KPZ ON TORUS: GAUSSIAN FLUCTUATIONS

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ABSTRACT. In this paper, we study the KPZ equation on the torus and derive Gaussian fluctuations in large time.

KEYWORDS: KPZ equation, directed polymer, invariant measure.

1. INTRODUCTION

1.1. **Main result.** We consider the stochastic heat equation (SHE) on the torus $\mathbb{T}^d$:

$$\partial_t \mathcal{U} = \tfrac{1}{2}\Delta\mathcal{U} + \mathcal{U}\xi, \tag{1.1}$$

where ξ is a generalized Gaussian random field over $\mathbb{R} \times \mathbb{T}^d$ with the covariance function

$$\mathbf{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)R(x-y).$$

Here $\mathbb{T}^d$ is the unit torus. We assume ξ is built on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and consider two cases in the paper: (i) $R : \mathbb{T}^d \to \mathbb{R}_{\geq 0}$ is a smooth bounded function and $d \geq 1$; (ii) $R(\cdot) = \delta(\cdot)$ is the Dirac function and $d = 1$, i.e., ξ is a $1+1$ spacetime white noise. The product between $\mathcal{U}$ and ξ is interpreted in the Itô's sense. Without loss of generality, we assume

$$\int_{\mathbb{T}^d} R(x)dx = 1 \tag{1.2}$$

in case (i). We assume that the initial condition is an arbitrary probability measure ν on $\mathbb{T}^d$.

Here is the main result of the paper:

Theorem 1.1. *There exists $\gamma, \sigma > 0$, which only depends on R, d and is defined in (2.22) and (5.58) below, such that for any $x \in \mathbb{T}^d$,*

$$\frac{\log \mathcal{U}(t,x) + \gamma t}{\sqrt{t}} \Rightarrow N(0, \sigma^2), \qquad \text{as } t \to \infty.$$

1.2. **Context.** Define $h(t,x) = \log \mathcal{U}(t,x)$, then h formally satisfies the nonlinear SPDE

$$\partial_t h = \tfrac{1}{2}\Delta h + \tfrac{1}{2}|\nabla h|^2 + \xi, \tag{1.3}$$

which is the KPZ equation and a default model for random surface growth [38]. The study of the equation in $d = 1$ with a spacetime white noise has witnessed important progress in recent years, including making sense of the equation [32, 33, 35, 36, 41], studying universal behaviors [1, 6, 11, 12, 28, 44, 47, 50, 51, 53], etc. For a detailed review of singular SPDEs and the KPZ universality class, we refer to [23, 24, 46, 48] and the references therein. For the equation (1.3) and other relevant models in statistical physics including directed polymers and exclusion processes, the study of the large scale behaviors is mostly focused on an infinite line and the limiting object is the so-called KPZ fixed point. It is expected that the result on the torus is different and Gaussian fluctuations prevail in large time. Indeed, Theorem 1.1 shows that as $t \to \infty$,

$$\frac{h(t,x)+\gamma t}{\sqrt{t}} \Rightarrow N(0,\sigma^2). \tag{1.4}$$

The result also holds in $d \geq 2$ with a smooth noise.

The recent work on periodic totally asymmetric simple exclusion process [2, 3, 4] considered the case where the size of the torus is scaled with time, and studied the fluctuations of the integrated particle current for several special initial data. A transition from the Tracy-Widom distribution in the small time limit to the Gaussian distribution in the large time limit was derived in [4, Theorem 1.5]. In our problem, the size of the torus is fixed, so it corresponds to their super-relaxation time scale.

There are recent results deriving the Edwards-Wilkinson limit of the KPZ equation in $d \geq 2$ in a weak disorder regime, with the equation posed on the whole space and driven by a smooth noise, see [17, 18, 21, 25, 29, 31, 42, 43]. Although the limit is also of Gaussian distribution, the mixing mechanism is rather different from our setting, and it is ultimately the result of the weak disorder and is related to the diffusive behaviors of directed polymers in high dimensions with high temperatures.

