cdots \widetilde{\mathcal{G}}^{i_mj_m}_{i_{m-1}j_{m-1}} \underbrace{\widetilde{\mathcal{G}}^{12}_{12}\cdots\widetilde{\mathcal{G}}^{12}_{12}}_{\ell_m} \widetilde{\mathcal{S}}_{i_mj_m}\mathcal{G}_z. \tag{3.8}$$

where the sum is over $\ell_1, \ldots, \ell_m \geqslant 0$, $\overrightarrow{(i,j)} \in \text{Dgm}(n,m)$, and $m = 1, 2, \ldots$. The sum converges absolutely in operator norm by our assumption on z. Since $\mathcal{G}_z$ acts symmetrically in the n components, we have $\widetilde{\mathcal{G}}^{12}_{12} = \widetilde{\mathcal{G}}^{ij}_{ij}$, for any pair $i < j$. Use this property to rewrite (3.8) as

$$\mathcal{R}_{\varepsilon,z} = \mathcal{G}_z + \sum \mathcal{G}_z\widetilde{\mathcal{S}}^{i_1j_1} \underbrace{\widetilde{\mathcal{G}}^{i_1j_1}_{i_1j_1}\cdots\widetilde{\mathcal{G}}^{i_1j_1}_{i_1j_1}}_{\ell_1} \widetilde{\mathcal{G}}^{i_2j_2}_{i_1j_1} \underbrace{\widetilde{\mathcal{G}}^{i_2j_2}_{i_2j_2}\cdots\widetilde{\mathcal{G}}^{i_2j_2}_{i_2j_2}}_{\ell_2} \widetilde{\mathcal{G}}^{i_3j_3}_{i_2j_2} \cdots \widetilde{\mathcal{G}}^{i_mj_m}_{i_{m-1}j_{m-1}} \underbrace{\widetilde{\mathcal{G}}^{i_mj_m}_{i_mj_m}\cdots\widetilde{\mathcal{G}}^{i_mj_m}_{i_mj_m}}_{\ell_m} \widetilde{\mathcal{S}}_{i_mj_m}\mathcal{G}_z. \tag{3.9}$$

The summation can be reorganized as $\sum_{m'=1}^\infty \sum_{i_1<j_1} \cdots \sum_{i_{m'}<j_{m'}} (\,\cdot\,)$. To see this, recall from (1.13) that $\overrightarrow{(i,j)} \in \text{Dgm}(n,m)$ consists of pairs $(i_k < j_k)$ under the constraint that consecutive pairs are non-repeating, i.e., $(i_{k-1} < j_{k-1}) \neq (i_k < j_k)$. The r.h.s. of (3.9) replenishes all possible repeatings of consecutive pairs, and hence lifts the constraints imposed by $\text{Dgm}(n,m)$. In the resulting sum, express $\widetilde{\mathcal{G}}^{ij}_{k\ell} = \widetilde{\mathcal{S}}^{ij}\mathcal{G}_z\widetilde{\mathcal{S}}_{k\ell}$ to get

$$\mathcal{R}_{\varepsilon,z} = \sum_{m=0}^{\infty} \mathcal{G}_z \Big(\sum_{i<j} \widetilde{\mathcal{S}}^{ij}\widetilde{\mathcal{S}}_{ij}\mathcal{G}_z \Big)^m.$$

From (3.5), we have $\sum_{i<j} \widetilde{\mathcal{S}}^{ij}\widetilde{\mathcal{S}}_{ij} = \beta_\varepsilon \mathcal{V}_\varepsilon$, hence $\mathcal{R}_{\varepsilon,z} = \mathcal{G}_z(\mathbf{I} - \beta_\varepsilon\mathcal{V}_\varepsilon\mathcal{G}_z)^{-1}$. Further $\mathcal{G}_z = (\mathcal{H}_{\mathrm{fr}} - z\mathbf{I})^{-1}$ gives

$$\mathcal{R}_{\varepsilon,z} = (\mathcal{H}_{\mathrm{fr}} - z\mathbf{I})^{-1}\big(\mathbf{I} - \beta_\varepsilon\mathcal{V}_\varepsilon(\mathcal{H}_{\mathrm{fr}} - z\mathbf{I})^{-1}\big)^{-1} = \big(\mathcal{H}_{\mathrm{fr}} - z\mathbf{I} - \beta_\varepsilon\mathcal{V}_\varepsilon\big)^{-1} = \big(\mathcal{H}_\varepsilon - z\mathbf{I}\big)^{-1}.$$

This completes the proof. □

The resolvent identity (3.6) is the gateway to the $\varepsilon \to 0$ limit. Roughly speaking, we will show that all terms in (3.6) converge to their limiting counterparts in the expression of $\mathcal{R}_z$ given in (1.23). The expression (1.23), however, does not expose such a convergence very well. This is so because some operators in (1.23) map one function space to a different one, (e.g., $\mathcal{S}_{ij}$ maps functions of n components to $n-1$ components), while the operators in (3.6) always maps $\mathscr{L}^2(\mathbb{R}^{2n})$ to $\mathscr{L}^2(\mathbb{R}^{2n})$. We next rewrite (1.23) in a way that better compares with (3.6). To this end, consider the operators

$$\Omega_\phi : \mathscr{L}^2(\mathbb{R}^{2n}) \to \mathscr{L}^2(\mathbb{R}^{2n-2}), \quad (\Omega_\phi v)(y_{2-n}) := \int_{\mathbb{R}^2} \phi(y_1)v(y_1, y_{2-n})\mathrm{d}y_1 \tag{3.10}$$

$$\phi \otimes \cdot : \mathscr{L}^2(\mathbb{R}^{2n-2}) \to \mathscr{L}^2(\mathbb{R}^{2n}), \quad (\phi \otimes v)(y_1, y_{2-n}) := \phi(y_1)v(y_{2-n}). \tag{3.11}$$

Given that $\phi \in \mathscr{C}_c^\infty(\mathbb{R}^2)$, it is readily checked that Ω_ϕ and $\phi \otimes \cdot$ are bounded operators. Note that from $\phi := \sqrt{\Phi}$, ϕ has unit norm, i.e., $\int_{\mathbb{R}^2} \phi^2 \mathrm{d}y = 1$. From this we obtain $\Omega_\phi(\phi \otimes \mathcal{Q}) = \mathcal{Q}$, for a generic $\mathcal{Q} : \mathscr{L}^2(\mathbb{R}^{2n}) \to \mathscr{L}^2(\mathbb{R}^{2n-2})$ or $\mathcal{Q} : \mathscr{L}^2(\mathbb{R}^{2n-2}) \to \mathscr{L}^2(\mathbb{R}^{2n-2})$. Using this property, we rewrite (1.23) as

$$\mathcal{R}_z = \mathcal{G}_z + \sum_{m=1}^{\infty} \sum_{\overrightarrow{(i,j)} \in \mathrm{Dgm}(n,m)} \big(\mathcal{G}_z\mathcal{S}^*_{i_1j_1}\Omega_\phi\big) \tag{3.12a}$$

$$\cdot \big(\phi \otimes 4\pi(\mathcal{J}_z - \beta_\star\mathbf{I})^{-1}\Omega_\phi\big) \prod_{s=2}^{m} \Big(\big(\phi \otimes \mathcal{S}_{i_{s-1}j_{s-1}}\mathcal{G}_z\mathcal{S}^*_{i_sj_s}\Omega_\phi\big)\big(\phi \otimes 4\pi(\mathcal{J}_z - \beta_\star\mathbf{I})^{-1}\Omega_\phi\big)\Big) \tag{3.12b}$$

$$\cdot \big(\phi \otimes \mathcal{S}_{ij}\mathcal{G}_z\big). \tag{3.12c}$$

That is, we augment the missing y_1 dependence (in the operators $\mathcal{S}_{ij}$, $\mathcal{S}^*_{ij}$, etc.) along the subspace $\mathbb{C}\phi \subset \mathscr{L}^2(\mathbb{R}^2)$. Equation (3.12) gives a better expression for comparison with (3.6).

For future references, let us setup some terminology for the operators in (3.6) and (3.12). We call the operators $\mathcal{S}_{ij}\mathcal{G}_z$ or $\phi \otimes \mathcal{S}_{ij}\mathcal{G}_z$ in (3.12c) the **limiting incoming operators**, and the operators $\mathcal{G}_z\mathcal{S}_{ij}$ or $\mathcal{G}_z\mathcal{S}^*_{ij}\Omega_\phi$ in (3.12c) the **limiting outgoing operators**. Slightly abusing language, we will use these phrases interchangeably to infer operators *with and without* the action by $\phi\otimes\cdot$ or Ω_ϕ. Similarly, we call the operators in (3.6c) the **pre-limiting incoming operators**, and the operators in (3.6a) the **pre-limiting outgoing operators**. Next, with $\mathcal{J}_z$ defined in (1.22) in the following, we refer to $(\mathcal{J}_z - \beta_\star\mathbf{I})$ and $(\beta_\varepsilon^{-1}\mathbf{I} - \mathcal{S}_{\varepsilon 12}\mathcal{G}_z\mathcal{S}^*_{\varepsilon 12})$ as the **limiting** and **pre-limiting diagonal mediating operators**, respectively, and refer to $\mathcal{S}_{ij}\mathcal{G}_z\mathcal{S}^*_{k\ell}$ and $\mathcal{S}_{\varepsilon ij}\mathcal{G}_z\mathcal{S}^*_{\varepsilon k\ell}$, with $(i<j) \neq (k<\ell)$, as the the **limiting** and **pre-limiting off-diagonal mediating operators**.

As we will show in Section 4, each pre-limiting incoming and outgoing operator converges to its limiting counterpart, and, as will show in Section 5, each off-diagonal mediating operator converges to its limiting counterpart. Diagonal mediating operators require a more delicate treatment because $\beta_\varepsilon^{-1}\mathbf{I}$ and $\mathcal{S}_{\varepsilon ij}\mathcal{G}_z\mathcal{S}^*_{\varepsilon ij}$ both diverge on their own, and we need to cancel the divergence (and also to take an inverse) to obtain a limit. This procedure, sometimes referred to as *renormalization* in the physics literature, will be carried out in Section 6.

4. Incoming and outgoing operators

In this section we obtain the $\varepsilon \to 0$ limit of $\phi\mathcal{S}_{\varepsilon ij}\mathcal{G}_z$ and $\mathcal{G}_z\mathcal{S}^*_{\varepsilon ij}\phi$ to $\phi \otimes (\mathcal{S}_{ij}\mathcal{G}_z)$ and $\mathcal{G}_z\mathcal{S}^*_{ij}\Omega_\phi$. The main result is stated in Lemma 4.4.

Recall the linear transformation S_{ij} and its induced operator $\mathcal{S}_{ij}$ from (1.10)–(1.11). Comparing (3.3) and (1.10), we see that $S_{\varepsilon ij}(y_1, \ldots, y_n) \to S_{ij}(y_2, \ldots, y_n)$ as $\varepsilon \to 0$. Namely, S_{ij} is the pointwise limit of $S_{\varepsilon ij}$. This observation hints that $\mathcal{S}_{ij}$ should be the limit of $\mathcal{S}_{\varepsilon ij}$, and the $\varepsilon \to 0$ limit of the incoming operator $\phi\mathcal{S}_{\varepsilon ij}\mathcal{G}_z$ should be obtained by replacing $\mathcal{S}_{ij}$ with $\mathcal{S}_{\varepsilon ij}$. Note that, however, the operator $\mathcal{S}_{ij}$ is *unbounded*, because, unlike $S_{\varepsilon ij}$, S_{ij}, maps between spaces of *different* dimensions; the y_1 dependence in $S_{\varepsilon ij}(y_1, \ldots, y_n)$ 'vanishes' as $\varepsilon \to 0$ (c.f., (3.3)).

As the first step of building the limiting operators, we construct the domain of $\mathcal{S}_{ij}$, along with its adjoint $\mathcal{S}_{ij}^*$. In the following we will often work in the Fourier domain. Let $\widehat{f}(q) := \int_{\mathbb{R}^d} e^{-\mathbf{i}y\cdot q} f(y) \frac{\mathrm{d}q}{(2\pi)^{d/2}}$ denote Fourier transform of functions on $\mathbb{R}^d$; the inverse Fourier transform then reads $f(y) = \int_{\mathbb{R}^d} e^{\mathbf{i}y\cdot q}\widehat{f}(q)\frac{\mathrm{d}q}{(2\pi)^{d/2}}$. Let $\mathscr{S}(\mathbb{R}^d)$ denote the space of Schwartz functions. In our subsequential analysis, d is typically $2n$ or $2(n-1)$. Consider the (invertible) linear transformation $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$:

$$M_{ij}q := (q_3, \ldots, \underbrace{\tfrac{1}{2}q_2 + q_1}_{i\text{-th}}, \ldots, \underbrace{\tfrac{1}{2}q_2 - q_1}_{j\text{-th}}, \ldots, q_n). \tag{4.1}$$

For $q \in \mathbb{R}^{2n}$, we write $q_{i-j} := (q_i, \ldots, q_j) \in \mathbb{R}^{2(j-i+1)}$, and recall that $q_{\overline{ij}} \in \mathbb{R}^{2n-4}$ is obtained from removing the i-th and j-th components of q.

Lemma 4.1.

(a) The operator $\mathcal{S}_{ij}$, given by equation (1.11), is unbounded from $\mathscr{L}^2(\mathbb{R}^{2n})$ to $\mathscr{L}^2(\mathbb{R}^{2n-2})$, with

$$\mathrm{Dom}(\mathcal{S}_{ij}) := \Big\{ f \in \mathscr{L}^2(\mathbb{R}^{2n}) : \int_{\mathbb{R}^2} \big|\widehat{f}(M_{ij}(q_1, \bullet))\big| \mathrm{d}q_1 \in \mathscr{L}^2(\mathbb{R}^{2n-2}) \Big\} \subset \mathscr{L}^2(\mathbb{R}^{2n}), \tag{4.2}$$

and for $f \in \mathrm{Dom}(\mathcal{S}_{ij})$, we have

$$\widehat{\mathcal{S}_{ij}f}(q_{2-n}) = \int_{\mathbb{R}^2} \widehat{f}(M_{ij}q)\frac{dq_1}{2\pi}. \tag{4.3}$$

In addition, for all $a > 1$, we have $\mathscr{H}^a(\mathbb{R}^{2n}) \subset \mathrm{Dom}(\mathcal{S}_{ij})$.

(b) The operator

$$\widehat{\mathcal{S}_{ij}^* g}(p) := \tfrac{1}{2\pi}\widehat{g}(p_i + p_j, p_{\overline{ij}}) \tag{4.4}$$

maps $\mathscr{L}^2(\mathbb{R}^{2n-2}) \to \cap_{a>1}\mathscr{H}^{-a}(\mathbb{R}^{2n})$, and is adjoint to $\mathcal{S}_{ij}$ in the sense that

$$\langle \mathcal{S}_{ij}^* g, f\rangle = \langle g, \mathcal{S}_{ij} f\rangle, \quad g \in \mathscr{L}^2(\mathbb{R}^{2n-2}),\ f \in \mathscr{H}^a(\mathbb{R}^{2n}),\ a > 1. \tag{4.5}$$

Proof. (a) Let us first show (4.3) for $f \in \mathscr{S}(\mathbb{R}^{2n})$. On the Fourier transform of f, perform the change of variables $x = S_{1ij}y$, where $S_{1ij} = S_{\varepsilon ij}|_{\varepsilon=1}$, and the substitute $p = M_{ij}q$. From (3.3), it is readily checked that $|\det(S_{1ij})| = 1$, and from (4.1), we have $(S_{1ij}y)\cdot(M_{ij}q) = y \cdot q$, so

$$\widehat{f}(M_{ij}q) = \int_{\mathbb{R}^{2n}} e^{-\mathbf{i}y\cdot q} f(S_{1ij}y)\frac{\mathrm{d}y}{(2\pi)^n}. \tag{4.6}$$

Our goal is to calculate the Fourier transform of $f(S_{ij}\bullet)$. Comparing (1.10) and (3.3) for $\varepsilon = 1$, we see that $(S_{1ij}y)|_{y_1=0} = S_{ij}(y_{2-n})$. It is hence desirable to 'remove' the y_1 variable on the r.h.s. of (4.6). To this end, apply the identity

$$\int_{\mathbb{R}^{2n-2}} g(0, y_{2-n}) e^{-\mathbf{i}q_{2-n}\cdot y_{2-n}} \frac{\mathrm{d}y_{2-n}}{(2\pi)^{n-1}} = \int_{\mathbb{R}^2} \widehat{g}(q)\frac{\mathrm{d}q_1}{2\pi}, \quad g \in \mathscr{S}(\mathbb{R}^{2n})$$

with $g(\bullet) = f(S_{1ij}\bullet)$ to obtain

$$\int_{\mathbb{R}^2} \widehat{f}(M_{ij}q)\frac{\mathrm{d}q_1}{2\pi} = \int_{\mathbb{R}^{2n-2}} e^{-\mathbf{i}y_{2-n}\cdot q_{2-n}} f(S_{1ij}y)\big|_{y_1=0}\frac{\mathrm{d}y_{2-n}}{(2\pi)^{n-1}} = \int_{\mathbb{R}^{2n-2}} e^{-\mathbf{i}y_{2-n}\cdot q_{2-n}} f(S_{ij}y_{2-n})\frac{\mathrm{d}y_{2-n}}{(2\pi)^{n-1}}.$$

The last expression is $\widehat{\mathcal{S}_{ij}f}(q_{2-n})$ by definition. We hence conclude (4.3) for $f \in \mathscr{S}(\mathbb{R}^{2n})$. By approximation, it follows that $\mathcal{S}_{ij}$ extends to an unbounded operator with domain (4.2), and the identity (4.3) extends to $f \in \mathrm{Dom}(\mathcal{S}_{ij})$.

Fix $a > 1$, we proceed to show $\mathscr{H}^a(\mathbb{R}^{2n}) \subset \mathrm{Dom}(\mathcal{S}_{ij})$. For $f \in \mathscr{H}^a(\mathbb{R}^{2n})$, it suffices to bound

$$\int_{\mathbb{R}^{2n-2}} \Big| \int_{\mathbb{R}^2} \big|\widehat{f}(M_{ij}q)\big|\mathrm{d}q_1 \Big|^2 \mathrm{d}q_{2-n}. \tag{4.7}$$

Within the integrals, multiply and divide by $(\frac{1}{2}|M_{ij}q|^2+1)^{\frac{a}{2}}$. Use $\frac{1}{2}|M_{ij}q|^2 \geqslant |q_1|^2$ (as readily checked from (4.1)) and apply the Cauchy–Schwart inequality over the integral in q_1. We then obtain

$$\begin{aligned}(4.7) &= \int_{\mathbb{R}^{2n-2}} \Big| \int_{\mathbb{R}^2} \frac{1}{(\frac{1}{2}|M_{ij}q|^2+1)^{\frac{a}{2}}} (\tfrac{1}{2}|M_{ij}q|^2+1)^{\frac{a}{2}} \big|\widehat{f}(M_{ij}q)\big| \mathrm{d}q_1 \Big|^2 \mathrm{d}q_{2-n} \\ &\leqslant \int_{\mathbb{R}^2} \Big(\frac{1}{|q_1|^2+1}\Big)^a \mathrm{d}q_1 \, \|f\|_{\mathscr{H}^a(\mathbb{R}^{2n})} \leqslant \frac{C}{a-1} \|f\|_{\mathscr{H}^a(\mathbb{R}^{2n})}.\end{aligned} \tag{4.8}$$

This verifies $\mathscr{H}^a(\mathbb{R}^{2n}) \subset \mathrm{Dom}(\mathcal{S}_{ij})$.
(b) That $\mathcal{S}^*_{ij}$ maps $\mathscr{L}^2(\mathbb{R}^{2n-2})$ to $\cap_{a>1}\mathscr{H}^{-a}(\mathbb{R}^{2n})$ is checked by similar calculations as in (4.8). To check (4.5), calculate the inner product $\langle \mathcal{S}^*_{ij} g, f\rangle$ in Fourier variables from (4.4). Within the resulting integral, perform a change of variable $p = M_{ij}q$, and use $|\det(M_{ij})| = 1$ and $(p_i + p_j, p_{\overline{ij}}) = [M^{-1}_{ij} p]_{2-n}$ (as readily checked from (4.1)). We then obtain

$$\langle \mathcal{S}^*_{ij} g, f\rangle = \int_{\mathbb{R}^{2n}} \overline{\widehat{g}(p_i+p_j, p_{\overline{ij}})} \widehat{f}(p) \frac{\mathrm{d}p}{2\pi} = \int_{\mathbb{R}^{2n}} \overline{\widehat{g}(q_{2-n})} \widehat{f}(M_{ij}q) \frac{\mathrm{d}q}{2\pi}.$$

From (4.3), we see that the last expression matches $\langle g, \mathcal{S}_{ij} f\rangle$. ☐

Recall that, for each $\mathrm{Re}(z) < 0$, $\mathcal{G}_z(\mathscr{L}^2(\mathbb{R}^{2n})) = \mathscr{H}^2(\mathbb{R}^{2n})$. This together with Lemma 4.1 implies that $\mathcal{S}_{ij}\mathcal{G}_z$ is defined on the entire $\mathscr{L}^2(\mathbb{R}^{2n})$, with image in $\mathscr{L}^2(\mathbb{R}^{2n-2})$, and that $\mathcal{G}_z\mathcal{S}^*_{ij}$ is defined on $\mathscr{L}^2(\mathbb{R}^{2n-2})$, with image in $\mathscr{L}^2(\mathbb{R}^{2n})$. Informally, $\mathcal{G}_z$ increases regularity by 2, while $\mathcal{S}_{ij}$ and $\mathcal{S}^*_{ij}$ both decrease regularity by $-(1^+)$, as seen from Lemma 4.1. In total $\mathcal{S}_{ij}\mathcal{G}_z$ and $\mathcal{G}_z\mathcal{S}^*_{ij}$ have regularity exponent $2-(1^+) = 1^- > 0$.

We now establish a quantitative bound on the operator norm of the limiting operators $\mathcal{S}_{ij}\mathcal{G}_z$ and $\mathcal{G}_z\mathcal{S}^*_{ij}$.

Lemma 4.2. *For $1 \leqslant i < j \leqslant n$ and $\mathrm{Re}(z) < 0$, $\|\mathcal{S}_{ij}\mathcal{G}_z\|_{\mathrm{op}} = \|\mathcal{G}_{\overline{z}}\mathcal{S}^*_{ij}\|_{\mathrm{op}} \leqslant C\,(\mathrm{Re}(-z))^{-1/2}$.*

Proof. That $\|\mathcal{S}_{ij}\mathcal{G}_z\|_{\mathrm{op}} = \|\mathcal{G}_{\overline{z}}\mathcal{S}^*_{ij}\|_{\mathrm{op}}$ follows by (4.5), so it is enough to bound $\|\mathcal{S}_{ij}\mathcal{G}_z\|_{\mathrm{op}}$. Fix $u \in \mathscr{L}^2(\mathbb{R}^{2n})$ and apply (4.3) for $u' = \mathcal{G}_z u$ to get

$$\widehat{\mathcal{S}_{ij}\mathcal{G}_z u}(q_{2-n}) = \int_{\mathbb{R}^2} \frac{\widehat{u}(M_{ij}q)}{\frac{1}{2}|M_{ij}q|^2 - z} \frac{\mathrm{d}q_1}{2\pi}. \tag{4.9}$$

Calculate the norm of $\mathcal{S}_{ij}\mathcal{G}_z u$ from (4.9) and by the same argument as in (4.8) we get

$$\|\mathcal{S}_{ij}\mathcal{G}_z u\|^2 = \int_{\mathbb{R}^{2n-2}} \Big| \int_{\mathbb{R}^2} \frac{\widehat{u}(M_{ij}q)}{\frac{1}{2}|M_{ij}q|^2 - z} \frac{\mathrm{d}q_1}{2\pi} \Big|^2 \mathrm{d}q_{2-n} \leqslant \int_{\mathbb{R}^{2n-2}} \Big| \int_{\mathbb{R}^2} \frac{\widehat{u}(M_{ij}q)}{|q_1|^2 - z} \frac{\mathrm{d}q_1}{2\pi} \Big|^2 \mathrm{d}q_{2-n},$$

Applying the Cauchy–Schwarz inequality over the q_1 integration, we conclude

$$\|\mathcal{S}_{ij}\mathcal{G}_z u\|^2 \leqslant \Big(\int_{\mathbb{R}^2} \Big| \frac{1}{|q_1|^2 - z} \Big|^2 \frac{\mathrm{d}q_1}{(2\pi)^2} \Big) \|u\|^2 \leqslant \Big(\int_{\mathbb{R}^2} \frac{1}{(|q_1|^2 + \mathrm{Re}(-z))^2} \frac{\mathrm{d}q_1}{(2\pi)^2} \Big) \|u\|^2.$$

The last integral over q_1 can be evaluated in polar coordinate to be $\frac{1}{4\pi}\mathrm{Re}(-z)$. This completes the proof. ☐

Having built the limiting operator, our next step is to show the convergence. In the course of doing so, we will often use a partial Fourier transform in the last $n-1$ components:

$$\overline{\overline{f}}(y_1, q_{2-n}) := \int_{\mathbb{R}^{2n-2}} e^{-\mathbf{i}(y_2,\ldots,y_d)\cdot(q_2,\ldots,q_n)} f(y_1,\ldots,y_n) \prod_{i=2}^{n} \frac{\mathrm{d}y_i}{2\pi}. \tag{4.10}$$

Recall $\mathcal{S}_{\varepsilon ij}$ from (3.4). To prepare for the proof of the convergence, we establish an expression of $\mathcal{S}_{\varepsilon ij} u$ in partial Fourier variables.

Lemma 4.3. *For every $1 \leqslant i < j \leqslant n$ and $u \in \mathscr{S}(\mathbb{R}^{2n})$, we have*

$$\overline{\overline{\mathcal{S}_{\varepsilon ij} u}}(y_1, q_{2-n}) = \int_{\mathbb{R}^2} e^{\mathbf{i}\varepsilon q_1\cdot y_1} \widehat{u}(M_{ij}q) \frac{dq_1}{2\pi}. \tag{4.11}$$

Proof. A partial Fourier transform can be obtained by inverting a full transform in the first component:

$$\overline{\overline{\mathcal{S}_{\varepsilon ij} f}}(y_1, q_{2-n}) = \int_{\mathbb{R}^2} \widehat{\mathcal{S}_{\varepsilon ij} f}(q) e^{\mathbf{i} y_1\cdot q_1} \frac{\mathrm{d}q_1}{2\pi}. \tag{4.12}$$

We write the full Fourier transform as $\widehat{\mathcal{S}_{\varepsilon ij}f}(q)=\int_{\mathbb{R}^{2n}}e^{-\mathbf{i}y\cdot q}f(S_{\varepsilon ij}y)\frac{\mathrm{d}y}{(2\pi)^n}$. We wish to perform a change of variable $x=S_{\varepsilon ij}y$. Doing so requires understanding how $(y\cdot q)$ transform accordingly. Defining

$$M_{\varepsilon ij}q:=(q_3,\ldots,\underbrace{\tfrac{1}{2}q_2+\varepsilon^{-1}q_1}_{i\text{-th}},\ldots,\underbrace{\tfrac{1}{2}q_2-\varepsilon^{-1}q_1}_{j\text{-th}},\ldots,q_n),$$

it is readily checked that $y\cdot q=(M_{\varepsilon ij}q)\cdot(S_{\varepsilon ij}y)$. Given this, we perform the change of variable $x=S_{\varepsilon ij}y$. With $|\det(S_{\varepsilon ij})|=\varepsilon^2$, we now have

$$\widehat{\mathcal{S}_{\varepsilon ij}f}(q)=\varepsilon^{-2}\int_{\mathbb{R}^{2n}}e^{-\mathbf{i}(M_{\varepsilon ij}q)\cdot x}f(x)\frac{\mathrm{d}x}{(2\pi)^n}=\varepsilon^{-2}\widehat{f}(M_{\varepsilon ij}q).\tag{4.13}$$

Inserting (4.13) into the r.h.s. of (4.12), and performing a change of variable $q_1\mapsto\varepsilon q_1$, under which $M_{\varepsilon ij}q\mapsto M_{ij}q$, we conclude the desired result (4.11). □

We now show the convergence. Recall Ω_ϕ from (3.10).

Lemma 4.4. *For each $i<j$ and $\mathrm{Re}(z)<0$, we have*

$$\Big\|\phi\mathcal{S}_{\varepsilon ij}\mathcal{G}_z-\phi\otimes(\mathcal{S}_{ij}\mathcal{G}_z)\Big\|_{\mathrm{op}}+\Big\|\mathcal{G}_z\mathcal{S}^*_{\varepsilon ij}\phi-\mathcal{G}_z\mathcal{S}^*_{ij}\Omega_\phi\Big\|_{\mathrm{op}}\leqslant C\,\varepsilon^{\frac{1}{2}}\,(-\mathrm{Re}(z))^{-1/4}\longrightarrow 0,\quad \textit{as } \varepsilon\to 0.$$

Proof. It suffices to consider $\phi\mathcal{S}_{\varepsilon ij}\mathcal{G}_z$ since $\mathcal{G}_z\mathcal{S}^*_{\varepsilon ij}\phi=(\phi\mathcal{S}_{\varepsilon ij}\mathcal{G}_{\overline{z}})^*$ and $\mathcal{G}_z\mathcal{S}^*_{ij}\Omega_\phi=(\phi\otimes(\mathcal{S}_{ij}\mathcal{G}_{\overline{z}}))^*$. Fix $u\in\mathscr{S}(\mathbb{R}^{2n})$, and, to simplify notation, let $u':=(\phi\mathcal{S}_{\varepsilon ij}\mathcal{G}_z-\phi\otimes(\mathcal{S}_{ij}\mathcal{G}_z))u$. We use (4.9) and (4.11) to calculate the partial Fourier transform of u' as

$$\overset{\sqcap}{u'}(y_1,q_{2-n})=\phi(y_1)\int_{\mathbb{R}^2}\frac{e^{\mathbf{i}\varepsilon y_1\cdot q_1}-1}{\frac{1}{2}|M_{ij}q|^2-z}\widehat{u}(M_{ij}q)\frac{\mathrm{d}q_1}{2\pi}.$$

From this we calculate the norm of u' as

$$\|u'\|^2=\int_{\mathbb{R}^{2n}}|\overset{\sqcap}{u'}(y_1,q_{2-n})|^2\mathrm{d}y_1\mathrm{d}q_{2-n}=\int_{\mathbb{R}^{2n}}\Big|\phi(y_1)\int_{\mathbb{R}^2}\frac{e^{\mathbf{i}\varepsilon y_1\cdot q_1}-1}{\frac{1}{2}|M_{ij}q|^2-z}\widehat{u}(M_{ij}q)\frac{\mathrm{d}q_1}{2\pi}\Big|^2\mathrm{d}y_1\mathrm{d}q_{2-n}.$$

Recall that, by assumption, $\phi\in\mathscr{C}^\infty_c(\mathbb{R}^2)$ is fixed, so $|\phi(y_1)|\leqslant C\mathbf{1}_{\{|y_1|\leqslant C\}}$. For $|y_1|\leqslant C$ we have $|e^{\mathbf{i}\varepsilon y_1\cdot q_1}-1|\leqslant C\,((\varepsilon|q_1|)\wedge 1)$. Using this and $|M_{ij}q|^2\geqslant 2|q_1|^2$ (as verified from (4.1)), we have

$$\|u'\|^2\leqslant C\int_{\mathbb{R}^{2n-2}}\Big(\int_{\mathbb{R}^2}\frac{(\varepsilon|q_1|)\wedge 1}{|q_1|^2-\mathrm{Re}(z)}|\widehat{u}(M_{ij}q)|\frac{\mathrm{d}q_1}{2\pi}\Big)^2\mathrm{d}q_{2-n}\leqslant C\|u\|^2\int_{\mathbb{R}^2}\Big(\frac{(\varepsilon|q_1|)\wedge 1}{|q_1|^2-\mathrm{Re}(z)}\Big)^2\mathrm{d}q_1.$$

Set $-\mathrm{Re}(z)=a>0$ to simplify notation. We perform a change of variable $q_1\mapsto\sqrt{a}q_1$ in the last integral to get $\frac{1}{a}\int_{\mathbb{R}^2}\frac{(\varepsilon\sqrt{a}|q_1|)^2\wedge 1}{(|q_1|^2+1)^2}\,\mathrm{d}q_1$. Decompose it according to $|q_1|<\varepsilon^{1/2}a^{1/4}$ and $|q_1|>\varepsilon^{1/2}a^{1/4}$. For the former use $\frac{(\varepsilon\sqrt{a}|q_1|)^2\wedge 1}{(|q_1|^2+1)^2}\leqslant 1$, and for the latter use $(\varepsilon\sqrt{a}|q_1|)^2\wedge 1\leqslant(\varepsilon\sqrt{a}|q_1|)^2$. It is readily checked that the integrals are both bounded by $C\varepsilon a^{-1/2}$. □

5. Off-diagonal mediating operators

To get a rough idea of how the mediating operators (those in (3.6b)) should behave as $\varepsilon\to 0$, we perform a regularity exponent count similar to the discussion just before Lemma 4.2. Recall that $\mathcal{G}_z$ increases regularity by 2, while $\mathcal{S}_{ij}$ and $\mathcal{S}^*_{k\ell}$ *decrease* regularity by $-(1^+)$. *Formally* the regularity of $\mathcal{S}_{ij}\mathcal{G}_z\mathcal{S}^*_{k\ell}$ adds up to $2-(1^+)-(1^+)=0^-<0$. This being the case, one might expect $\mathcal{S}_{\varepsilon ij}\mathcal{G}_z\mathcal{S}^*_{\varepsilon k\ell}$ to diverge, in a somewhat marginal way, as $\varepsilon\to 0$.

As we will show in the next section, the diagonal operator $\mathcal{S}_{\varepsilon 12}\mathcal{G}_z\mathcal{S}^*_{\varepsilon 12}$ diverges logarithmically in ε. This divergence, after a suitable manipulation, cancels the relevant, leading order divergence in $\beta_\varepsilon^{-1}\mathbf{I}$ (recall from (1.7) that $\beta_\varepsilon^{-1}\to\infty$). On the other hand, for each $(i<j)\neq(k<\ell)$, the off-diagonal operator $\mathcal{S}_{\varepsilon ij}\mathcal{G}_z\mathcal{S}^*_{\varepsilon k\ell}$ converges. This is not an obvious fact, cannot be teased out from the preceding regularity counting, and is ultimately due to an inequality derived in [DFT94, Equation (3.2)]. We treat the off-diagonal terms in this section.

We begin by building the limiting operator.

Lemma 5.1. *Fix* $(i<j)\neq(k<\ell)$ *and* $\mathrm{Re}(z)<0$. *We have that* $\mathcal{G}_z\mathcal{S}^*_{k\ell}(\mathscr{L}^2(\mathbb{R}^{2n-2}))\subset\mathrm{Dom}(\mathcal{S}_{ij})$, *so* $\mathcal{S}_{ij}\mathcal{G}_z\mathcal{S}^*_{k\ell}$ *maps* $\mathscr{L}^2(\mathbb{R}^{2n-2})$ *to* $\mathscr{L}^2(\mathbb{R}^{2n-2})$. *Furthermore,* $\|\mathcal{S}_{ij}\mathcal{G}_z\mathcal{S}^*_{k\ell}\|_{\mathrm{op}}\leqslant C$ *and*

$$\big\langle g,\mathcal{S}_{ij}\mathcal{G}_z\mathcal{S}^*_{k\ell}f\big\rangle=\int_{\mathbb{R}^{2n}}\overline{\widehat{g}(p_i+p_j,p_{\overline{ij}})}\,\frac{1}{\frac12|p|^2-z}\widehat{f}(p_k+p_\ell,p_{\overline{k\ell}})\;\frac{\mathrm{d}p}{(2\pi)^2},\tag{5.1}$$

for $f,g\in\mathscr{L}^2(\mathbb{R}^{2n-2})$.

Proof. The inequalities derived in [DFT94, Equations (3.1), (3.3), (3.4), (3.6)] translate, under our notation, into

$$\sup_{\alpha>0}\int_{\mathbb{R}^{2n}}\frac{|\widehat{g}(p_i+p_j,p_{\overline{ij}})|\,|\widehat{f}(p_k+p_\ell,p_{\overline{k\ell}})|}{|p|^2+\alpha}\mathrm{d}p\leqslant C\,\|g\|\,\|f\|,\tag{5.2}$$

for all $(i<j)\neq(k<\ell)$ and $f,g\in\mathscr{L}^2(\mathbb{R}^{2n-2})$. Also, from (4.4) we have

$$\widehat{\mathcal{S}^*_{ij}g}(p)=\tfrac{1}{2\pi}\widehat{g}(p_i+p_j,p_{\overline{ij}}),\qquad \widehat{\mathcal{S}^*_{k\ell}f}(p)=\tfrac{1}{2\pi}\widehat{f}(p_k+p_\ell,p_{\overline{k\ell}}).\tag{5.3}$$

A priori, we only have $\mathcal{G}_z\mathcal{S}^*_{k\ell}f\in\mathscr{L}^2(\mathbb{R}^{2n})$ from Lemma 4.1. Given (5.2)–(5.3) together with $\mathrm{Re}(z)<0$, we further obtain

$$\int_{\mathbb{R}^{2n}}\Big|\widehat{g}(p_i+p_j,p_{\overline{ij}})\frac{1}{\frac12|p|^2-z}\widehat{\mathcal{S}^*_{k\ell}f}(p)\Big|\,\mathrm{d}p=\int_{\mathbb{R}^{2n}}\Big|\widehat{g}(q_{2-n})\frac{1}{\frac12|M_{ij}q|^2-z}\widehat{\mathcal{S}^*_{k\ell}f}(M_{ij}q)\Big|\,\mathrm{d}q\leqslant C\,\|g\|\,\|f\|,\tag{5.4}$$

where, in deriving the equality, we apply a change of variable $q=M_{ij}^{-1}p$, together with $(p_i+p_j,p_{\overline{ij}})=[M_{ij}^{-1}p]_{2-n}$ and $|\det(M_{ij})|=1$ (as readily verified from (4.1)). Referring to the definition (4.2) of $\mathrm{Dom}(\mathcal{S}_{ij})$, since (5.4) holds for all $g\in\mathscr{L}^2(\mathbb{R}^{2n-2})$, we conclude $\mathcal{G}_z\mathcal{S}^*_{k\ell}f\in\mathrm{Dom}(\mathcal{S}_{ij})$ and further that $|\langle g,\mathcal{S}_{ij}\mathcal{G}_z\mathcal{S}^*_{k\ell}f\rangle|=|\langle\mathcal{S}^*_{ij}g,\mathcal{G}_z\mathcal{S}^*_{k\ell}f\rangle|\leqslant C\|g\|\,\|f\|$. The desired identity (5.1) now follows from (5.3). ☐

We next derive the $\varepsilon>0$ analog of (5.1). Recall that $\overset{\frown}{v}(y_1,q_{2-n})$ denotes partial Fourier transform in the last $n-1$ components.