Our proof of (1.4) goes through the Hopf-Cole transformation so that we directly study $\mathcal{U} = \exp(h)$ and avoid dealing with the small scale singularity appearing in (1.3). The recent work [34] studied the generators of the stochastic Burgers equation, the singular SPDE satisfied by $\nabla h(t,x) = \frac{\nabla \mathcal{U}(t,x)}{\mathcal{U}(t,x)}$, and their result implies the geometric ergodicity of ∇h. Here we choose to work on a different quantity, the endpoint distribution of the directed polymer in random environment:

$$u(t,x) = \frac{\mathcal{U}(t,x)}{\int_{\mathbb{T}^d} \mathcal{U}(t,x')dx'}.$$

One of the main ingredients in our proof is the geometric ergodicity of the Markov process $\{u(t,\cdot)\}_{t\geq 0}$, see Theorem 2.3. The proof is inspired by the classical work of Sinai [52], which is actually on the Burgers equation. Another key quantity is the intersection time of two independent polymer paths, similar to the so-called *replica overlap*, which is used to distinguished different temperature regimes and plays a crucial role in the study of thermodynamic limits of directed polymers, and in our case corresponds to

$$\int_0^t \int_{\mathbb{T}^{2d}} R(x-y)u(s,x)u(s,y)dxdyds.$$

It is an additive functional of $\{u(t,\cdot)\}_{t\geq 0}$, and the idea is to utilize the aforementioned geometric ergodicity to construct a corrector and extract a martingale from the above term. The geometric ergodicity is then used to drive a martingale central limit theorem which eventually leads to (1.4). Another closely related work is [49], where a one force - one solution principle (modulo constants) for the KPZ equation was established.

For a more detailed discussion on directed polymers in random environment, we refer to the book [20] and the references therein. For the endpoint distribution we are interested in, our approach is more related to that of [5, 7, 13]. For the polymer lying on the torus, we refer to the physics literature [14, 15, 16] for some relevant discussions.

1.3. **Organization of the paper.** The paper is organized as follows. In Section 2, we sketch the main ingredients used in the proof, including the geometric ergodicity of $\{u(t,\cdot)\}_{t\geq 0}$ and a martingale decomposition for the free energy of the directed polymer. In Section 3, we derive a nonlocal SPDE satisfied by u. Section 4 is devoted to proving the geometric ergodicity of $\{u(t,\cdot)\}_{t\geq 0}$ by following the approach of [52]. In Section 5, we prove the main theorem through constructing the corrector and applying the martingale central limit theorem. In the appendix we recall some facts concerning the Fortet-Mourier metric on the space of probability measures on polish metric spaces, see Section A and recall some fairly standard facts concerning the stochastic heat equation, see Section B.

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2. Ideas of the proof

2.1. **Endpoint distribution of directed polymer.** To study the SHE (1.1), a closely related object is the directed polymer in random environment. We introduce some notations. For any $y\in\mathbb{T}^d$, Let $\mathcal{U}(t,x;y)$ be the solution to

$$\begin{aligned}&\partial_t\mathcal{U}(t,x;y)=\tfrac12\Delta_x\mathcal{U}(t,x;y)+\mathcal{U}(t,x;y)\xi(t,x), \qquad t>0,\\ &\mathcal{U}(0,x;y)=\delta(x-y).\end{aligned}\tag{2.1}$$

In other words, $\mathcal{U}$ is the Green's function of the SHE. For any probability measure ν, denote the solution to (1.1) with the initial data $\nu(dx)$ by $\mathcal{U}(t,x;\nu)$, then with the Green's function, we can write

$$\mathcal{U}(t,x;\nu)=\int_{\mathbb{T}^d}\mathcal{U}(t,x;y)\nu(dy).\tag{2.2}$$

If $\nu(dx)=f(x)dx$ for some $0\le f\in L^1(\mathbb{T}^d)$, we write instead $\mathcal{U}(t,x;f)=\mathcal{U}(t,x;\nu)$. For a directed polymer of length t, with the starting point distributed according to ν, its endpoint distribution is given by

$$u(t,x;\nu)=\frac{\mathcal{U}(t,x;\nu)}{\int_{\mathbb{T}^d}\mathcal{U}(t,x';\nu)dx'}.\tag{2.3}$$

Throughout the paper, we will omit the dependence of either $\mathcal{U}$, or u on the initial data ν when there is no danger of confusion. Let $\mathcal{M}_1(\mathbb{T}^d)$ be the space of probability measures on $\mathbb{T}^d$, then for $t\ge 0$ and each realization of the noise ξ, we can view $u(t)=u(t,\cdot)$ as the density of some probability measure on $\mathbb{T}^d$ and sometimes, with some abuse of notations, we write $u(t)\in\mathcal{M}_1(\mathbb{T}^d)$.