Lemma 5.2. *For (not necessarily distinct)* $(i<j),(k<\ell)$, $\mathrm{Re}(z)<0$, *and* $v,w\in\mathscr{S}(\mathbb{R}^{2n})$,

$$\big\langle w,\mathcal{S}_{\varepsilon ij}\mathcal{G}_z\mathcal{S}^*_{\varepsilon k\ell}v\big\rangle=\int_{\mathbb{R}^{2n}}\overline{\widehat{w}(\tfrac{\varepsilon}{2}(p_i-p_j),p_i+p_j,p_{\overline{ij}})}\frac{1}{\frac12|p|^2-z}\widehat{v}(\tfrac{\varepsilon}{2}(p_k-p_\ell),p_k+p_\ell,p_{\overline{k\ell}})\;\mathrm{d}p\tag{5.5a}$$

$$=\int_{\mathbb{R}^{2+2+2n}}\overline{\overset{\frown}{w}(y_1',p_i+p_j,p_{\overline{ij}})}\frac{e^{\frac12\mathbf{i}\varepsilon((p_i-p_j)\cdot y_1'-(p_k-p_\ell)\cdot y_1)}}{\frac12|p|^2-z}\,\overset{\frown}{v}(y_1,p_k+p_\ell,p_{\overline{k\ell}})\;\frac{\mathrm{d}y_1\mathrm{d}y_1'\mathrm{d}p}{(2\pi)^2}.\tag{5.5b}$$

Proof. Fixing $v,w\in\mathscr{S}(\mathbb{R}^{2n})$, we write $\langle w,\mathcal{S}_{\varepsilon ij}\mathcal{G}_z\mathcal{S}^*_{\varepsilon k\ell}v\rangle=\langle\mathcal{S}^*_{\varepsilon ij}w,\mathcal{G}_z\mathcal{S}^*_{\varepsilon k\ell}v\rangle$. Our goal is to expressing the last quantity in Fourier variables, which amounts to expressing $\mathcal{S}^*_{\varepsilon k\ell}v$ and $\mathcal{S}^*_{\varepsilon ij}w$ in Fourier variables. Recall (from (3.4)) that $\mathcal{S}^*_{\varepsilon ij}$ acts on $\mathscr{L}(\mathbb{R}^{2n})$ by $v(\bullet)\mapsto\varepsilon^{-2}v(T_{\varepsilon ij}\bullet)$, where $T_{\varepsilon ij}$ is the invertible linear transformation defined in (3.2). Write

$$\widehat{\mathcal{S}^*_{\varepsilon ij}w}(p)=\int_{\mathbb{R}^{2n}}e^{-\mathbf{i}p\cdot x}\varepsilon^{-2}w(T_{\varepsilon ij}x)\frac{\mathrm{d}x}{(2\pi)^n}.$$

We wish to perform a change of variable $T_{\varepsilon ij}x=y$. Doing so requires understanding how $(p\cdot x)$ transform accordingly. Defining $\widetilde{M}_{\varepsilon ij}p:=(\frac{\varepsilon}{2}(p_i-p_j),p_i+p_j,p_{\overline{ij}})$, it is readily checked that $p\cdot x=\widetilde{M}_{\varepsilon ij}p\cdot(T_{\varepsilon ij}x)$. Given this, we perform the change of variable $T_{\varepsilon ij}x=y$. With $|\det(T_{\varepsilon ij})|=\varepsilon^{-2}$, we now have

$$\widehat{\mathcal{S}^*_{\varepsilon ij}w}(p)=\int_{\mathbb{R}^{2n}}e^{-\mathbf{i}(\widetilde{M}_{\varepsilon ij}p)\cdot y}w(y)\frac{\mathrm{d}y}{(2\pi)^n}=\widehat{w}(\widetilde{M}_{\varepsilon ij}p)=\widehat{w}(\tfrac{\varepsilon}{2}(p_i-p_j),p_i+p_j,p_{\overline{ij}}),$$

and similarly $\widehat{\mathcal{S}^*_{\varepsilon k\ell}v}(p)=\widehat{v}(\frac{\varepsilon}{2}(p_k-p_\ell),p_k+p_\ell,p_{\overline{k\ell}})$. From these expressions of $\mathcal{S}^*_{\varepsilon k\ell}v$ and $\mathcal{S}^*_{\varepsilon ij}w$ we conclude (5.5a). The identity (5.5b) follows from (5.5a) by writing $\overset{\frown}{v}(y_1,p_{2-n})=\int_{\mathbb{R}^2}e^{\mathbf{i}y_1\cdot p_1}\widehat{v}(p)\frac{\mathrm{d}p_1}{2\pi}$ (and similarly for $\overset{\frown}{w}$). ☐

A useful consequence of Lemma 5.2 is the following norm bound.

Lemma 5.3. *For distinct* $(i<j)\neq(k<\ell)$, $\mathrm{Re}(z)<0$, *and* $\varepsilon\in(0,1)$, $\|\phi\mathcal{S}_{\varepsilon ij}\mathcal{G}_z\mathcal{S}^*_{\varepsilon k\ell}\phi\|_{\mathrm{op}}\leqslant C$.

Proof. In (5.5b), apply (5.2) with $f(\bullet) = \phi(y_1)\,\overline{v}(y_1,\bullet)$ and $g(\bullet) = \phi(y_1')\,\overline{w}(y_1',\bullet)$. and integrate the result over y_1, y_1'. We have

$$|\langle \phi w, \mathcal{S}_{\varepsilon ij}\mathcal{G}_z\mathcal{S}^*_{\varepsilon k\ell}(\phi v)\rangle| \leqslant C \int_{\mathbb{R}^2} \|v(y_1,\bullet)\|\phi(y_1)\mathrm{d}y_1 \int_{\mathbb{R}^2} \|w(y_1',\bullet)\|\phi(y_1')\mathrm{d}y_1'.$$

The last expression, upon an application of the Cauchy–Swchwarz inequality in y_1 and in y_1', is bounded by $C\|v\|\,\|w\|$. From this we conclude $\|\phi\mathcal{S}_{\varepsilon ij}\mathcal{G}_z\mathcal{S}^*_{\varepsilon k\ell}\phi\|_{\mathrm{op}} \leqslant C$. □

We are now ready to establish the convergence of the operator $\phi\,\mathcal{S}_{\varepsilon ij}\mathcal{G}_z\mathcal{S}^*_{\varepsilon k\ell}\,\phi$ for distinct pairs. Recall Ω_ϕ from (3.10).

Lemma 5.4. *For each $(i<j) \neq (k<\ell)$, and* $\mathrm{Re}(z) < 0$, *we have* $\phi\,\mathcal{S}_{\varepsilon ij}\mathcal{G}_z\mathcal{S}^*_{\varepsilon k\ell}\,\phi \to \phi\otimes(\mathcal{S}_{ij}\mathcal{G}_z\mathcal{S}^*_{k\ell}\Omega_\phi)$ *strongly as $\varepsilon \to 0$.*

Proof. Our goal is to show $\phi\,\mathcal{S}_{\varepsilon ij}\mathcal{G}_z\mathcal{S}^*_{\varepsilon k\ell}\phi v \to \phi\otimes\mathcal{S}_{ij}\mathcal{G}_z\mathcal{S}^*_{k\ell}\Omega_\phi v$, for each $v \in \mathscr{L}^2(\mathbb{R}^{2n})$. As shown in Lemmas 5.1 and 5.3, the operators $(\mathcal{S}_{\varepsilon ij}\mathcal{G}_z\mathcal{S}^*_{\varepsilon k\ell})$ and $(\mathcal{S}_{ij}\mathcal{G}_z\mathcal{S}^*_{k\ell})$ are norm-bounded, uniformly in ε. Hence it suffices to consider Schwartz v. Fix $v \in \mathscr{S}^2(\mathbb{R}^{2n})$, and, to simplify notation, set $u_\varepsilon := (\phi\,\mathcal{S}_{\varepsilon ij}\mathcal{G}_z\mathcal{S}^*_{\varepsilon k\ell}\phi)v$ and $u := (\phi\otimes\mathcal{S}_{ij}\mathcal{G}_z\mathcal{S}^*_{k\ell}\Omega_\phi)v$. The strategy of the proof is to express $\|u_\varepsilon - u\|^2$ as an integral, and use the dominated convergence theorem.

The first step is to obtain expressions for the partial Fourier transforms of $u_\varepsilon = (\phi\,\mathcal{S}_{\varepsilon ij}\mathcal{G}_z\mathcal{S}^*_{\varepsilon k\ell}\phi)v$ and $u = (\phi\otimes\mathcal{S}_{ij}\mathcal{G}_z\mathcal{S}^*_{k\ell}\Omega_\phi)v$. To this end, fix $v, w \in \mathscr{S}(\mathbb{R}^{2n})$, in (5.1), set $(f(\bullet), g(\bullet)) = (\phi(y_1)v(y_1,\bullet), \phi(y_1')w(y_1',\bullet))$, and integrate over y_1, y_1'. Note that $\widehat{f}(p_{2-n}) = \phi(y_1)\,\overline{v}(y_1, p_{2-n})$ (and similar for g). We have

$$\big\langle w, u \big\rangle = \int_{\mathbb{R}^{2+2+2n}} \overline{\overline{w}(y_1', p_i+p_j, p_{\overline{ij}})}\phi(y_1')\frac{1}{\frac{1}{2}|p|^2 - z}\phi(y_1)\,\overline{v}(y_1, p_k+p_\ell, p_{\overline{k\ell}})\;\frac{\mathrm{d}y_1\mathrm{d}y_1'\mathrm{d}p}{(2\pi)^2}. \tag{5.1'}$$

Similarly, in (5.5b), substitute $(v,w) = (\phi v, \phi w)$ to get

$$\big\langle w, u_\varepsilon \big\rangle = \int_{\mathbb{R}^{2+2+2n}} \overline{\overline{w}(y_1', p_i+p_j, p_{\overline{ij}})}\phi(y_1')\frac{e^{\frac{1}{2}\mathbf{i}\varepsilon((p_i-p_j)\cdot y_1' - (p_k-p_\ell)\cdot y_1)}}{\frac{1}{2}|p|^2 - z}\phi(y_1)\,\overline{v}(y_1, p_k+p_\ell, p_{\overline{k\ell}})\;\frac{\mathrm{d}y_1\mathrm{d}y_1'\mathrm{d}p}{(2\pi)^2}. \tag{5.5b'}$$

Equations (5.1') and (5.5b') express the inner product (against a generic w) of u_ε and u in partial Fourier variables. From these expressions we can read off $\overline{u_\varepsilon}(y_1', q_{2-n})$ and $\overline{u}(y_1', q_{2-n})$. Specifically, we perform a change of variable $q = M_{ij}^{-1}p = (\frac{1}{2}(p_i - p_j), p_i + p_j, p_{\overline{ij}})$ in (5.1') and (5.5b'), so that $\overline{w}$ takes variables (y_1', q_{2-n}) instead of $(y_1', p_i+p_j, p_{\overline{ij}})$. From the result we read off

$$\overline{u}\,(y_1', q_{2-n}) = \int_{\mathbb{R}^4} f_{z,v}\,\mathrm{d}y_1\mathrm{d}q_1, \quad \overline{u_\varepsilon}\,(y_1', q_{2-n}) = \int_{\mathbb{R}^4} E_\varepsilon\, f_{z,v}\,\mathrm{d}y_1\mathrm{d}q_1. \tag{5.6}$$

Here E_ε and $f_{z,v}$ are (rather complicated-looking) functions of q, y_1, y_1', given in the following. The precisely functional forms of $f_{z,v}$ and E_ε will be irrelevant. Instead, we will explicitly signify what properties of these functions we are using whenever doing so. We have $E_\varepsilon := e^{\mathbf{i}\varepsilon q_1\cdot y_1' - \mathbf{i}\varepsilon[M_{k\ell}^{-1}M_{ij}q]_1\cdot y_1}$ and

$$f_{z,v} := \phi(y_1')\frac{1}{\frac{1}{2}|M_{ij}q|^2 - z}\phi(y_1)\,\overline{v}(y_1, [M_{k\ell}^{-1}M_{ij}q]_{2-n})\;\frac{1}{(2\pi)^2}.$$

Additionally, we will need an auxiliary function $v' \in \mathscr{L}^2(\mathbb{R}^{2n})$ such that $\overline{v'}(y_1, \widetilde{p}) = |\overline{v}(y_1, \widetilde{p})|$. Such a function $v' = v'(y)$ is obtained by taking inverse Fourier of $|\overline{v}(y_1, q_{2-n})|$ in q_{2-n}. Note that $\|v'\| = \|v\| < \infty$. Set $a := -\mathrm{Re}(z) > 0$ and $u' := (\phi\otimes\mathcal{S}_{ij}\mathcal{G}_{-a}\mathcal{S}^*_{k\ell}\Omega_\phi)v'$. We have

$$\overline{u'}\,(y_1', q_{2-n}) = \int_{\mathbb{R}^4} f_{-a,v'}\,\mathrm{d}y_1\mathrm{d}q_1, \qquad f_{-a,v'} \geqslant |f_{z,v}| \geqslant 0. \tag{5.7}$$

Now, use (5.6) and (5.7) to write

$$\|u_\varepsilon - u\|^2 \leqslant \int_{\mathbb{R}^{2n}} \Big(\int_{\mathbb{R}^4} |f_{z,v}|\,|E_\varepsilon - 1|\mathrm{d}y_1\mathrm{d}q_1\Big)^2 \mathrm{d}y_1'\mathrm{d}q_{2-n}, \tag{5.8}$$

$$\|u'\|^2 = \int_{\mathbb{R}^{2n}} \Big(\int_{\mathbb{R}^4} f_{-a,v'}\mathrm{d}y_1\mathrm{d}q_1\Big)^2 \mathrm{d}y_1'\mathrm{d}q_{2-n}. \tag{5.9}$$

View (5.8)–(5.9) as integrals over $\mathbb{R}^{8+2n}$, i.e.,

$$\text{r.h.s. of (5.8)} := \int_{\mathbb{R}^{8+2n}} g_\varepsilon \, \mathrm{d}(\ldots), \qquad \text{r.h.s. of (5.9)} := \int_{\mathbb{R}^{8+2n}} g \, \mathrm{d}(\ldots).$$

We now wish to apply the dominated convergence theorem on g_ε and g. To check the relevant conditions, note that: since $|E_\varepsilon - 1| \leqslant 1$ and $|f_{z,v}| \leqslant f_{-a,v'}$, we have $0 \leqslant g_\varepsilon \leqslant g$; since $|E_\varepsilon - 1| \to 0$ pointwisely on $\mathbb{R}^{8+2n}$, we have $g_\varepsilon \to 0$ pointwise on $\mathbb{R}^{8+2n}$; the integral of g over $\mathbb{R}^{8+2n}$ evaluates to $\|u'\|^2 = \|(\phi \otimes \mathcal{S}_{ij} \mathcal{G}_z \mathcal{S}^*_{k\ell} \Omega_\phi) v'\|^2$, which is *finite* since the operators $\mathcal{S}_{ij}\mathcal{G}_z\mathcal{S}^*_{k\ell}$, $(\phi \otimes \bullet)$, and Ω_ϕ are bounded. The desired result $\int_{\mathbb{R}^{8+2n}} g_\varepsilon \, \mathrm{d}(\ldots) = \|w_\varepsilon - w\|^2 \to 0$ follows. □

6. Diagonal mediating operators

The main task here is to analyze the asymptotic behavior of the diagonal part $\phi\, \mathcal{S}_{\varepsilon 12} \mathcal{G}_z \mathcal{S}^*_{\varepsilon 12} \phi$, which diverges logarithmically. We begin by deriving an expression for $\langle w, \phi\, \mathcal{S}_{\varepsilon 12} \mathcal{G}_z \mathcal{S}^*_{\varepsilon 12} \phi\, v\rangle$ that exposes such $\varepsilon \to 0$ behavior. Let $\mathsf{G}_z(x)$, $x \in \mathbb{R}^2$ denote the Green's function in two dimensions. Recall that $|p|^2_{2-n} := \frac{1}{2}|p_2|^2 + |p_3|^2 + \ldots + |p_n|^2$.

Lemma 6.1. *For $v, w \in \mathscr{L}^2(\mathbb{R}^{2n})$, we have*

$$\langle w, \phi\, \mathcal{S}_{\varepsilon 12} \mathcal{G}_z \mathcal{S}^*_{\varepsilon 12} \phi v\rangle = \int_{\mathbb{R}^{2n}} \overline{w(y_1', p_{2-n})}\, \phi(y_1')\, \tfrac{1}{2} \mathsf{G}_{\varepsilon^2(\frac{1}{2}z - \frac{1}{4}|p|^2_{2-n})}\big(y_1' - y_1\big)\, \phi(y_1)\, \overline{v}(y_1, p_{2-n})\, \mathrm{d}y_1 \mathrm{d}y_1' \mathrm{d}p_{2-n}. \tag{6.1}$$

Proof. Apply Lemma 5.2 for $(i < j) = (k < \ell) = (1 < 2)$ and for $(v, w) \mapsto (\phi v, \phi w)$, and perform a change of variable $(\frac{p_1 - p_2}{2}, p_1 + p_2) \mapsto (p_1, p_2)$ in the result. We obtain

$$\big\langle w, \phi \mathcal{S}_{\varepsilon 12} \mathcal{G}_z \mathcal{S}^*_{\varepsilon 12} \phi v\big\rangle = \int_{\mathbb{R}^{2+2+2n}} \overline{\overline{w}(y_1', p_{2-n})} \phi(y_1') \frac{e^{\mathbf{i}\varepsilon p_1 \cdot (y_1' - y_1)}}{|p_1|^2 + \frac{1}{2}|p|^2_{2-n} - z} \phi(y_1)\, \overline{v}(y_1, p_{2-n})\, \frac{\mathrm{d}y_1 dy_1' \mathrm{d}p}{(2\pi)^2}, \tag{6.2}$$

and we recognize $\int_{\mathbb{R}^2} \frac{e^{\mathbf{i}p_1 \cdot x_1}}{\frac{1}{2}|p_1|^2 - z} \frac{\mathrm{d}p_1}{(2\pi)^2}$ as the Fourier transform of the two-dimensional Green's function G_z. □

Lemma 6.1 suggests analyzing the behavior of $\mathsf{G}_z(x)$ for small $|z|$:

Lemma 6.2. *Take the branch cut of the complex-variable functions $\sqrt{z}$ and $(\log z)$ to be $(-\infty, 0]$, let γ_{EM} denote the Euler–Mascheroni constant. For all $x \neq 0$ and $z \in \mathbb{C} \setminus [0, \infty)$, we have*

$$\mathsf{G}_z(x) = -\tfrac{1}{\pi} \log \tfrac{\sqrt{-z}|x|}{\sqrt{2}} - \tfrac{1}{\pi}\gamma_{\mathrm{EM}} + A(\sqrt{-z}x), \tag{6.3}$$

for some $A(\bullet)$ that grows linearly near the origin, i.e., $\sup_{|z| \leqslant a}(|z|^{-1}|A(z)|) \leqslant C(a)$, for all $a < \infty$.

The proof follows from classical special function theory. We present it here for the convenience of the readers.

Proof. Write the equation $(-\frac{1}{2}\nabla^2 - z)\mathsf{G}_z(x) = 0$, $x \neq 0$, in radial coordinate, compare the result to the modified Bessel equation [AS65, 9.6.1], and note that $\mathsf{G}_z(x)$ vanishes at $|x| \to \infty$. We see that $\mathsf{G}_z(x) = cK_0(\sqrt{-2z}|x|)$, for some constant c, where K_ν denotes the modified Bessel function of second kind. To fix c, compare the know expansion of $K_0(r)$ around $r = 0$ [AS65, 9.6.54] (noting that $I_0(0) = 1$ therein), and use $-\pi r \frac{\mathrm{d}}{\mathrm{d}r} G_z(|r|) = 1$ (because $(-\frac{1}{2}\nabla^2 G_z(x) - z) = \delta(x)$) for $r \to 0$. We find $c = \frac{1}{\pi}$. The claim now follows from [AS65, 9.6.54]. □

For subsequent analysis, it is convenient to decompose $\mathscr{L}^2(\mathbb{R}^{2n})$ into a 'projection onto ϕ' and its orthogonal compliment. More precisely, recall Ω_ϕ from (3.10), and that $\int \phi^2 = 1$, we define the projection

$$\Pi_\phi := \phi \otimes \Omega_\phi : \mathscr{L}^2(\mathbb{R}^{2n}) \to \mathscr{L}^2(\mathbb{R}^{2n}), \quad (\phi \otimes \Omega_\phi v)(y) := \phi(y_1) \int_{\mathbb{R}^2} \phi(y_1') v(y_1', y_{2-n})\, \mathrm{d}y_1'. \tag{6.4}$$

Returning to the discussion about the $\varepsilon \to 0$ behavior of $\phi\, \mathcal{S}_{\varepsilon 12} \mathcal{G}_z \mathcal{S}^*_{\varepsilon 12} \phi$, inserting (6.3) into (6.1), we see that $(\phi \mathcal{S}_{\varepsilon 12} \mathcal{G}_z \mathcal{S}^*_{\varepsilon 12} \phi)$ has a divergent part $(\frac{1}{2\pi}|\log \varepsilon|)\Pi_\phi$. The coefficient $(\frac{1}{2\pi}|\log\varepsilon|)$ matches the leading order of β_ε^{-1} (see (1.7)), so $(\frac{1}{2\pi}|\log\varepsilon|)\Pi_\phi$ cancels the divergence $\beta_\varepsilon^{-1}\mathbf{I}$ on the subspace $\mathrm{Img}(\Pi_\phi)$, but still leaves the remaining part $\beta_\varepsilon^{-1}\mathbf{I}|_{\mathrm{Img}(\Pi_\phi)^\perp} = \beta_\varepsilon^{-1}(\mathbf{I} - \Pi_\phi)$ divergent. However, recall that $(\beta_\varepsilon^{-1}\mathbf{I} - \phi\, \mathcal{S}_{\varepsilon 12} \mathcal{G}_z \mathcal{S}^*_{\varepsilon 12} \phi)$ appears as an *inverse* in the resolvent identity (3.6). Upon taking inverse, the divergent part on $\mathrm{Img}(\Pi_\phi)^\perp$ becomes a vanishing term.

We now begin to show the convergence of $(\beta_\varepsilon^{-1}\mathbf{I} - \phi\,\mathcal{S}_{\varepsilon 12}\mathcal{G}_z\mathcal{S}^*_{\varepsilon 12}\phi)^{-1}$. Doing so requires a technical lemma. To setup the lemma, consider a collection of bounded operators $\{\mathcal{T}_{\varepsilon,p} : \mathscr{L}^2(\mathbb{R}^2) \to \mathscr{L}^2(\mathbb{R}^2)\}$, indexed by $\varepsilon \in (0,1)$ and $p \in \mathbb{R}^{2n-2}$, such that for each $\varepsilon > 0$, $\sup_{p\in\mathbb{R}^{2n-2}} \|\mathcal{T}_{\varepsilon,p}\|_{\mathrm{op}} < \infty$. Note that here, unlike in the preceding, here $p = (p_2,\ldots,p_n) \in \mathbb{R}^{2n-2}$ denotes a vector of $n-1$ components. For each $\varepsilon \in (0,1)$, construct a bounded operator $\mathcal{T}_\varepsilon$ as

$$\mathcal{T}_\varepsilon : \mathscr{L}^2(\mathbb{R}^{2n}) \to \mathscr{L}^2(\mathbb{R}^{2n}), \quad \widehat{\mathcal{T}_\varepsilon u}(\,\cdot\,,p) := \mathcal{T}_{\varepsilon,p}\,\widehat{u}(\,\cdot\,,p).$$

Roughly speaking, we are interested in an operator $\mathcal{T}_\varepsilon$ that acts on $y_1 \in \mathbb{R}^2$ in a way that depends on the partial Fourier components $p = (p_2,\ldots,p_n) \in \mathbb{R}^{2n-2}$. The operator $\mathcal{T}_{\varepsilon,p}$ records the action of $\mathcal{T}_\varepsilon$ on y_1 per *fixed* $p \in \mathbb{R}^{2n-2}$. We are interested in obtaining the inverse $\mathcal{T}_\varepsilon^{-1}$ and its strong convergence (as $\varepsilon \downarrow 0$). The following lemma gives the suitable criteria in terms of each $\mathcal{T}_{\varepsilon,p}$.

Lemma 6.3. *Let $\{\mathcal{T}_{\varepsilon,p}\}$ and $\mathcal{T}_\varepsilon$ be as in the preceding. If each $\mathcal{T}_{\varepsilon,p}$ is invertible with*

$$\sup\big\{\|\mathcal{T}_{\varepsilon,p}^{-1}\|_{\mathrm{op}} : \varepsilon \in (0,1), p \in \mathbb{R}^{2n-2}\big\} := b < \infty,$$

and if each $\mathcal{T}_{\varepsilon,p}^{-1}$ permits a norm limit, i.e., there exists $\mathcal{T}'_p : \mathscr{L}^2(\mathbb{R}^2) \to \mathscr{L}^2(\mathbb{R}^2)$ such that

$$\mathcal{T}_{\varepsilon,p}^{-1} \longrightarrow \mathcal{T}'_p \ \text{ in norm as } \varepsilon \to 0, \qquad \text{for each fixed } p \in \mathbb{R}^{2n-2},$$

then $\mathcal{T}_\varepsilon$ is invertible with $\sup_{\varepsilon\in(0,1)} \|\mathcal{T}_\varepsilon^{-1}\|_{\mathrm{op}} \leqslant b < \infty$,

$$\mathcal{T}_\varepsilon^{-1} \longrightarrow \mathcal{T}', \quad \text{strongly, as } \varepsilon \to 0,$$

and $\|\mathcal{T}'\|_{\mathrm{op}} \leqslant b < \infty$, where the operator $\mathcal{T}' : \mathscr{L}^2(\mathbb{R}^{2n}) \to \mathscr{L}^2(\mathbb{R}^{2n})$ is built from the limit of each $\mathcal{T}_{\varepsilon,p}^{-1}$ as $\widehat{\mathcal{T}'u}(\,\cdot\,,p) := \mathcal{T}'_p\,\widehat{u}(\,\cdot\,,p)$.

Proof. We begin by constructing the inverse of $\mathcal{T}_\varepsilon$. By assumption each $\mathcal{T}_{\varepsilon,p}$ has inverse $\mathcal{T}_{\varepsilon,p}^{-1}$, from which we define $\widehat{\mathcal{T}'_\varepsilon u}(\,\cdot\,,p) := \mathcal{T}_{\varepsilon,p}^{-1}\,\widehat{u}(\,\cdot\,,p)$. It is readily checked that $\|\mathcal{T}'_\varepsilon\|_{\mathrm{op}} \leqslant \sup_{\varepsilon,p} \|\mathcal{T}_{\varepsilon,p}^{-1}\| \leqslant b$, and the operator $\mathcal{T}'_\varepsilon$ actually gives the inverse of $\mathcal{T}_\varepsilon$, i.e., $\mathcal{T}'_\varepsilon\mathcal{T}_\varepsilon = \mathcal{T}_\varepsilon\mathcal{T}'_\varepsilon = \mathbf{I}$. Note that, for each $p \in \mathbb{R}^{2n-2}$, the operator $\mathcal{T}'_p$ inherits a bound from $\mathcal{T}_{\varepsilon,p}^{-1}$, i.e., $\sup_p \|\mathcal{T}'_p\|_{\mathrm{op}} \leqslant \sup_{\varepsilon,p} \|\mathcal{T}_{\varepsilon,p}^{-1}\|_{\mathrm{op}} \leqslant b$. Together with the definition of $\mathcal{T}'$ we also have $\|\mathcal{T}'\|_{\mathrm{op}} \leqslant b$.

It remains to check the strong convergence. For each $u \in \mathscr{L}^2(\mathbb{R}^{2n})$ we have

$$\begin{aligned}\|\mathcal{T}_\varepsilon^{-1}u - \mathcal{T}'u\|^2 &= \int_{\mathbb{R}^{2n-2}} \Big(\int_{\mathbb{R}^2} |\mathcal{T}_{\varepsilon,p}^{-1}\,\widehat{u}(y_1,p) - \mathcal{T}'_p\,\widehat{u}(y_1,p)|^2\,\mathrm{d}y_1\Big)\mathrm{d}p \\ &\leqslant \int_{\mathbb{R}^{2n-2}} \bigg(\|\mathcal{T}_{\varepsilon,p}^{-1} - \mathcal{T}'_p\|^2_{\mathrm{op}} \int_{\mathbb{R}^2} |\,\widehat{u}(y_1,p)|^2\,\mathrm{d}y_1\bigg)\,\mathrm{d}p.\end{aligned}$$

The integrand within the last integral converges to zero pointwisely, and is dominated by $4b^2|\widehat{u}(y_1,p)|^2$, which is integrable over $\mathbb{R}^{2n}$. Hence by the dominated convergence theorem $\|\mathcal{T}_\varepsilon^{-1}u - \mathcal{T}'u\|^2 \to 0$. ☐

With Lemma 6.3, we next establish the norm boundedness and strong convergence of $(\beta_\varepsilon^{-1}\mathbf{I} - \phi\,\mathcal{S}_{\varepsilon 12}\mathcal{G}_z\mathcal{S}^*_{\varepsilon 12}\phi)^{-1}$ in two steps, first for *fixed* $p \in \mathbb{R}^{2n-2}$. Slightly abusing notation, in the following lemma, we also treat Π_ϕ (defined in (6.4)) as its analog on $\mathscr{L}^2(\mathbb{R}^2)$, namely the projection operator $\Pi_\phi f(y_1) := \phi(y_1)\int_{\mathbb{R}^2}\phi(y'_1)f(y'_1)\,\mathrm{d}y'_1$.

Lemma 6.4. *For each $p \in \mathbb{R}^{2n-2}$, define an operator $\mathcal{T}_{\varepsilon,p} : \mathscr{L}^2(\mathbb{R}^2) \to \mathscr{L}^2(\mathbb{R}^2)$,*

$$\mathcal{T}_{\varepsilon,p} f(y_1) := \beta_\varepsilon^{-1} f(y_1) - \phi(y_1)\int_{\mathbb{R}^2} \tfrac{1}{2}\mathsf{G}_{\varepsilon^2(\frac{1}{2}z - \frac{1}{4}|p|^2_{2-n})}(y_1 - y'_1)\phi(y'_1)f(y'_1)\,\mathrm{d}y'_1. \tag{6.5}$$

Then, there exist constants $C_1 < \infty, C_2(\beta_{\mathrm{fine}}) > 0$ such that, for all $\mathrm{Re}(z) < -e^{\beta_\star + C_1}$ and $\varepsilon \in (0, 1/C_2(\beta_{\mathrm{fine}}))$,

$$\begin{aligned}&\big\|(\mathcal{T}_{\varepsilon,p})^{-1}\big\|_{\mathrm{op}} \leqslant C\,(\log(-\mathrm{Re}(z)) - \beta_\star)^{-1}, \\ &\big(\mathcal{T}_{\varepsilon,p}\big)^{-1} \longrightarrow \frac{4\pi}{\log(\frac{1}{2}|p|^2_{2-n} - z) - \beta_\star}\Pi_\phi, \quad \text{in norm as } \varepsilon \to 0, \ \text{for each fixed } p \in \mathbb{R}^{2n-2}.\end{aligned}$$

Proof. Through out the proof, we say a statement holds for $-\mathrm{Re}(z)$ large enough, if the statement holds for all $-\mathrm{Re}(z) > e^{\beta_\star + C}$, for some fixed constant $C < \infty$, and we say a statement holds for all ε small enough, if the statement holds for all $\varepsilon < 1/C(\beta_{\mathrm{fine}})$, for some constant $C(\beta_{\mathrm{fine}}) < \infty$ that depends only on β_{fine}.

Our first goal is to show $\mathcal{T}_{\varepsilon,p}$ is invertible and establish bounds on $\|\mathcal{T}_{\varepsilon,p}^{-1}\|_{\text{op}}$. We do this in two separate cases: *i)* $|\frac{1}{2}|p|^2_{2-n} - z| \leqslant \varepsilon^{-2}$ and *ii)* $|\frac{1}{2}|p|^2_{2-n} - z| > \varepsilon^{-2}$.

i) The first step here is to derive a suitable expansion of $\mathcal{T}_{\varepsilon,p}$. Recall that, we have abused notation to write Π_ϕ (defined in (6.4)) for the projection operator $\Pi_\phi f(y_1) := \phi(y_1)\int_{\mathbb{R}^2}\phi(y_1')f(y_1')\,\mathrm{d}y_1'$. Applying Lemma 6.2 yields

$$\mathcal{T}_{\varepsilon,p} = \beta_\varepsilon^{-1}\mathbf{I} + \Big(-\tfrac{1}{2\pi}|\log\varepsilon| + \tfrac{1}{4\pi}\log(\tfrac{1}{2}|p|^2_{2-n} - z) - \tfrac{1}{2\pi}\log 2 + \tfrac{1}{2\pi}\gamma_{\text{EM}}\Big)\Pi_\phi + \mathcal{L}_\phi - \mathcal{A}_{\varepsilon,z,p}, \tag{6.6}$$

where $\mathcal{L}_\phi$ and $\mathcal{A}_{\varepsilon,z,p}$ are integral operators $\mathscr{L}^2(\mathbb{R}^2) \to \mathscr{L}^2(\mathbb{R}^2)$ defined as

$$(\mathcal{L}_\phi f)(y_1) := \frac{1}{2\pi}\phi(y_1)\int_{\mathbb{R}^2}\log|y_1 - y_1'|\phi(y_1')f(y_1')\,\mathrm{d}y_1', \tag{6.7}$$

$$(\mathcal{A}_{\varepsilon,z,p} f)(y_1) := \frac{1}{2}\phi(y_1)\int_{\mathbb{R}^2} A\Big(\tfrac{1}{2}|y_1 - y_1'|\varepsilon\sqrt{\tfrac{1}{2}|p|^2_{2-n} - z}\Big)\phi(y_1')f(y_1')\,\mathrm{d}y_1', \tag{6.8}$$

and the function $A(\bullet)$ is the remainder term in Lemma 6.2. Let $\Pi_\perp := \mathbf{I} - \Pi_\phi$ denote the orthogonal projection onto $(\mathbb{C}\phi)^\perp$ in $\mathscr{L}^2(\mathbb{R}^2)$ and recall β_ε from (1.7). In (6.6), decomposing $\beta_\varepsilon^{-1}\mathbf{I} = \beta_\varepsilon^{-1}\Pi_\perp + \frac{1}{2\pi}(|\log\varepsilon| - \beta_{\varepsilon,\text{fine}})\Pi_\phi$, where $\beta_{\varepsilon,\text{fine}} := |\log\varepsilon| - |\log\varepsilon|(1 + \frac{\beta_{\text{fine}}}{|\log\varepsilon|})^{-1}$, we rearrange terms to get

$$\mathcal{T}_{\varepsilon,p} = \beta_\varepsilon^{-1}\Pi_\perp + \tfrac{1}{4\pi}\big(\log(\tfrac{1}{2}|p|^2_{2-n} - z) - \beta'_{\star,\varepsilon}\big)\Pi_\phi + \mathcal{L}_\phi - \mathcal{A}_{\varepsilon,z,p}, \tag{6.9}$$

where $\beta'_{\star,\varepsilon} := 2(\log 2 + \beta_{\varepsilon,\text{fine}} - \gamma_{\text{EM}})$. We next take the inverse of $\mathcal{T}_{\varepsilon,p}$ from (6.9), utilizing

$$\big(\mathcal{Q} - \widetilde{\mathcal{Q}}\big)^{-1} = \sum_{m=0}^\infty \mathcal{Q}^{-1}\big(\widetilde{\mathcal{Q}}\mathcal{Q}^{-1}\big)^m, \qquad \|(\mathcal{Q} - \widetilde{\mathcal{Q}})^{-1}\|_{\text{op}} \leqslant \|\mathcal{Q}^{-1}\|_{\text{op}}/(1 - \|\mathcal{Q}^{-1}\|_{\text{op}}\|\widetilde{\mathcal{Q}}\|_{\text{op}}), \tag{6.10}$$

valid for operators $\mathcal{Q}, \widetilde{\mathcal{Q}}$ such that $\mathcal{Q}$ is invertible with $\|\mathcal{Q}^{-1}\|_{\text{op}}\|\widetilde{\mathcal{Q}}\|_{\text{op}} < 1$. Our choice will be $\mathcal{Q} := \beta_\varepsilon^{-1}\Pi_\perp + \frac{1}{4\pi}\big(\log(\frac{1}{2}|p|^2_{2-n} - z) - \beta'_{\star,\varepsilon}\big)\Pi_\phi$ and $\widetilde{\mathcal{Q}} := -\mathcal{L}_\phi + \mathcal{A}_{\varepsilon,z,p}$.