Remark 2.1. In the case when R is smooth, by the Feynman-Kac formula we have

$$\mathcal{U}(t,x;y)=\mathbb{E}_x\big[e^{\int_0^t\xi(t-s,B_s)ds-\frac12R(0)t}\delta(B_t-y)\big],$$

where $\mathbb{E}_x$ is the expectation with respect to the Wiener measure $\mathbb{P}_x$ on $C([0,+\infty);\mathbb{R}^d)$, with the paths starting at x, i.e. $\mathbb{P}_x[B_0=x]=1$. Through a change of variable and the time reversal property of a Brownian bridge, we have

$$\begin{aligned}\mathcal{U}(t,x;y)=&\mathbb{E}_x\big[e^{\int_0^t\xi(s,B_{t-s})ds-\frac12R(0)t}\delta(B_t-y)\big]\\ =&\mathbb{E}_y\big[e^{\int_0^t\xi(s,B_s)ds-\frac12R(0)t}\delta(B_t-x)\big],\end{aligned}$$

which implies

$$\mathcal{U}(t,x;\nu)=\int_{\mathbb{T}^d}\nu(dy)\mathbb{E}_y\big[e^{\int_0^t\xi(s,B_s)ds-\frac12R(0)t}\delta(B_t-x)\big].$$

Therefore, one can write

$$u(t,x;\nu) = \frac{\int_{\mathbb{T}^d} \nu(dy)\mathbb{E}_y\big[e^{\int_0^t \xi(s,B_s)ds}\delta(B_t - x)\big]}{\int_{\mathbb{T}^d} \nu(dy)\mathbb{E}_y\big[e^{\int_0^t \xi(s,B_s)ds}\big]},$$

which is usually how the endpoint distribution of the directed polymer is defined.

Sometimes to indicate the dependence of ξ on the random event we shall write

$$\xi = \big\{\xi(t,x;\omega)\big\}_{(t,x;\omega)\in\mathbb{R}\times\mathbb{T}^d\times\Omega}.$$

In addition, since the noise is space-time homogeneous, we may and shall assume that there is an additive group $\{\theta_{t,x}\}_{(t,x)\in\mathbb{R}\times\mathbb{T}^d}$ of $\mathcal{F}/\mathcal{F}$ measurable maps $\theta_{t,x}:\Omega\to\Omega$ such that $\mathbf{P}\circ\theta_{t,x}^{-1} = \mathbf{P}$ for all $(t,x)\in\mathbb{R}\times\mathbb{T}^d$ and

$$\xi(t,x;\theta_{s,y}(\omega)) = \xi(t+s,x+y;\omega), \quad (t,x),(s,y)\in\mathbb{R}\times\mathbb{T}^d,\ \omega\in\Omega. \tag{2.4}$$

As a consequence of the Markov property of $\{\mathcal{U}(t)\}_{t\ge0}$ and the white-in-time nature of the noise, we conclue the following.

Lemma 2.2. *$\{u(t)\}_{t\ge0}$ is a Markov process taking values in $\mathcal{M}_1(\mathbb{T}^d)$.*

The proof of the lemma can be found in Section B.1 of the Appendix.

One of the main technical parts of the paper is to prove the geometric ergodicity of the Markov process $\{u(t;\nu)\}_{t\ge0}$. To state the result, we introduce some notations.

For any $p\ge1$ we let

$$\mathcal{M}_1^{(p)}(\mathbb{T}^d) := \Big\{\nu\in\mathcal{M}_1(\mathbb{T}^d) : \nu(dx) = f(x)dx,\ f\in L^p(\mathbb{T}^d)\Big\},$$

equipped with the relative topology from $L^p(\mathbb{T}^d)$. We shall identify $\mathcal{M}_1^{(p)}(\mathbb{T}^d)$ with the appropriate subset of densities, i.e.

$$D^p(\mathbb{T}^d) = D(\mathbb{T}^d)\cap L^p(\mathbb{T}^d) \quad \text{and } D^\infty(\mathbb{T}^d) = D(\mathbb{T}^d)\cap C(\mathbb{T}^d). \tag{2.5}$$