From (6.7), we have $\|\mathcal{L}_\phi\|_{\text{op}} < \infty$. Under our current assumption $|\frac{1}{2}|p|^2_{2-n} - z| \leqslant \varepsilon^{-2}$, from (6.8) and the property of $A(\bullet)$ stated in Lemma 6.2, we have $\|\mathcal{A}_{\varepsilon,z,p}\|_{\text{op}} \leqslant C < \infty$. Hence

$$\big\| -\mathcal{L}_\phi + \mathcal{A}_{\varepsilon,z,p}\big\|_{\text{op}} \leqslant C. \tag{6.11}$$

With $\Pi_\perp$ and Π_ϕ being projection operators orthogonal to each other, we calculate

$$\big(\beta_\varepsilon^{-1}\Pi_\perp + \tfrac{1}{4\pi}\big(\log(\tfrac{1}{2}|p|^2_{2-n} - z) - \beta'_{\star,\varepsilon}\big)\Pi_\phi\big)^{-1} = \beta_\varepsilon\Pi_\perp + 4\pi\big(\log(\tfrac{1}{2}|p|^2_{2-n} - z) - \beta'_{\star,\varepsilon}\big)^{-1}\Pi_\phi. \tag{6.12}$$

The operator norm of this inverse is thus bounded by $\max\{\beta_\varepsilon\,,\ \frac{4\pi}{\log(-\mathrm{Re}(z)) - \beta'_{\star,\varepsilon}}\}$. Since $\beta'_{\star,\varepsilon} \to \beta_\star + 2\beta_\Phi$ and $\beta_\varepsilon \to 0$, this allows us to get a convergent series (6.10) for $-\mathrm{Re}(z)$ large enough and ε small enough, with $\|\mathcal{T}_{\varepsilon,p}^{-1}\|_{\text{op}} \leqslant C(\log(-\mathrm{Re}(z)) - \beta_\star)^{-1}$.

ii) We apply (6.10) again to (6.5) with $\mathcal{Q} = \beta_\varepsilon^{-1}\mathbf{I}$. To check the relevant condition, we write the operator $\mathcal{T}_{\varepsilon,p}$ (in (6.5)) in a coordinate-free form as $\mathcal{T}_{\varepsilon,p} = \beta_\varepsilon^{-1}\mathbf{I} - \phi\frac{1}{2}\mathcal{G}^{(n=1)}_{\varepsilon^2(\frac{1}{2}z - \frac{1}{4}|p|^2_{2-n})}\phi$, where $\mathcal{G}^{(n=1)}_z$ denotes the two-dimensional Laplace resolvent. Recall that $\mathrm{Re}(z) < -e^{-\beta_\star + C_1} < 0$, so $\mathrm{Re}(\frac{1}{2}z - \frac{1}{4}|p|^2_{2-n}) < 0$, which gives $\|\mathcal{G}^{(n=1)}_{\varepsilon^2(\frac{1}{2}z - \frac{1}{4}|p|^2_{2-n})}\|_{\text{op}} = |\varepsilon^2(\frac{1}{2}z - \frac{1}{4}|p|^2_{2-n})|^{-1}$. Under the current assumption $|\frac{1}{2}|p|^2_{2-n} - z| > \varepsilon^{-2}$, this is bounded by 2, so

$$\big\|\phi\tfrac{1}{2}\mathcal{G}^{(n=1)}_{\varepsilon^2(\frac{1}{2}z - \frac{1}{4}|p|^2_{2-n})}\phi\big\| \leqslant C.$$

Since $\beta_\varepsilon^{-1} \to \infty$, (6.10) applied to (6.5) with $\mathcal{Q} = \beta_\varepsilon^{-1}\mathbf{I}$, show that $\mathcal{T}_{\varepsilon,p}^{-1}$ exists with $\|\mathcal{T}_{\varepsilon,p}^{-1}\|_{\text{op}} \leqslant C\,(\log\varepsilon)^{-1}$, for all ε small enough.

Having obtained $\mathcal{T}_{\varepsilon,p}^{-1}$ and its bound, we next show the norm convergence. The condition $|\frac{1}{2}|p|^2_{2-n} - z| \leqslant \varepsilon^{-2}$ holds for all $\varepsilon \leqslant C(p)$, whence we have from (6.10) that

$$\mathcal{T}_{\varepsilon,p}^{-1} = \Big(\beta_\varepsilon\Pi_\perp + \frac{4\pi}{\log(\frac{1}{2}|p|^2_{2-n} - z) - \beta'_{\star,\varepsilon}}\Pi_\phi\Big)\sum_{m=0}^\infty\Big((-\mathcal{L}_\phi + \mathcal{A}_{\varepsilon,z,p})\Big(\beta_\varepsilon\Pi_\perp + \frac{4\pi}{\log(\frac{1}{2}|p|^2_{2-n} - z) - \beta'_{\star,\varepsilon}}\Pi_\phi\Big)\Big)^m. \tag{6.13}$$

We now take termwise limit in (6.13). Referring to (6.8), with $p \in \mathbb{R}^{2n-2}$ being *fixed*, the linear growth property of $A(\bullet)$ in Lemma 6.2 gives that $\mathcal{A}_{\varepsilon,z,p}$ converges to 0 in norm. Since $\beta_\varepsilon \to 0$,

$$\beta_\varepsilon \Pi_\perp + \frac{4\pi}{\log(\frac{1}{2}|p|^2_{2-n} - z) - \beta'_{\star,\varepsilon}} \Pi_\phi \longrightarrow \frac{4\pi}{\log(\frac{1}{2}|p|^2_{2-n} - z) - \beta_\star - 2\beta_\Phi} \Pi_\phi, \quad \text{in norm.}$$

Further, the bound (6.11) guarantees that, for all $-\mathrm{Re}(z)$ large enough, the series (6.13) converges absolutely in norm, uniformly for all ε small enough. From this we conclude $\mathcal{T}^{-1}_{\varepsilon,p} \to \mathcal{T}'_p$ in norm, where

$$\mathcal{T}'_p := \frac{4\pi}{\log(\frac{1}{2}|p|^2_{2-n} - z) - \beta'_\star - 2\beta_\Phi} \sum_{m=0}^{\infty} \Pi_\phi \Big(\frac{4\pi}{\log(\frac{1}{2}|p|^2_{2-n} - z) - \beta_\star - 2\beta_\Phi} (-\mathcal{L}_\phi) \Pi_\phi \Big)^m. \tag{6.14}$$

This expression can be further simplified using $\Pi_\phi^m = \Pi_\phi$ and $\Pi_\phi \mathcal{L}_\phi \Pi_\phi = \frac{\beta_\Phi}{2\pi} \Pi_\phi$,

$$\mathcal{T}'_p = \frac{4\pi}{\log(\frac{1}{2}|p|^2_{2-n} - z) - \beta_\star - 2\beta_\Phi} \sum_{m=0}^{\infty} \Pi_\phi \Big(\frac{-2\beta_\Phi}{\log(\frac{1}{2}|p|^2_{2-n} - z) - \beta_\star - 2\beta_\Phi} \Pi_\phi \Big)^m = \frac{4\pi}{\log(\frac{1}{2}|p|^2_{2-n} - z) - \beta_\star} \Pi_\phi.$$

This completes the proof. ☐

Recall $\mathcal{J}_z$ from (1.22). Combining Lemmas 6.3–6.4 immediately gives the main result of this section:

Lemma 6.5. *There exist constants* $C_1 < \infty, C_2(\beta_{\text{fine}}) > 0$ *such that, for all* $\mathrm{Re}(z) < -e^{\beta_\star + C_1}$*, and for all* $\varepsilon \in (0, 1/C_2(\beta))$*, the inverse* $(\beta_\varepsilon^{-1}\mathbf{I} - \phi\, \mathcal{S}_{\varepsilon 12} \mathcal{G}_z \mathcal{S}^*_{\varepsilon 12} \phi)^{-1} : \mathscr{L}^2(\mathbb{R}^{2n}) \to \mathscr{L}^2(\mathbb{R}^{2n})$ *exists, with*

$$\big\| \big(\beta_\varepsilon^{-1}\mathbf{I} - \phi\, \mathcal{S}_{\varepsilon 12} \mathcal{G}_z \mathcal{S}^*_{\varepsilon 12} \phi\big)^{-1} \big\|_{\text{op}} \leqslant C \left(\log(-\mathrm{Re}(z)) - \beta_\star \right)^{-1}, \tag{6.15}$$

$$\big(\beta_\varepsilon^{-1}\mathbf{I} - \phi\, \mathcal{S}_{\varepsilon 12} \mathcal{G}_z \mathcal{S}^*_{\varepsilon 12} \phi\big)^{-1} \longrightarrow 4\pi\phi \otimes \big((\mathcal{J}_z - \beta_\star \mathbf{I})^{-1} \Omega_\phi \big), \text{ strongly, as } \varepsilon \to 0. \tag{6.16}$$

7. Convergence of the resolvent

In this section we collect the results of Sections 3–6 to prove Proposition 1.4(a)–(b) and Theorem 1.5(a)–(b) and the convergence part of Theorem 1.1(b).

Proposition 1.4(a) and Theorem 1.5(a) follow from the bounds obtained in Lemmas 4.2–4.4, 5.3, and 6.5. We now turn to Theorem 1.5(b). Recall that Lemma 3.2, as stated, applies only for $\mathrm{Re}(z) < -C(n,\varepsilon)$, for some threshold $C(n,\varepsilon)$ that depends on ε. Here we argue that the threshold can be improved to be independent of ε. Given the bounds from Lemmas 4.2–4.4, 5.3, and 6.5, we see that the r.h.s. of (3.6) defines an analytic function (in operator norm) in $B := \{\mathrm{Re}(z) < -e^{Cn^2 + \beta_\star}\}$. On the other hand, we also know that the l.h.s. $(\mathcal{R}_{\varepsilon,z} - \mathcal{G}_z)$ is analytic in z off $\sigma(\mathcal{H}_\varepsilon) \cup [0,\infty)$, where $\sigma(\mathcal{H}_\varepsilon) \subset \mathbb{R}$ denotes the spectrum of $\mathcal{H}_\varepsilon$. Consequently, both sides must match on $B \setminus \sigma(\mathcal{H}_\varepsilon)$. We now argue $B \cap \sigma(\mathcal{H}_\varepsilon) = \varnothing$, so the matching actually holds on the entire B. Assuming the contrary, we fix $z_0 \in B \cap \sigma(\mathcal{H}_\varepsilon)$, take a sequence $z_k \in B$ and approaches $z_k \to z_0$ along the vertical axis. Along this sequence $(\mathcal{R}_{\varepsilon,z_k} - \mathcal{G}_{z_k})$ is bounded, contradicting $z_0 \in \sigma(\mathcal{H}_\varepsilon)$.

We now show the convergence of the resolvent, i.e. (3.6) to (3.12). As argued previously, both series (3.6) and (3.12) converge absolutely in operator norm, uniformly over ε. It hence suffices to show termwise convergence. By Lemmas 4.4, 5.4, and 6.5, each factor in (3.6a)–(3.6c) converges to its limiting counterparts in (3.12a)–(3.12c), strongly or in norm. Using this in conjunctionwith the elementary, readily checked fact

$$\mathcal{Q}_\varepsilon \mathcal{Q}'_\varepsilon \to \mathcal{Q}\mathcal{Q}' \text{ strongly if } \mathcal{Q}_\varepsilon, \mathcal{Q}'_\varepsilon \text{ are uniformly bounded and } \mathcal{Q}_\varepsilon \to \mathcal{Q}, \mathcal{Q}'_\varepsilon \to \mathcal{Q}' \text{ strongly,}$$

we conclude the desired convergence of the resolvent, Theorem 1.5(b).

Next we prove Proposition 1.4(b). First, given the bounds from Lemmas 4.2–4.4, 5.3, and 6.5, we see that $\mathcal{R}^{\text{sym}}_z$ in (1.24) defines a bounded operator on $\mathscr{L}^2(\mathbb{R}^{2n})$ for all $\mathrm{Re}(z) < -e^{\beta_\star + n^2 C}$. Our goal is to match $\mathcal{R}^{\text{sym}}_z$ to $\mathcal{R}_z$ on $\mathscr{L}^2_{\text{sym}}(\mathbb{R}^{2n})$, for these values of z. Apply (6.10) with $\mathcal{Q} = \frac{1}{4\pi}(\mathcal{J}_z - \beta_\star \mathbf{I})$ and with $\widetilde{\mathcal{Q}} = \frac{2}{n(n-1)} \sum^{\mathrm{d}} \mathcal{S}_{ij} \mathcal{G}_z \mathcal{S}^*_{k\ell}$ for the prescribed values of z (so that the condition for (6.10) to apply checks). We obtain

$$\mathcal{R}^{\text{sym}}_z = \mathcal{G}_z + \sum \mathcal{S}^*_{i_1 j_1} \mathcal{G}_z \Big(4\pi (\mathcal{J}_z - \beta_\star \mathbf{I})^{-1} \prod_{s=2}^{m} \Big(\tfrac{2}{n(n-1)} \mathcal{S}_{k_s k_s} \mathcal{G}_z \mathcal{S}^*_{i_s j_s}\, 4\pi (\mathcal{J}_z - \beta_\star \mathbf{I})^{-1} \Big) \Big) \tfrac{2}{n(n-1)} \mathcal{G}_z \mathcal{S}_{k_{m+1} k_{m+1}}, \tag{7.1}$$

where the sum is over all pairs $(i_1 < j_1)$, $(k_2 < \ell_2) \neq (i_2 < j_2)$, $\ldots$, $(k_m < \ell_m) \neq (i_m < j_m)$, $(k_{m+1} < k_{m+1})$, and all m.

At this point we need to use the symmetry of $\mathscr{L}^2_{\mathrm{sym}}(\mathbb{R}^{2n})$. Given that $\mathcal{G}_z$ acts symmetrically in the n components, for the incoming operator we have

$$\mathcal{S}_{ij}\mathcal{G}_z\big|_{\mathscr{L}^2_{\mathrm{sym}}(\mathbb{R}^{2n})} = \mathcal{S}_{i'j'}\mathcal{G}_z\big|_{\mathscr{L}^2_{\mathrm{sym}}(\mathbb{R}^{2n})}, \qquad \forall\,(i<j),\ (i'<j'). \tag{7.2}$$

A similar symmetry holds for the off-diagonal mediating operator. To state it, note that $\mathcal{G}_z$ maps $\mathscr{L}^2_{\mathrm{sym}}(\mathbb{R}^{2n})$ into a subspace of $\mathscr{L}^2(\mathbb{R}^{2n-2})$ that consists of functions $v(y_2,\ldots,y_n)$ symmetric in the last $(n-2)$ components. More explicitly,

$$\mathcal{G}_z\big(\mathscr{L}^2_{\mathrm{sym}}(\mathbb{R}^{2n})\big) \subset \mathscr{L}^2_{\mathrm{sym}'}(\mathbb{R}^{2n-2}) := \big\{ v\in\mathscr{L}^2(\mathbb{R}^{2n-2}) : v(y_2,y_{\sigma(3)},\ldots,y_{\sigma(n)}) = y(y_2,y_3,\ldots,y_n)\ \sigma\in\mathbb{S}_{n-2}\big\}.$$

Also, this space $\mathscr{L}^2_{\mathrm{sym}'}(\mathbb{R}^{2n-2})$ is invariant under the action of mediating operators (both diagonal and off-diagonal). From (5.1) we have

$$\mathcal{S}_{k\ell}\mathcal{G}_z\mathcal{S}^*_{ij}\big|_{\mathscr{L}^2_{\mathrm{sym}'}(\mathbb{R}^{2n-2})} = \mathcal{S}_{\sigma(k)\sigma(\ell)}\mathcal{G}_z\mathcal{S}^*_{\sigma(i)\sigma(j)}\big|_{\mathscr{L}^2_{\mathrm{sym}'}(\mathbb{R}^{2n-2})}, \qquad \forall\,(i<j)\neq(k<\ell),\ \sigma\in\mathbb{S}_n. \tag{7.3}$$

In (7.1), use (7.3) to rearrange the sum over $(k_2<\ell_2)\neq(i_2<j_2)$ as

$$\frac{2}{n(n-1)}\sum_{(k_2<\ell_2)\neq(i_2<j_2)} \mathcal{S}_{k_2\ell_2}\mathcal{G}_z\mathcal{S}^*_{i_2j_2}\big|_{\mathscr{L}^2_{\mathrm{sym}'}(\mathbb{R}^{2n-2})} = \sum_{i_2<j_2}\mathcal{S}_{i_1j_1}\mathcal{G}_z\mathcal{S}^*_{i_2j_2}\big|_{\mathscr{L}^2_{\mathrm{sym}'}(\mathbb{R}^{2n-2})}\mathbf{1}_{\{(i_2<j_2)\neq(i_1<i_1)\}}.$$

That is, we use (7.3) for some $\sigma\in\mathbb{S}_n$ such that $(\sigma(k_2)<\sigma(k_2))=(i_1<j_1)$. Doing so reduces the sum over double pairs $(k_2<\ell_2)\neq(i_2<j_2)$ into a sum over a single pair $(i_2<j_2)$ with $(i_2<j_2)\neq(i_1<j_1)$, and the counting in this reduction cancels the prefactor $2/(n(n-1))$. Continue this procedure inductively from $s=2$ through $s=m$, and then, at the $m+1$ step, similarly use (7.2) to write

$$\frac{2}{n(n-1)}\sum_{k_{m+1}<\ell_{m+1}} \mathcal{S}_{k_{m+1}\ell_{m+1}}\mathcal{G}_z\big|_{\mathscr{L}^2_{\mathrm{sym}}(\mathbb{R}^{2n})} = \mathcal{S}_{i_mj_m}\mathcal{G}_z\big|_{\mathscr{L}^2_{\mathrm{sym}}(\mathbb{R}^{2n})}.$$

We then conclude Proposition 1.4(b),

$$\mathcal{R}^{\mathrm{sym}}_z|_{\mathscr{L}^2_{\mathrm{sym}}(\mathbb{R}^{2n})} = \mathcal{R}_z|_{\mathscr{L}^2_{\mathrm{sym}}(\mathbb{R}^{2n})}. \tag{7.4}$$

We now turn to the convergence of the fixed time correlation functions in Theorem 1.1(b). Given Theorem 1.5, applying the Trotter–Kato Theorem, c.f., [RS72, Theorem VIII.22], we know that there exists an (unbounded) self-adjoint operator $\mathcal{H}$ on $\mathscr{L}^2(\mathbb{R}^{2n})$, such that $\mathcal{R}_z$ (in (1.23)) is the resolvent for $\mathcal{H}$, i.e., $\mathcal{R}_z=(\mathcal{H}-z\mathbf{I})^{-1}$, for all $\mathrm{Im}(z)\neq 0$. Theorem 1.5 also guarantees that the spectra of $\mathcal{H}_\varepsilon$ and $\mathcal{H}$ are bounded below, uniformly in ε. More precisely, $\sigma(\mathcal{H}_\varepsilon),\sigma(\mathcal{H})\subset(-C_1(n,\beta_\star),\infty)$, for all $\varepsilon\in(0,1/C_2(\beta_{\mathrm{fine}}))$, for some $C_1(n,\beta_\star)<\infty$ and $C_2(\beta_{\mathrm{fine}})>0$. Fix $t\in\mathbb{R}_+$. We now apply [RS72, Theorem VIII.20], which says that if self-adjoint operators $\mathcal{H}_\varepsilon\to\mathcal{H}$ in the strong resolvent sense, and f is bounded and continuous on $\mathbb{R}$ then $f(\mathcal{H}_\varepsilon)\to f(\mathcal{H})$ strongly. We use $f(\lambda)=e^{(-t\lambda)\wedge C_1(n,\beta_\star)}$, which is bounded and continuous, and from what we have proved, $f(\mathcal{H}_\varepsilon)=e^{-t\mathcal{H}_\varepsilon}$ and $f(\mathcal{H})=e^{-t\mathcal{H}}$. Hence

$$e^{-t\mathcal{H}_\varepsilon}\longrightarrow e^{-t\mathcal{H}} \quad \text{strongly on } \mathscr{L}^2(\mathbb{R}^{2n}),\ \text{for each fixed } t\in\mathbb{R}_+. \tag{7.5}$$

For Theorem 1.1(b), we wish to upgrade this convergence to be *uniform* over finite intervals in t. Given the lower bound on the spectra, we have the uniform (in ε) norm continuity:

$$\big\|e^{-t\mathcal{H}_\varepsilon}-e^{-s\mathcal{H}_\varepsilon}\big\|_{\mathrm{op}}+\big\|e^{-t\mathcal{H}}-e^{-s\mathcal{H}}\big\|_{\mathrm{op}} \leqslant C_2(n,\beta_\star)|t-s|e^{C_2(n,\beta)(t\vee s)},$$

for all $\varepsilon\in(0,1/C_2(\beta_{\mathrm{fine}}))$ and $s,t\in[0,\infty)$. This together with (7.5) gives

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,\tau]}\big\|e^{-t\mathcal{H}_\varepsilon}u-e^{-t\mathcal{H}}u\big\|=0, \quad u\in\mathscr{L}^2(\mathbb{R}^{2n}),\ \tau<\infty.$$

Comparing this with (1.4), we now have, for each fixed $g\in\mathscr{L}^2(\mathbb{R}^{2n})$,

$$\mathbb{E}\big[\langle Z^{\otimes n}_{\varepsilon,t},g\rangle\big]\longrightarrow\big\langle Z^{\otimes n}_{\mathrm{ic}},e^{-t\mathcal{H}}g\big\rangle, \quad \text{uniformly over finite intervals in } t. \tag{7.6}$$

What is missing for the proof of Theorem 1.1 is the identification of the semigroup $e^{-t\mathcal{H}}$ with the explicit operators defined in (1.16), (1.17). This is the subject of the next section.

8. Identification of the limiting semigroup

The remaining task is to match $e^{-t\mathcal{H}}$ to the operator $\mathcal{P}_t + \mathcal{D}_t^{\mathrm{Dgm}(n)}$ on r.h.s. of (1.18). This matching can be *heuristically* seen by taking the inverse Laplace transform of $\mathcal{R}_z$ in (1.23) in z. At a formal level, doing so turns the operators $\mathcal{G}_\bullet$ and $(\mathcal{J}_\bullet - \beta_\star \mathbf{I})^{-1}$ into $\mathcal{P}_\bullet$ and $\mathcal{P}^{\mathcal{J}}_\bullet$, and the products of operators in z become the convolutions in t.

To rigorously perform this matching, it is more convenient to operate in the forward Laplace transform, i.e., going from t to z. Doing so requires establishing bounds on the relevant operators in (1.17), and verifying the semigroup property of $\mathcal{P}_t + \mathcal{D}_t^{\mathrm{Dgm}(n)}$, defined in (1.16). The bounds will be established in Section 8.1, and, as the major step toward verifying the semigroup property, we establish an identity in Section 8.2.

8.1. **Bounds and Laplace transforms.** We begin with the incoming and outgoing operators. We now establish a quantitative bound on the norms of $\mathcal{S}_{ij}\mathcal{P}_t$ and $\mathcal{P}_t\mathcal{S}_{ij}^*$, and match them to the corresponding Laplace transform.

Lemma 8.1.

(a) For each pair $i<j$ and $t \in \mathbb{R}_+$, $\mathcal{S}_{ij}\mathcal{P}_t : \mathscr{L}^2(\mathbb{R}^{2n}) \to \mathscr{L}^2(\mathbb{R}^{2n-2})$ and $\mathcal{P}_t\mathcal{S}_{ij}^* : \mathscr{L}^2(\mathbb{R}^{2n-2}) \to \mathscr{L}^2(\mathbb{R}^{2n})$ are bounded with

$$\|\mathcal{S}_{ij}\mathcal{P}_t\|_{\mathrm{op}} + \|\mathcal{P}_t\mathcal{S}_{ij}^*\|_{\mathrm{op}} \leqslant Ct^{-1/2}.$$

(b) For each pair $i<j$, $\mathrm{Re}(z)<0$, $u \in \mathscr{L}^2(\mathbb{R}^{2n})$, and $v \in \mathscr{L}^2(\mathbb{R}^{2n-2})$,

$$\int_{\mathbb{R}_+} e^{tz}\langle v, \mathcal{S}_{ij}\mathcal{P}_t u\rangle\,\mathrm{d}t = \int_{\mathbb{R}_+\times\mathbb{R}^{4n-2}} e^{tz}\overline{v(y)}P(t, S_{ij}y - x)u(x)\,\mathrm{d}t\mathrm{d}y\mathrm{d}x = \langle u, \mathcal{S}_{ij}\mathcal{G}_z v\rangle,$$
$$\int_{\mathbb{R}_+} e^{tz}\langle u, \mathcal{P}_t\mathcal{S}_{ij}^* v\rangle\,\mathrm{d}t = \int_{\mathbb{R}_+\times\mathbb{R}^{4n-2}} e^{tz}\overline{u(x)}P(t, x - S_{ij}y)v(y)\,\mathrm{d}t\mathrm{d}x\mathrm{d}y = \langle u, \mathcal{G}_z\mathcal{S}_{ij}^* v\rangle,$$

where the integrals converge absolutely (over $\mathbb{R}_+$ and over $\mathbb{R}_+\times\mathbb{R}^{2n-4}$).

Proof. It suffices to consider $\mathcal{S}_{ij}\mathcal{P}_t$ since $\mathcal{P}_t\mathcal{S}_{ij}^* = (\mathcal{S}_{ij}\mathcal{P}_t)^*$.
(a) Fix $u \in \mathscr{L}^2(\mathbb{R}^{2n})$, we use (4.3) to bound

$$\|\mathcal{S}_{ij}\mathcal{P}_t u\|^2 = \int_{\mathbb{R}^{2n-2}} \Big(\int_{\mathbb{R}^2} \widehat{\mathcal{P}_t u}(M_{ij}q)\frac{\mathrm{d}q_1}{2\pi}\Big)^2 \mathrm{d}q_{2-n} = \int_{\mathbb{R}^{2n-2}} \Big(\int_{\mathbb{R}^2} e^{-\frac{1}{2}t|M_{ij}q|^2}\widehat{u}(M_{ij}q)\frac{\mathrm{d}q_1}{2\pi}\Big)^2 \mathrm{d}q_{2-n}.$$

On the r.h.s., bound $|M_{ij}q|^2 \geqslant \frac{1}{2}|q_1|^2$ (as checked from (4.1)), and applying the Cauchy–Schwarz inequality in the q_1 integral. We conclude the desired result

$$\|\mathcal{S}_{ij}\mathcal{P}_t u\|^2 \leqslant C\int_{\mathbb{R}^2}\Big(e^{-\frac{1}{4}t|q_1|^2}\Big)^2\mathrm{d}q_1\ \|u\|^2 \leqslant \frac{C}{t}\|u\|^2.$$

(b) Fix $\mathrm{Re}(z)<0$, integrate $\langle v, \mathcal{S}_{ij}\mathcal{P}_t u\rangle$ against e^{zt} over $t\in(0,\infty)$, and use (4.3) to get

$$\int_0^\infty e^{zt}\langle v, \mathcal{S}_{ij}\mathcal{P}_t u\rangle\,\mathrm{d}t = \int_0^\infty \int_{\mathbb{R}^{2n}} \overline{\widehat{v}(q_{2-n})}e^{tz - \frac{t}{2}|M_{ij}q|^2}\widehat{u}(M_{ij}q)(2\pi)^{-1}\,\mathrm{d}t\mathrm{d}q.$$

This integral converges absolutely since $\|\mathcal{S}_{ij}\mathcal{P}_t\|_{\mathrm{op}} \leqslant Ct^{-1/2}$ and $\mathrm{Re}(z)<0$. This being the case, we swap the integrals and evaluate the integral over t to get

$$\int_0^\infty e^{zt}\langle v, \mathcal{S}_{ij}\mathcal{P}_t u\rangle\,\mathrm{d}t = \int_{\mathbb{R}^{2n}} \overline{\widehat{v}(q_{2-n})}\frac{1}{\frac{1}{2}|M_{ij}q|^2 - z}\widehat{u}(M_{ij}q)\,\frac{\mathrm{d}q}{2\pi}.$$

The last expression matches $\langle v, \mathcal{S}_{ij}\mathcal{G}_z u\rangle$, as seen from (4.9). ☐

Lemma 8.2.

(a) For distinct pairs $(i<j)\neq(k<\ell)$, $t\in\mathbb{R}_+$, $\mathcal{P}_t\mathcal{S}_{k\ell}^*(\mathscr{L}^2(\mathbb{R}^{2n-2})) \subset \mathrm{Dom}(\mathcal{S}_{ij})$, so the operator $\mathcal{S}_{ij}\mathcal{P}_t\mathcal{S}_{k\ell}^*$ maps $\mathscr{L}^2(\mathbb{R}^{2n-2}) \to \mathscr{L}^2(\mathbb{R}^{2n-2})$. Further

$$\|\mathcal{S}_{ij}\mathcal{P}_t\mathcal{S}_{k\ell}^*\|_{\mathrm{op}} \leqslant Ct^{-1}.$$

(b) For distinct pairs $(i<j)\neq(k<\ell)$, $v,w\in\mathscr{L}^2(\mathbb{R}^{2n-2})$, *and* $\mathrm{Re}(z)<0$,

$$\int_{\mathbb{R}_+\times\mathbb{R}^{4n-4}} e^{zt}\overline{w}(y)P(t,S_{ij}y-S_{k\ell}y')v(y')\,\mathrm{d}t\mathrm{d}y\mathrm{d}y' = \langle w,\mathcal{S}_{ij}\mathcal{G}_z\mathcal{S}^*_{k\ell}v\rangle, \tag{8.1}$$

where the integral converges absolutely.

Remark 8.3. Unlike in the case for incoming and outgoing operators, here our bound on Ct^{-1} on the mediating operator does not ensure the integrability of $\|\mathcal{S}_{ij}\mathcal{P}_t\mathcal{S}^*_{k\ell}\|_{\mathrm{op}}$ near $t=0$. Nevertheless, the integral in (8.1) still converges absolutely.

Proof. Fix distinct pairs $(i<j)\neq(k<\ell)$ and $v,w\in\mathscr{L}(\mathbb{R}^{2n-2})$.
(a) As argued just before Lemma 8.1, we have $\mathcal{P}_t\mathcal{S}^*_{k\ell}v\in\mathscr{L}^2(\mathbb{R}^{2n})$. To check the condition $\mathcal{P}_t\mathcal{S}^*_{k\ell}v\in\mathrm{Dom}(\mathcal{S}_{ij})$, consider

$$\int_{\mathbb{R}^{2n}}\Big|\widehat{w}(q_{2-n})e^{-\frac{t}{2}|M_{ij}q|^2}\widehat{\mathcal{S}^*_{k\ell}v}(M_{ij}q)\Big|\frac{\mathrm{d}q}{2\pi} = \int_{\mathbb{R}^{2n}}\Big|\widehat{w}(p_i+p_j,p_{\overline{ij}})e^{-\frac{t}{2}|p|^2}\widehat{\mathcal{S}^*_{k\ell}v}(p)\Big|\frac{\mathrm{d}p}{2\pi}, \tag{8.2}$$

where the equality follows by a change of variable $q=M_{ij}^{-1}p$, together with $(p_i+p_j,p_{\overline{ij}})=[M_{ij}^{-1}p]_{2-n}$ and $|\det(M_{ij})|=1$ (as readily verified from (4.1)). In (8.2), bound $e^{-\frac{t}{2}|p|^2}\leqslant C\,(t|p|^2)^{-1}$ and use (5.2) to get

$$(8.2)\leqslant C\,t^{-1}\|v\|\,\|w\|. \tag{8.3}$$

Referring to the definition (4.2) of $\mathrm{Dom}(\mathcal{S}_{ij})$, since (5.4) holds for all $w\in\mathscr{L}^2(\mathbb{R}^{2n-2})$, we conclude $\mathcal{P}_t\mathcal{S}^*_{k\ell}v\in\mathrm{Dom}(\mathcal{S}_{ij})$ and $|\langle w,\mathcal{S}_{ij}\mathcal{P}_t\mathcal{S}^*_{k\ell}v\rangle|=|\langle\mathcal{S}^*_{ij}w,\mathcal{P}_t\mathcal{S}^*_{k\ell}v\rangle|\leqslant Ct^{-1}\|w\|\,\|v\|$.

(b) To prove (8.1), assume for a moment $z=-\lambda\in(-\infty,0)$ is real, and $v(y),w(y)\geqslant0$ are positive. In (8.1), express the integral over y,y' as $\langle w,\mathcal{S}_{ij}\mathcal{P}_t\mathcal{S}^*_{k\ell}v\rangle=\langle\mathcal{S}^*_{ij}w,\mathcal{P}_t\mathcal{S}^*_{k\ell}v\rangle$, and use (5.3) to get

$$\int_{\mathbb{R}_+\times\mathbb{R}^{4n-4}} e^{zt}\overline{w}(y)P(t,S_{ij}y-S_{k\ell}y')v(y')\,\mathrm{d}t\mathrm{d}y\mathrm{d}y' = \int_0^\infty e^{-\lambda t}\Big(\int_{\mathbb{R}^{2n}}\overline{\widehat{w}(p_i+p_j,p_{\overline{ij}})}e^{-\frac{t}{2}|p|^2}\widehat{v}(p_k+p_\ell,p_{\overline{k\ell}})\,\mathrm{d}p\Big)\,\mathrm{d}t.$$

The integral on the r.h.s. converges absolutely over $\mathbb{R}_+\times\mathbb{R}^{2n}$, i.e., jointly in t,p. This follows by using (5.2) together with $\int_0^\infty e^{-\lambda t-\frac{t}{2}|p|^2}\mathrm{d}t=\frac{1}{\lambda+\frac12|p|^2}$. Given the absolute convergence, we swap the integrals over t and over p, and evaluate the former to get the expression for $\langle w,\mathcal{S}_{ij}\mathcal{G}_z\mathcal{S}^*_{k\ell}v\rangle$ on the right hand side of (5.1). For general $v(y),w(y)$, the preceding calculation done for $(v(y),w(y))\mapsto(|v(y)|,|w(y)|)$ and for $z\mapsto\mathrm{Re}(z)$ guarantees the relevant integrability. ☐

Recall $\mathsf{j}(t,\beta_\star)$ from (1.9). For the diagonal mediating operator, let us first settle some properties of j.

Lemma 8.4. *For each* $\mathrm{Re}(z)<-e^{\beta_\star}$, *the Laplace transform of* $\mathsf{j}(t,\beta_\star)$ *evaluates to*

$$\int_0^\infty e^{zt}\mathsf{j}(t,\beta_\star)\,\mathrm{d}t=\frac{1}{\log(-z)-\beta_\star}, \tag{8.4}$$

where the integral converges absolutely, and $\mathsf{j}(t,\beta_\star)$ *has the following pointwise bound*

$$\mathsf{j}(t,\beta_\star)=|\mathsf{j}(t,\beta_\star)|\leqslant C\,t^{-1}\,|\log(t\wedge\tfrac12)|^{-2}e^{(\beta_\star+1)Ct},\quad t\in\mathbb{R}_+. \tag{8.5}$$

Proof. To evaluate the Laplace transform, assume for a moment that $z\in(-\infty,-e^{\beta_\star})$ is real. Integrate (1.9) against e^{zt} over t. Under the current assumption that z is real, the integrand therein is positive, so we apply Fubini's theorem to swap the t and α integrals to get

$$\int_0^\infty e^{zt}\mathsf{j}(t,\beta_\star)\,\mathrm{d}t=\int_0^\infty\frac{e^{\beta_\star\alpha}}{\Gamma(\alpha)}\Big(\int_0^\infty t^{\alpha-1}e^{-(-zt)}\mathrm{d}t\Big)\mathrm{d}\alpha.$$

The integral over t, upon a change of variable $-zt\mapsto t$, evaluates to $\Gamma(\alpha)/(-z)^\alpha$. Canceling the $\Gamma(\alpha)$ factors and evaluating the remaining integral over α yields (8.4) for $z\in(-\infty,-e^{\beta_\star})$. For general $z\in\mathbb{C}$ with $\mathrm{Re}(z)<-e^{\beta_\star}$, since $|e^{zt}|=e^{\mathrm{Re}(z)t}$, the preceding result guarantees integrability of $|e^{-zt+\alpha\beta_\star}t^{\alpha-1}\Gamma(\alpha)^{-1}|$ over $(t,\alpha)\in\mathbb{R}^2_+$. Hence Fubini's theorem still applies, and (8.4) follows.

To show (8.5), in (1.9), we separate the integral (over $\alpha\in\mathbb{R}_+$) into two integrals over $\alpha>1$ and over $\alpha<1$, denoted by I_+ and I_-, respectively. For I_+, we use the bound $\exp(-\log\Gamma(\alpha))\leqslant\frac{\alpha}{2}\log\alpha-C\alpha$ (c.f., [AS65, 6.1.40]) to write $I_+\leqslant\int_1^\infty\exp(-\alpha(\frac12\log\alpha-(C+\beta_\star)-\log t))\mathrm{d}\alpha$. It is now straightforward to check that $I_+\leqslant e^{(\beta_\star+1)Ct}$. Using $|\frac{1}{\Gamma(\alpha)}|\leqslant C\alpha$, $\alpha\in(0,1)$ (c.f., [AS65, 6.1.34]), we bound I_- as $I_-\leqslant Ct^{-1}e^{\beta_\star}\int_0^1\alpha t^\alpha\mathrm{d}\alpha$.

For all $t \geqslant \frac{1}{2}$, the last integral is indeed bounded by $e^{(\beta_\star+1)Ct}$. For $t < \frac{1}{2}$, we write $t^\alpha = e^{-\alpha|\log t|}$ we perform a change of variable $\alpha|\log t| \to t$ to get $I_- \leqslant C\,t^{-1}e^{\beta_\star}|\log t|^{-2}\int_0^{|\log t|}\alpha e^{-\alpha}\mathrm{d}\alpha \leqslant C\,t^{-1}e^{\beta_\star}|\log t|^{-2}$. Collecting the preceding bounds and adjusting the constant C gives (8.5). ☐

Referring to the definition (1.12) of $\mathcal{P}_t^{\mathcal{J}}$, we see that this operator has an integral kernel

$$\big(\mathcal{P}_t^{\mathcal{J}}v\big)(y) = \int_{\mathbb{R}^{2n-2}} P^{\mathcal{J}}(t,y,y')v(y')\,\mathrm{d}y', \quad P^{\mathcal{J}}(t,y,y') := \mathsf{j}(t,\beta_\star)\mathsf{p}(\tfrac{t}{2},y_2-y_2')\prod_{i=3}^{n}\mathsf{p}(t,y_i-y_i'). \tag{8.6}$$

Lemma 8.5.