Here $D(\mathbb{T}^d)$ is the set of densities w.r.t. the Lebesgue measure on $\mathbb{T}^d$. We will consider the Fortet-Mourier metric on $\mathcal{M}_1(\mathbb{T}^d)$, see Section A for a brief recollection of the facts concerning this metric. For any $F:\mathcal{M}_1(\mathbb{T}^d)\to\mathbb{R}$, we define

$$\|F\|_{\mathrm{Lip}} := \|F\|_\infty + \sup_{\mu\ne\nu,\mu,\nu\in\mathcal{M}_1(\mathbb{T}^d)} \frac{|F(\mu)-F(\nu)|}{\mathrm{d_{TV}}(\mu,\nu)}. \tag{2.6}$$

Denote by $\mathrm{Lip}(\mathcal{M}_1(\mathbb{T}^d))$ the space of all functions F for which $\|F\|_{\mathrm{Lip}} < +\infty$. Let $\mathcal{B}(\mathcal{M}_1(\mathbb{T}^d))$ and $\mathcal{B}(D^p(\mathbb{T}^d))$ be the Borel σ-algebra of $\mathcal{M}_1(\mathbb{T}^d)$ and $D^p(\mathbb{T}^d)$ respectively.

Define the transition probability densities

$$\mathcal{P}_t(\nu,A) := \mathbf{E}1_A(u(t;\nu)), \quad A\in\mathcal{B}(\mathcal{M}_1(\mathbb{T}^d)),\ \nu\in\mathcal{M}_1(\mathbb{T}^d). \tag{2.7}$$

By the property of SHE, for any $t>0$, $u(t,\cdot;\nu)$ is a continuous function almost surely, thus, for any $t>0$, $\mathcal{P}_t(\nu,\cdot)$ is actually supported on $D^p(\mathbb{T}^d)$ for any $p\in[1,+\infty]$, and we can consider transition probabilities $\mathcal{P}_t(\nu,\cdot)$ on $\mathcal{B}(D^p(\mathbb{T}^d))$.

Define the transition probability operator by

$$\mathcal{P}_tF(\nu) := \int_{\mathcal{M}_1(\mathbb{T}^d)} F(v)\mathcal{P}_t(\nu,dv), \quad F\in B_b(\mathcal{M}_1(\mathbb{T}^d)),\ \nu\in\mathcal{M}_1(\mathbb{T}^d). \tag{2.8}$$

We shall also consider a semigroup $\{\mathcal{P}_t\}_{t\ge0}$ on $B_b(D_p(\mathbb{T}^d))$ for any $p\in[1,+\infty]$.

Theorem 2.3. *There exist a Borel probability measure π_∞ on $\mathcal{M}_1(\mathbb{T}^d)$ and constants $C,\lambda>0$, which only depend on the covariance function $R(\cdot)$ and dimension d such that*

(i)

$$\pi_\infty\big(D^p(\mathbb{T}^d)\big) = 1, \quad p \in [1,+\infty], \tag{2.9}$$

(ii) *for any $F \in \mathrm{Lip}(\mathcal{M}_1(\mathbb{T}^d))$ we have*

$$\left\| \mathcal{P}_t F - \int_{\mathcal{M}_1(\mathbb{T}^d)} F(u)\pi_\infty(du) \right\|_\infty \le Ce^{-\lambda t}\|F\|_{\mathrm{Lip}}, \quad t \ge 0. \tag{2.10}$$

The above result also holds for $D^p(\mathbb{T}^d)$ in place of $\mathcal{M}_1(\mathbb{T}^d)$ for any $p \in [1,+\infty]$.

The proof of the theorem is given in Section 4.

2.2. **Martingale decomposition.** We will first prove Theorem 1.1 for the partition of the directed polymer, defined as

$$Z_t = \int_{\mathbb{T}^d} \mathcal{U}(t,x)dx. \tag{2.11}$$

By the mild formulation of SHE, one can write

$$\mathcal{U}(t,x) = \int_{\mathbb{T}^d} p_t(x-y)\nu(dy) + \int_0^t \int_{\mathbb{T}^d} p_{t-s}(x-y)\mathcal{U}(s,y)\xi(s,y)dyds, \tag{2.12}$$

where

$$p_t(x) = \frac{1}{(2\pi t)^{d/2}} \sum_{n\in\mathbb{Z}^d} \exp\left\{-\frac{|x+n|^2}{2t}\right\} \tag{2.13}$$

is the heat kernel on $\mathbb{T}^d$. Thus, recalling that ν is a probability measure, we have

$$Z_t = 1 + \int_0^t \int_{\mathbb{T}^d} \mathcal{U}(s,y)\xi(s,y)dyds, \tag{2.14}$$

which is a positive local martingale, whose quadratic variation equals

$$\langle Z\rangle_t = \int_0^t \mathcal{R}\big(\mathcal{U}(s)\big)ds. \tag{2.15}$$