(a) For each $t\in\mathbb{R}_+$, $\mathcal{P}_t^{\mathcal{J}} : \mathscr{L}^2(\mathbb{R}^{2n-2}) \to \mathscr{L}^2(\mathbb{R}^{2n-2})$ is a bounded operator with

$$\|\mathcal{P}_t^{\mathcal{J}}\|_{\mathrm{op}} \leqslant C\,(t\wedge\tfrac{1}{2})^{-1}\,|\log(t\wedge\tfrac{1}{2})|^{-2}e^{(\beta_\star+1)Ct}. \tag{8.7}$$

(b) Further, for each $v,w\in\mathscr{L}^2(\mathbb{R}^{2n-2})$ and $\mathrm{Re}(z) < -e^{\beta_\star}$,

$$\int_{\mathbb{R}_+} e^{zt}\,\langle w,\mathcal{P}_t^{\mathcal{J}}v\rangle\,\mathrm{d}t = \int_{\mathbb{R}_+\times\mathbb{R}^{4n-4}} e^{zt}\,\overline{w(y)}P^{\mathcal{J}}(t,y,y')v(y')\,\mathrm{d}t\mathrm{d}y\mathrm{d}y' = \langle w,(\mathcal{J}_z-\beta_\star\mathbf{I})^{-1}v\rangle, \tag{8.8}$$

where the integrals converge absolutely (over $\mathbb{R}_+$ and over $\mathbb{R}_+\times\mathbb{R}^{4n-4}$).

Proof. Part (a) follows from (8.5) and the fact that heat semigroups have unit norm, i.e., $\|e^{-at\nabla_i^2}\|_{\mathrm{op}} = 1$, $a\geqslant 0$. For part (b), we work in Fourier domain and write

$$\int_{\mathbb{R}^{4n-4}}\overline{w(y)}P^{\mathcal{J}}(t,y,y')v(y')\,\mathrm{d}t\mathrm{d}y\mathrm{d}y' = \mathsf{j}(t,\beta_\star)\int_{\mathbb{R}^{2n-2}}\overline{\widehat{w}(p)}e^{-\frac{1}{2}t|p|^2_{2-n}}\widehat{v}(p)\mathrm{d}p,$$

where, recall that $|p|^2_{2-n} = \frac{1}{2}|p_2|^2 + |p_3|^2 + \ldots + |p_n|^2$. Integrate both sides against e^{zt} over $t\in\mathbb{R}_+$, and exchange the integrals over p and over t. The swap of integrals are justified the same way as in the proof of Lemma 8.2, so we do not repeat it here. We now have

$$\int_{\mathbb{R}_+\times\mathbb{R}^{4n-4}} e^{zt}\overline{w(y)}P^{\mathcal{J}}(t,y,y')v(y')\,\mathrm{d}t\mathrm{d}y\mathrm{d}y' = \int_{\mathbb{R}^{2n-2}}\Big(\int_0^\infty e^{zt-\frac{1}{2}t|p|^2_{2-n}}\mathsf{j}(t,\beta_\star)\mathrm{d}t\Big)\overline{\widehat{w}(p)}\widehat{v}(p)\mathrm{d}p.$$

Applying (8.4) to evaluate the integral over t yields the expression in (1.22) for $\langle w,(\mathcal{J}_z-\beta_\star\mathbf{I})^{-1}v\rangle$. ☐

8.2. **An identity for the semigroup property.** Our goal is to prove Lemma 8.8 in the following. Key to the proof is the identity (8.12). It depends on a cute fact about the Γ function. Set

$$p_k(\alpha) := \frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+1)} = (\alpha+k)\cdots(\alpha+1)\alpha, \quad \alpha\geqslant 0. \tag{8.9}$$

with the convention $p_{-1} := 1$.

Lemma 8.6. *For $m\in\mathbb{Z}_{\geqslant 0}$,*

$$p_m(\alpha) = \int_0^\alpha\sum_{k=0}^{m}\binom{m+1}{m-k+1}(m-k)!\,p_{k-1}(\alpha_1)\,\mathrm{d}\alpha_1.$$

Proof. Taking derivative gives $\frac{\mathrm{d}}{\mathrm{d}\alpha}p_m(\alpha) = \sum_{j=0}^m\prod_{j^c}^m(\alpha+i)$, where $\prod_{j^c}^m$ denotes a product over $i\in\{0,\ldots,m\}\setminus\{j\}$. Our goal is to express this derivative in terms of $p_{m-1}(\alpha), p_{m-2}(\alpha),\ldots$. The $j=m$ term skips the $(\alpha+m)$ factor, and is hence exactly $p_{m-1}(\alpha)$. For other values of j, we use $(\alpha-m)$ to compensate the missing $(\alpha+j)$ factor. Namely, writing $(\alpha+m) = (\alpha+j+(m-j))$, we have

$$\prod\nolimits_{j^c}^m(\alpha+i) = p_{m-1}(\alpha) + (m-j)\prod\nolimits_{j^c}^{m-1}(\alpha+i). \tag{8.10}$$

This gives

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}p_m(\alpha) = \sum_{j=0}^m\prod\nolimits_{j^c}^m(\alpha+i) = \sum_{j=0}^m p_{m-1}(\alpha) + \sum_{j=0}^m(m-j)\prod\nolimits_{j^c}^{m-1}(\alpha+i).$$

In (8.10), we have reduced $\prod_{j^c}^{m}(\alpha+i)$ to $\prod_{j^c}^{m-1}(\alpha+i)$, i.e., the same expression but with m decreased by 1. Repeating this procedure yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\alpha}p_m(\alpha) &= \sum_{\ell=1}^{m} p_{m-\ell}(\alpha)\Big(\sum_{j=0}^{m}(m-j)_+(m-j-1)_+\cdots(m-j-\ell)_+\Big) \\ &= \sum_{\ell=1}^{m} p_{m-\ell-1}(\alpha)\sum_{j=0}^{m}\prod_{i=0}^{\ell-1}(j-i)_+ = \sum_{\ell=1}^{m} p_{m-\ell-1}(\alpha)\binom{m+1}{\ell+1}\ell!\,,\end{aligned} \tag{8.11}$$

where $\prod_{i\in\varnothing}(\cdot):=1$. Within the last equality, we have used the identity $\sum_{j=0}^{m}\prod_{i=0}^{\ell-1}(j-i)_+ = \binom{m+1}{\ell+1}\ell!$. In (8.11), perform a change of variable $m-\ell:=k$, and integrate in α, using $p_m(0)=0$ to get the result. □

Lemma 8.7. *For $s<t\in\mathbb{R}_+$, $i<j$, we have*

$$\mathsf{j}(t,\beta_\star) = \int_{0<t_1<s}\int_{s<t_2<t}\mathsf{j}(t_1,\beta_\star)(t_2-t_1)^{-1}\mathsf{j}(t-t_2,\beta_\star)\,\mathrm{d}t_1\mathrm{d}t_2. \tag{8.12}$$

Proof. Write $\mathsf{j}(t,\beta_\star)=\mathsf{j}(t)$ to simplify notation. Let the r.h.s. of (8.12) be denoted by $F(s,t)$. It is standard to check that $F(s,t)$ is continuous on $0<s<t<\infty$. Hence it suffices to show

$$\int_0^t F(s,t)s^m\,\mathrm{d}s = \mathsf{j}(t)\int_0^t s^m\,\mathrm{d}s = \mathsf{j}(t)\,(m+1)^{-1}t^{m+1},\quad m\in\mathbb{Z}_{\geqslant 0}. \tag{8.13}$$

From (8.5), it is readily checked both sides of (8.13) grow at most exponentially in t. Taking Laplace transform on both sides of (8.5), the problem is further reduced to showing, for some $C(m,\beta_\star)<\infty$,

$$\int_0^\infty\int_0^t e^{-\lambda t}F(s,t)s^k\,\mathrm{d}t\mathrm{d}s = \int_0^\infty e^{-\lambda t}\mathsf{j}(t)\,(m+1)^{-1}t^{m+1}\mathrm{d}t,\quad \lambda>C(m,\beta_\star). \tag{8.14}$$

The left hand side can be computed

$$\text{l.h.s. of (8.14)} = \int_0^\infty \frac{e^{\beta_\star\alpha}\lambda^{-\alpha-m-1}}{m+1}\Big(\int_0^\alpha\sum_{k=0}^{m}\binom{m+1}{m-k+1}(m-k)!\,p_{k-1}(\alpha_1)\mathrm{d}\alpha_1\Big)\mathrm{d}\alpha. \tag{8.15}$$

The integral (8.15) is indeed finite for large enough $\lambda\geqslant C(\beta_\star,m)$. The right hand side is given by

$$\text{r.h.s. of (8.14)} = \int_0^\infty\frac{e^{\beta_\star\alpha}\lambda^{-\alpha-m-1}}{m+1}p_m(\alpha)\,\mathrm{d}\alpha. \tag{8.16}$$

By Lemma 8.6 the two coincide. □

Lemma 8.8. *For $t'<s<t\in\mathbb{R}_+$, $i<j$, we have*

$$\int_{t'<t_1<s}\int_{s<t_2<t}(4\pi\mathcal{P}^{\mathcal{J}}_{t_1-t'})\mathcal{S}_{ij}\mathcal{P}_{t_2-t_1}\mathcal{S}^*_{ij}(4\pi\mathcal{P}^{\mathcal{J}}_{t-\tau_2})\,\mathrm{d}t_1\mathrm{d}t_2 = \mathcal{P}^{\mathcal{J}}_{t-t'}. \tag{8.17}$$

Remark 8.9. The integral (8.17) converges absolutely in operator norm. This is seem by writing $\mathcal{S}_{ij}\mathcal{P}_{t_2-t_1}\mathcal{S}^*_{ij} = (\mathcal{S}_{ij}\mathcal{P}_{s-t_1})(\mathcal{P}_{t_2-s}\mathcal{S}^*_{ij})$, and by using the bounds from Lemmas 8.1(a) and 8.5(a).

Proof. For $\tau>0$, the operator $\mathcal{S}_{ij}\mathcal{P}_\tau\mathcal{S}^*_{ij}$ has an integral kernel $P(\tau,S_{ij}(y-y'))=(\mathsf{p}(\tau,y_2-y_2))^2\prod_{i=3}^{n}\mathsf{p}(\tau,y_i-y_i)$, where p denotes the two-dimensional heat kernel. From this and $(\mathsf{p}(\tau,y))^2=\frac{1}{4\pi\tau}\mathsf{p}(\frac{\tau}{2},y)$, we have $\mathcal{S}_{ij}\mathcal{P}_\tau\mathcal{S}^*_{ij}=\frac{1}{4\pi\tau}\exp(-\frac{\tau}{4}\nabla_2^2-\frac{\tau}{2}\sum_{i=3}^{n}\nabla_i^2)$. Recall that $\mathcal{P}^{\mathcal{J}}_\tau:=\mathsf{j}(\tau,\beta_\star)\exp(-\frac{\tau}{4}\nabla_2^2-\frac{\tau}{2}\sum_{i=3}^{n}\nabla_i^2)$. We obtain

$$\text{l.h.s. of (8.17)} = 4\pi e^{-\frac{t-t'}{4}\nabla_2^2-\frac{t-t'}{2}\sum_{i=3}^{n}\nabla_i^2}\int_{t'<t_1<s}\int_{s<t_2<t}\mathsf{j}(t_1-t')(t_2-t_1)^{-1}\mathsf{j}(t-t_2)\,\mathrm{d}t_1\mathrm{d}t_2.$$

The desired result now follows from Lemma 8.12. □

8.3. **Proof of Theorem 1.1.** We begin with a quantitative bound on $\mathcal{D}_t^{\overrightarrow{(i,j)}}$.

Lemma 8.10. *For* $\overrightarrow{(i,j)} = ((i_k, j_k))_{k=1}^m \in \mathrm{Dgm}(n,m)$, $t \in \mathbb{R}_+$ *and* $\lambda \geqslant 2$, *we have*

$$\|\mathcal{D}_t^{\overrightarrow{(i,j)}}\|_{\mathrm{op}} \leqslant C(\log(\tfrac{t}{2m+1} \wedge \tfrac{1}{2}))^{-1} m^2 e^{\lambda C(\beta_\star+1)t} (C/\log\lambda)^{m-1}. \tag{8.18}$$

Proof. To simplify notation, we index the incoming and outgoing operators by 0 and by m: $\mathcal{Q}_{\tau_0}^{(0)} := \mathcal{P}_{\tau_0}\mathcal{S}_{i_1j_1}^*$, $\mathcal{Q}_{\tau_m}^{(m)} := \mathcal{S}_{i_mj_m}\mathcal{P}_{\tau_m}$, index the diagonal mediating operators by half integers: $\mathcal{Q}_{\tau_a}^{(a)} := 4\pi\mathcal{P}_{\tau_a}^{\mathcal{J}}$, $a \in (\frac{1}{2}+\mathbb{Z}) \cap (0,m)$, and index the off-diagonal mediating operators by integers: $\mathcal{Q}_{\tau_a}^{(a)} := \mathcal{S}_{i_aj_a}\mathcal{P}_{\tau_a}\mathcal{S}_{i_{a+1}j_{a+1}}^*$, $a \in \mathbb{Z}\cap(0,m)$. Under these notation

$$\mathcal{D}_t^{\overrightarrow{(i,j)}} = \int_{\Sigma_m(t)} \mathcal{Q}_{\tau_0}^{(0)}\,\mathcal{Q}_{\tau_{1/2}}^{(1/2)} \cdots \mathcal{Q}_{\tau_m}^{(m)}\,\mathrm{d}\vec{\tau}. \tag{1.17'}$$

Recall that the incoming, outgoing, and mediating operators all have positive integral kernels, c.f., (2.1)–(2.2), (2.3), (8.6). Accordingly, we *interpret* the r.h.s. of (1.17') as an integral operator, with integrand consisting of a convolution of the aforementioned kernels. We seek to bound

$$\int_{\Sigma_m(t)} \Big|\Big\langle u', \prod_{a\in A} \mathcal{Q}_{\tau_a}^{(a)} u\Big\rangle\Big|\,\mathrm{d}\vec{\tau} = \int_{\Sigma_m(t)} \Big\langle |u'|, \prod_{a\in A} \mathcal{Q}_{\tau_a}^{(a)}|u|\Big\rangle\,\mathrm{d}\vec{\tau}, \tag{8.19}$$

for $u, u' \in \mathscr{L}^2(\mathbb{R}^{2n})$.

An undesirable feature of (8.19) is the constraint $\tau_0 + \tau_{1/2} + \ldots + \tau_m = t$ from $\Sigma_m(t)$. To break such a constraint, fix $\lambda \geqslant 2$. In (8.19), multiply and divide by $e^{\lambda\beta_\star t}$, and use $\Sigma_m(t) \subset (\cup_{a\in A}\{\tau_a \geqslant \frac{t}{2m+1}\}) \cap (0,t)^{2m+1}$ to obtain

$$\|\mathcal{D}_t^{\overrightarrow{(i,j)}}\|_{\mathrm{op}} \leqslant e^{\lambda\beta_\star t}\sum_{a\in A} F_a, \quad F_a := \Big(\sup_{\tau\in[\frac{t}{2m+1},t]} e^{-\lambda\beta_\star\tau}\|\mathcal{Q}_\tau^{(a)}\|_{\mathrm{op}}\Big) \prod_{a'\in A\setminus\{a\}} \Big\|\int_0^t e^{-\lambda\beta_\star\tau}\mathcal{Q}_\tau^{(a')}\mathrm{d}\tau\Big\|_{\mathrm{op}}. \tag{8.20}$$

To bound the 'sup' term in (8.20), forgo the exponential factor (i.e., $e^{-\lambda\beta_\star\tau} \leqslant 1$), and use the bound on $\|\mathcal{Q}_\tau^{(a)}\|_{\mathrm{op}}$ from Lemmas 8.1(a), 8.2(a), and 8.5(a). We have

$$\begin{aligned}\sup_{\tau\in[\frac{t}{2m+1},t]} e^{-\lambda\beta_\star\tau}\|\mathcal{Q}_\tau^{(a)}\|_{\mathrm{op}} &\leqslant C\left\{\begin{array}{ll}(t/m)^{-1/2} & \text{, for } a=0,m,\\ (t/m)^{-1} & \text{, for } a\in\mathbb{Z}\cap(0,m),\\ (t/m)^{-1}\,(\log(\frac{t}{2m+1}\wedge\frac12))^{-2}e^{C(1+\beta_\star)t}, & \text{for } a\in(\frac12+\mathbb{Z})\cap(0,m),\end{array}\right\}\\ &\leqslant Cme^{C(1+\beta_\star)t}\left\{\begin{array}{ll}t^{-1/2} & \text{, for } a=0,m,\\ t^{-1} & \text{, for } a\in\mathbb{Z}\cap(0,m),\\ t^{-1}\,(\log(\frac{t}{2m+1}\wedge\frac12))^{-2}, & \text{for } a\in(\frac12+\mathbb{Z})\cap(0,m).\end{array}\right.\end{aligned} \tag{8.21}$$

Moving on, to bound the integral terms in (8.20), for $a' \in \{0,m\}\cup((\frac12\mathbb{Z})\cap(0,m))$, we forgo the exponential factor, and use the bound from Lemma 8.1(a) to get

$$\Big\|\int_0^t e^{\lambda\beta_\star\tau}\mathcal{Q}_\tau^{(a')}\mathrm{d}\tau\Big\|_{\mathrm{op}} \leqslant \int_0^t \|\mathcal{Q}_\tau^{(a')}\|_{\mathrm{op}}\mathrm{d}\tau \leqslant Ct^{1/2}, \quad \text{for } a'=0,m, \tag{8.22}$$

$$\Big\|\int_0^t e^{\lambda\beta_\star\tau}\mathcal{Q}_\tau^{(a')}\mathrm{d}\tau\Big\|_{\mathrm{op}} \leqslant \int_0^t \|\mathcal{Q}_\tau^{(a')}\|_{\mathrm{op}}\mathrm{d}\tau \leqslant C\,(\log(\tfrac{t}{2m+1}\wedge\tfrac12))^{-1}e^{C(1+\beta_\star)t}, \quad \text{for } a'\in(\tfrac12+\mathbb{Z})\cap(0,m). \tag{8.23}$$