Here for $R \in L^\infty(\mathbb{T}^d)$ we let

$$\begin{aligned} &\mathcal{R}(u,v) := \int_{\mathbb{T}^{2d}} R(x-y)u(x)v(y)dxdy, \\ &\mathcal{R}(u) := \mathcal{R}(u,u) = \int_{\mathbb{T}^{2d}} R(x-y)u(x)u(y)dxdy, \quad u,v \in L^1(\mathbb{T}^d). \end{aligned} \tag{2.16}$$

If $R(\cdot) = \delta(\cdot)$, the above definition holds for $u,v \in L^2(\mathbb{T}^d)$.

By Lemma B.4, we know that $\mathbf{E}\langle Z\rangle_t^p \le C(t,p)$, for some constant $C(t,p) < +\infty$ depending only on t and $p \in [1,+\infty)$. Hence $\big(Z_t\big)_{t\ge0}$ is a martingale. Define a local martingale $(M_t)_{t\ge0}$ through the SDE

$$dM_t = Z_t^{-1}dZ_t, \qquad M_0 = 0. \tag{2.17}$$

By the Itô formula

$$Z_t = \exp\left\{M_t - \tfrac{1}{2}\langle M\rangle_t\right\}. \tag{2.18}$$

We can write M_t and its quadratic variation in terms of the endpoint distribution of the directed polymer:

$$M_t = \int_0^t \int_{\mathbb{T}^d} Z_s^{-1}\mathcal{U}(s,y)\xi(s,y)dyds = \int_0^t \int_{\mathbb{T}^d} u(s,y)\xi(s,y)dyds, \tag{2.19}$$

and

$$\langle M\rangle_t = \int_0^t \mathcal{R}(u(s))ds. \tag{2.20}$$

The following lemma shows that M is a square-integrable martingale.

Lemma 2.4. *For any $t > 0$ and $p \in [1, +\infty)$ we have* $\mathbf{E}[\langle M\rangle_t^p] \le C(t, p, d, R)$.

Proof. For the case of R being bounded, we have $\langle M\rangle_t \le t\|R\|_{L^\infty(\mathbb{T}^d)}$. For the case of $R(\cdot) = \delta(\cdot)$, we have $\langle M\rangle_t = \int_0^t \|u(s)\|^2_{L^2(\mathbb{T}^d)} ds$. Applying the triangle inequality we further derive

$$\mathbf{E}[\langle M\rangle_t^p]^{1/p} \le C \int_0^t \mathbf{E}[\|u(s)\|^{2p}_{L^2(\mathbb{T}^d)}]^{1/p} ds,$$

then the lemma is a direct consequence of Lemma B.3. □

Then we can write

$$\begin{aligned} \log Z_t =& M_t - \tfrac{1}{2}\langle M\rangle_t \\ =& \int_0^t \int_{\mathbb{T}^d} u(s,y)\xi(s,y)dyds - \tfrac{1}{2}\int_0^t \mathcal{R}(u(s))ds. \end{aligned} \tag{2.21}$$

The term $\int_0^t \mathcal{R}(u(s))ds$ is an additive functional of the Markov process $(u(s))_{s\ge 0}$, which will be the key object to study in the paper. Define

$$\gamma := \tfrac{1}{2}\int_{\mathcal{M}_1(\mathbb{T}^d)} \mathcal{R}(u)\pi_\infty(du), \tag{2.22}$$

where π_∞ is the unique invariant measure for the process (Theorem 2.3), and

$$\tilde{\mathcal{R}} = \mathcal{R} - 2\gamma. \tag{2.23}$$

It will become clear later (see Lemma 5.30 below) that $\gamma \in (0, +\infty)$.