The bound (8.23) gives a useful logarithmic decay in $t\to 0$, but has an undesirable exponential growth in $t\to\infty$. We will also need a bound that does not exhibit the exponential growth. For $a'\in(\frac12\mathbb{Z})\cap(0,m)$, we use the fact that $\mathcal{Q}_\tau^{(a')}$ is an integral operator with a *positive* kernel to write

$$\Big\|\int_0^t e^{-\lambda\beta_\star\tau}\mathcal{Q}_\tau^{(a')}\mathrm{d}\tau\Big\|_{\mathrm{op}} \leqslant \Big\|\int_0^\infty e^{-\lambda\beta_\star\tau}\mathcal{Q}_\tau^{(a')}\mathrm{d}\tau\Big\|_{\mathrm{op}}.$$

The last expression is a Laplace transform, and has been evaluated in Lemmas 8.2(b) and 8.5(b), whereby

$$\Big\|\int_0^t e^{-\lambda\beta_\star\tau}\mathcal{Q}_\tau^{(a')}\mathrm{d}\tau\Big\|_{\mathrm{op}} \leqslant \left\{\begin{array}{ll}\|\mathcal{S}_{ij}\mathcal{G}_{-\lambda\beta_\star}\mathcal{S}_{k\ell}^*\|_{\mathrm{op}} & \text{, for } a'\in(0,m)\cap\mathbb{Z},\\ \|(\mathcal{J}_{-\lambda\beta_\star}-\beta_\star)^{-1}\|_{\mathrm{op}}, & \text{for } a'\in(0,m)\cap(\frac12+\mathbb{Z}).\end{array}\right.$$

Here $(i<j)\neq(k<\ell)$ corresponds to the index a'. Using the bounds on $\|\mathcal{S}_{ij}\mathcal{G}_z\mathcal{S}^*_{k\ell}\|_{\mathrm{op}}$ from Lemma 5.1 and the bound $\|(\mathcal{J}_{-\lambda\beta_\star}-\beta_\star)^{-1}\|\leqslant 1/\log\lambda$ (c.f., (1.22)) we have

$$\Big\|\int_0^t e^{-\lambda\beta_\star\tau}\mathcal{Q}^{(a')}_\tau \mathrm{d}\tau\Big\|_{\mathrm{op}} \leqslant C\left\{\begin{array}{ll} 1 & \text{, for } a'\in(0,m)\cap\mathbb{Z}, \\ (\log\lambda)^{-1}, & \text{for } a'\in(0,m)\cap(\frac{1}{2}+\mathbb{Z}). \end{array}\right. \tag{8.24}$$

For $a\in\frac{1}{2}\mathbb{Z}$, inserting the bounds (8.21)–(8.22), (8.24) into (8.20) gives

$$F_a\leqslant Cme^{\lambda C\,(\beta_\star+1)t}(\log(\tfrac{t}{2m+1}\wedge\tfrac{1}{2}))^{-2}\,t^{-1+\frac{1}{2}+\frac{1}{2}}\,(\log\lambda)^{m-1}C^{2m+1}.$$

For $a\notin\frac{1}{2}\mathbb{Z}$, in (8.20), use the bound (8.21) for the sup term, use (8.23) for $a'=\frac{1}{2}$, and use (8.22) and (8.24) for other a'. This gives

$$F_a\leqslant Cme^{\lambda C\,(\beta_\star+1)t}(\log(\tfrac{t}{2m+1}\wedge\tfrac{1}{2}))^{-1}\left\{\begin{array}{ll} t^{-1/2+1/2} & \text{, for } a\in\{0,m\} \\ t^{-1+1/2+1/2}, & \text{for } a\in\mathbb{Z}\cap(0,m) \end{array}\right\}(\log\lambda)^{m-1}C^{2m+1}.$$

Inserting these bounds on F_a into (8.20), we conclude the desired result (8.18). □

Proof of Theorem 1.1(a). Sum the bound (8.18) over $\overrightarrow{(i,j)}\in\mathrm{Dgm}(n)$, and note that $|\mathrm{Dgm}(n,m)|\leqslant(n(n-1)/2)^m$ (c.f., (1.13)). In the result, choose $\lambda=Cn^2$ for some large but fixed $C<\infty$, we have

$$\big\|\mathcal{D}^{\mathrm{Dgm}(n)}_t\big\|_{\mathrm{op}}\leqslant\sum_{m=1}^\infty m^2n^2(\log(\tfrac{t}{2m+1}\wedge\tfrac{1}{2}))^{-1}\,2^{-(m-1)}\exp\big(Ce^{Cn^2}(\beta_\star+1)t\big) \tag{8.25}$$

$$\leqslant C\,n^2\exp\big(e^{Cn^2}(\beta_\star+1)Ct\big). \tag{8.26}$$

This verifies that $\mathcal{D}^{\mathrm{Dgm}(n)}_t$ defines a bounded operator on $\mathscr{L}^2(\mathbb{R}^{2n})$.

To show the semigroup property, we fix $s<t\in\mathbb{R}_+$ and calculate $(\mathcal{P}_s+\mathcal{D}^{\mathrm{Dgm}(n)}_s)(\mathcal{P}_{t-s}+\mathcal{D}^{\mathrm{Dgm}(n)}_{t-s})$, which boils down to calculating $\mathcal{P}_s\mathcal{P}_{t-s}$, $\mathcal{P}_s\mathcal{D}^{\overrightarrow{(i',j')}}_{t-s}$, $\mathcal{D}^{\overrightarrow{(i,j)}}_s\mathcal{P}_{t-s}$, $\mathcal{D}^{\overrightarrow{(i,j)}}_s\mathcal{D}^{\overrightarrow{(i',j')}}_{t-s}$, for $\overrightarrow{(i,j)}\in\mathrm{Dgm}(n,m)$ and $\overrightarrow{(i',j')}\in\mathrm{Dgm}(n,m')$. To streamline notation, we relabel time variables as $t_k:=\tau_0+\ldots+\tau_{k/2-1}$, and set

$$B^{\overrightarrow{(i,j)}}(\vec{t}):=\mathcal{P}_{t_1}\mathcal{S}^*_{i_1j_1}\big(4\pi\mathcal{P}^{\mathcal{J}}_{t_2-t_1}\big)\Big(\prod_{k=1}^{m-1}\mathcal{S}_{i_kj_k}\mathcal{P}_{t_{2k+1}-t_{2k}}\mathcal{S}^*_{i_{k+1}j_{k+1}}\big(4\pi\mathcal{P}^{\mathcal{J}}_{\tau_{2k+2}-t_{2k+1}}\big)\Big)\mathcal{S}_{i_mj_m}\mathcal{P}_{t-t_{2m}}.$$

Using (1.17') and the semigroup property of $\mathcal{P}_\bullet$, we have $\mathcal{P}_s\mathcal{P}_{t-s}=\mathcal{P}_t$,

$$\mathcal{P}_s\mathcal{D}^{\overrightarrow{(i',j')}}_{t-s}=\int_{(s,t)^{2m'}_<}B^{\overrightarrow{(i',j')}}(\vec{t})\,\mathrm{d}\vec{t}, \tag{8.27}$$

$$\mathcal{D}^{\overrightarrow{(i,j)}}_s\mathcal{P}_{t-s}=\int_{(0,s)^{2m}_<}B^{\overrightarrow{(i,j)}}(\vec{t})\,\mathrm{d}\vec{t}, \tag{8.28}$$

$$\mathcal{D}^{\overrightarrow{(i,j)}}_s\mathcal{D}^{\overrightarrow{(i',j')}}_{t-s}=\int_{\Omega_{2m,2m'}(s,t)}B^{\overrightarrow{(i'',j'')}}(\vec{t})\,\mathrm{d}\vec{t}, \tag{8.29}$$

where $(a,b)^k_<:=\{\vec{t}\in(a,b)^k: a<t_1<\ldots<t_k<b\}$, $\Omega_{k,\ell}(s,t):=\{\vec{t}\in(0,t)^{k+\ell}:\ldots<t_k<s<t_{k+1}<\ldots<t_{k+\ell}<t\}$, and $\overrightarrow{(i'',j'')}$ is obtained by concatenating $\overrightarrow{(i,j)}$ and $\overrightarrow{(i',j')}$, i.e.,

$$\overrightarrow{(i'',j'')}=(i''_k,j''_k)_{k=1}^{m+m'}:=\big((i_1<j_1),\ldots,(i_m<j_m),(i'_1<j'_1),\ldots,(i_{m'}<j_{m'})\big).$$

Such an index is not necessarily in $\mathrm{Dgm}(n)$, because we could have $(i_m<j_m)=(i'_1<j'_1)$. When this happens, applying Lemma 8.8 with $(i,j)=(i_m,j_m)$ and with $(t',t)\mapsto(t_{2m-1},t_{2m+2})$ gives

$$\mathcal{D}^{\overrightarrow{(i,j)}}_s\mathcal{D}^{\overrightarrow{(i',j')}}_{t-s}=\int_{\Omega_{2m-1,2m'-1}(s,t)}B^{\overrightarrow{(i''',j''')}}(\vec{t})\,\mathrm{d}\vec{t}, \tag{8.29'}$$

where $\overrightarrow{(i''',j''')}$ is obtained by removing $(i'_1<j'_1)$ from $\overrightarrow{(i'',j'')}$, i.e.,

$$\overrightarrow{(i''',j''')}:=\big((i_1<j_1),\ldots,(i_m<j_m),(i_2<j_2)\ldots,(i_{m'}<j_{m'})\big)\in\mathrm{Dgm}(n).$$

Summing (8.27)–(8.29), (8.29') over $\overrightarrow{(i,j)},\overrightarrow{(i',j')}\in\mathrm{Dgm}(n)$ verifies the desired semigroup property:

$$\mathcal{P}_s\mathcal{P}_{t-s}+\big(\mathcal{P}_s\mathcal{D}^{\mathrm{Dgm}(n)}_{t-s}+\mathcal{D}^{\mathrm{Dgm}(n)}_s\mathcal{P}_{t-s}+\mathcal{D}^{\mathrm{Dgm}(n)}_s\mathcal{D}^{\mathrm{Dgm}(n)}_{t-s}\big)=\mathcal{P}_t+\mathcal{D}^{\mathrm{Dgm}(n)}_{t-s}.$$

We now turn to norm continuity. Given the semigroup property, it suffices to show continuity at $t = 0$. The heat semigroup $\mathcal{P}_t$ is indeed continuous at $t = 0$. As for $\mathcal{D}_t^{\mathrm{Dgm}(n)}$, we have $\mathcal{D}_0^{\mathrm{Dgm}(n)} := 0$, and from (8.25) $\lim_{t\to 0} \|\mathcal{D}_t^{\mathrm{Dgm}(n)}\|_{\mathrm{op}} = 0$.

Proof of Theorem 1.1(b). Given (7.6), proving Part (b) amounts to showing $\mathcal{P}_t + \mathcal{D}_t^{\mathrm{Dgm}(n)} = e^{-t\mathcal{H}}$ or, what is the same $\langle u', (\mathcal{P}_t + \mathcal{D}_t^{\mathrm{Dgm}(n)})u\rangle = \langle u', e^{-t\mathcal{H}}u\rangle$ for $u, u' \in \mathscr{L}^2(\mathbb{R}^{2n})$, $t \geqslant 0$. Both functions of t are continuous since $\mathcal{P}_t + \mathcal{D}_t^{\mathrm{Dgm}(n)}$ and $e^{-t\mathcal{H}}$ are norm-continuous and have exponential growth since from (8.26) and from $\sigma(\mathcal{H}) \subset [-C(n\beta_\star), \infty)$, we have $\|\mathcal{P}_t + \mathcal{D}_t^{\mathrm{Dgm}(n)}\|_{\mathrm{op}} + \|e^{-t\mathcal{H}}\|_{\mathrm{op}} \leqslant C(n, \beta_\star)e^{C(n,\beta_\star)t}$, so it suffices to match their Laplace transforms for sufficiently large values $\lambda > C(\beta_\star, n)$ of the Laplace variable, i.e., to show that the Laplace transform of $\langle u', (\mathcal{P}_t + \mathcal{D}_t^{\mathrm{Dgm}(n)})u\rangle$ is the resolvent $\big\langle u', \mathcal{R}_{-\lambda}u\big\rangle$.

To evaluate the Laplace transform of $\langle u', (\mathcal{P}_t + \mathcal{D}_t^{\mathrm{Dgm}(n)})u\rangle$, assume for a moment $u(x), u'(x) \geqslant 0$, we integrate (1.17') (viewed as in integral operator) against $e^{-\lambda t}\overline{u'}(x)u(x')$ over $t \in \mathbb{R}_+$ and $x, x' \in \mathbb{R}^{2n}$, and sum the result over all $\overrightarrow{(i,j)} \in \mathrm{Dgm}(n)$. This gives

$$\int_0^\infty e^{-\lambda t}\langle u', \mathcal{D}_t^{\mathrm{Dgm}(n)}u\rangle\,\mathrm{d}t = \sum_{\overrightarrow{(i,j)}\in\mathrm{Dgm}(n)} \Big\langle u', \Big(\prod_{a\in A}\int_0^\infty e^{-\lambda t}\mathcal{Q}_t^{(a)}\,\mathrm{d}t\Big)u\Big\rangle, \tag{8.30}$$

where, the operator $\mathcal{Q}_t^{(a)}$ are indexed as described in the preceding. In deriving (8.30), we have exchanged sums and integrals, which is justified because each $\mathcal{Q}_t^{(a)}$ has a positive kernel, and $u(x'), u'(x) \geqslant 0$ under the current assumption. On the r.h.s. of (8.30), the Laplace transforms $\int_0^\infty e^{-t\lambda}\mathcal{Q}_t^{(a)}\,\mathrm{d}t$ are evaluated as in Lemmas 8.1(b), 8.2(b), and 8.5(b). Putting together the expressions from these lemmas, and comparing the result to (3.12), we now have

$$\int_0^\infty e^{-\lambda t}\langle u', (\mathcal{P}_t + \mathcal{D}_t^{\mathrm{Dgm}(n)})u\rangle\,\mathrm{d}t = \big\langle u', \big(\text{r.h.s. of (1.23)}\big|_{z=-\lambda}\big)u\big\rangle = \big\langle u', \mathcal{R}_{-\lambda}u\big\rangle.$$

For general $u, u' \in \mathscr{L}^2(\mathbb{R}^{2n})$, the preceding calculation done for $(u(x), u'(x')) \mapsto (|u(x)|, |u'(x')|)$ guarantees the relevant integrability, and justifies the exchange of sums and integrals.

References

[AGHKH88] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, and H. Holden. *Solvable models in quantum mechanics.* Springer-Verlag, 1988.

[AS65] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions: with formulas, graphs, and mathematical tables*, volume 55. Courier Corporation, 1965.

[BC98] L. Bertini and N. Cancrini. The two-dimensional stochastic heat equation: renormalizing a multiplicative noise. *J Phys A: Math Gen*, 31(2):615, 1998.

[CCM18] F. Comets, C. Cosco, and C. Mukherjee. Fluctuation and rate of convergence for the stochastic heat equation in weak disorder. *arXiv:1807.03902*, 2018.

[CCM19] F. Comets, C. Cosco, and C. Mukherjee. Renormalizing the Kardar–Parisi–Zhang equation in $d \geqslant 3$ in weak disorder. *arXiv:1902.04104*, 2019.

[CD18] S. Chatterjee and A. Dunlap. Constructing a solution of the (2+1)-dimensional KPZ equation. *arXiv:1809.00803*, 2018.

[CL17] F. Comets and Q. Liu. Rate of convergence for polymers in a weak disorder. *J Math Anal Appl*, 455(1):312–335, 2017.

[CN19] C. Cosco and S. Nakajima. Gaussian fluctuations for the directed polymer partition function for $d \geqslant 3$ and in the whole L^2-region. *arXiv:1903.00997*, 2019.

[Com17] F. Comets. Directed polymers in random environments. In *Lecture Notes in Mathematics, vol. 2175, Springer, Cham, 2017, Lecture notes from the 46th Probability Summer School held in Saint-Flour, 2016*, 2017.

[Cor12] I. Corwin. The Kardar–Parisi–Zhang equation and universality class. *Random matrices: Theory and applications*, 1(01):1130001, 2012.

[CSZ17] F. Caravenna, R. Sun, and N. Zygouras. Universality in marginally relevant disordered systems. *Ann Appl Probab*, 27(5):3050–3112, 2017.

[CSZ18a] F. Caravenna, R. Sun, and N. Zygouras. The Dickman subordinator, renewal theorems, and disordered systems. *arXiv:1805.01465*, 2018.

[CSZ18b] F. Caravenna, R. Sun, and N. Zygouras. On the moments of the (2+1)-dimensional directed polymer and stochastic heat equation in the critical window. *arXiv:1808.03586*, 2018.

[CSZ18c] F. Caravenna, R. Sun, and N. Zygouras. The two-dimensional KPZ equation in the entire subcritical regime. *arXiv:1812.03911*, 2018.

[DFT94] G.-F. Dell'Antonio, R. Figari, and A. Teta. Hamiltonians for systems of N particles interacting through point interactions. In *Annales de l'IHP Physique théorique*, volume 60, pages 253–290, 1994.

[DGRZ18] A. Dunlap, Y. Gu, L. Ryzhik, and O. Zeitouni. Fluctuations of the solutions to the KPZ equation in dimensions three and higher. *arXiv:1812.05768*, 2018.

[DR04] J. Dimock and S. Rajeev. Multi-particle Schrödinger operators with point interactions in the plane. *J Phys A: Math Gen*, 37(39):9157, 2004.

[Fen15] Z. S. Feng. *Rescaled directed random polymer in random environment in dimension 1+ 2*. PhD thesis, University of Toronto (Canada), 2016. 2, 2015.

[GIP15] M. Gubinelli, P. Imkeller, and N. Perkowski. Paracontrolled distributions and singular PDEs. In *Forum of Mathematics, Pi*, volume 3. Cambridge University Press, 2015.

[GRZ18] Y. Gu, L. Ryzhik, and O. Zeitouni. The Edwards–Wilkinson limit of the random heat equation in dimensions three and higher. *Commun Math Phys*, 363(2):351–388, 2018.

[Gu18] Y. Gu. Gaussian fluctuations of the 2D KPZ equation. *arXiv:1812.07467*, 2018.

[Hai14] M. Hairer. A theory of regularity structures. *Invent Math*, 198(2):269–504, 2014.

[Kal97] O. Kallenberg. *Foundations of modern probability*. Springer, 1997.

[Kho14] D. Khoshnevisan. *Analysis of stochastic partial differential equations*, volume 119. American Mathematical Soc., 2014.

[MSZ16] C. Mukherjee, A. Shamov, and O. Zeitouni. Weak and strong disorder for the stochastic heat equation and continuous directed polymers in $d \geqslant 3$. *Electron Commun Probab*, 21, 2016.

[MU18] J. Magnen and J. Unterberger. The scaling limit of the KPZ equation in space dimension 3 and higher. *J Stat Phys*, pages 1–56, 2018.

[Raj99] S. Rajeev. A condensation of interacting Bosons in two dimensional space. *arXiv: hep-th/9905120*, 1999.

[RS72] M. Reed and B. Simon. *Methods of modern mathematical physics*, volume 1. Elsevier, 1972.

[Wal86] J. B. Walsh. An introduction to stochastic partial differential equations. In *École d'Été de Probabilités de Saint Flour XIV-1984*, pages 265–439. Springer, 1986.

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