Now we write

$$\log Z_t + \gamma t = \int_0^t \int_{\mathbb{T}^d} u(s,y)\xi(s,y)dyds - \tfrac{1}{2}\int_0^t \tilde{\mathcal{R}}(u(s))ds. \tag{2.24}$$

The idea is to utilize Theorem 2.3 to solve the Poisson equation associated with $\tilde{\mathcal{R}}$ and extract the martingale part from $\int_0^t \tilde{\mathcal{R}}(u(s))ds$, which combines with the martingale M_t to converge to the limiting Gaussian random variable, after a rescaling:

Theorem 2.5. *As $t \to \infty$, we have*

$$\frac{\log Z_t + \gamma t}{\sqrt{t}} \Rightarrow N(0, \sigma^2).$$

In addition, we have

$$\sigma^2 \ge \int_{\mathbb{T}^d} R(x)dx. \tag{2.25}$$

Remark 2.6 (Process-level convergence). Combining Theorems 2.3 and 2.5, we actually have the following picture: as $t \to \infty$,

$$\log u(t,x) = \log \tfrac{\mathcal{U}(t,x)}{Z_t} = h(t,x) - \log Z_t \Rightarrow \log \rho(x) \tag{2.26}$$

in distribution in $C(\mathbb{T}^d)$, where the limit ρ is sampled from the invariant measure π_∞. Therefore, in large time, the random height function h is approximately $\log\rho$ shifted by the random constant $\log Z_t$, and this random constant is roughly Gaussian distributed after a centering and rescaling. We also have

$$h(t,x) - h(t,0) \Rightarrow \log\rho(x) - \log\rho(0)$$

in distribution in $C(\mathbb{T}^d)$.

According to the results of [30], see Theorems 1.1 and 2.1, in the case of $1+1$ spacetime white noise, for each $t \ge 0$ and in statinarity the law of $(h(t,x) - h(t,0))_{x\in\mathbb{T}}$

over $C(\mathbb{T})$ coincides with that of $(B(x))_{x\in\mathbb{T}}$ - a Brownian bridge satisfying $B(0) = B(1) = 0$. The invariant measure π_∞ in this case is given by the law of

$$\rho(x) = \frac{e^{B(x)}}{\int_{\mathbb{T}} e^{B(x')}dx'}, \quad x \in \mathbb{T}$$

over $D^\infty(\mathbb{T})$, see also [9, 33].

3. SPDE for endpoint distribution of directed polymer

Define the Fourier coefficients of the spatial covariance of the noise

$$\hat{r}_k = \int_{\mathbb{T}^d} R(x)e^{-i2\pi k\cdot x}dx, \quad k \in \mathbb{Z}^d. \tag{3.1}$$

They are non-negative.

We can write formally that

$$\xi(t,x) = \sum_{k\in\mathbb{Z}^d} e_k(x)\dot{w}_k(t), \qquad e_k(x) = \sqrt{\hat{r}_k}e^{i2\pi k\cdot x}, \tag{3.2}$$

where $(w_k)_{k\in\mathbb{Z}^d}$ are one dimensional complex-valued zero mean Wiener processes satisfying

$$w_k^\star(t) = w_{-k}(t), \quad \mathbf{E}[w_k(t)w_\ell^\star(s)] = \delta_{k,\ell}\min(t,s), \quad k,\ell \in \mathbb{Z}^d,\, t,s \ge 0. \tag{3.3}$$

Here $\delta_{k,\ell}$ is the Kronecker symbol. We also define

$$W(t,x) = \sum_{k\in\mathbb{Z}^d} e_k(x)w_k(t) \tag{3.4}$$

and write

$$W(t) = W(t,\cdot), \qquad dW(t,x) = \xi(t,x)dt.$$

We formulate the following assumption:

Assumption 3.1. *There exists $N > 0$ such that $\hat{r}_k = 0$ for $|k| > N$.*

Under the above assumption, the summations in (3.2) and (3.4) are finite.

The main result of this section is to derive an SPDE satisfied by the endpoint distribution of the directed polymer. Recall that $u(t,x;\nu)$ has been defined in (2.3) for any $\nu \in \mathcal{M}_1(\mathbb{T}^d)$. In this section the initial data is assumed to be of the form $\nu(dx) = v(x)dx$, where $v \in D(\mathbb{T}^d)\cap C^\infty(\mathbb{T}^d)$. To simplify the notation, we omit the initial data and write $u(t) = u(t,\cdot)$ and $\mathcal{U}(t) = \mathcal{U}(t,\cdot)$.

Proposition 3.2. *Suppose that Assumption 3.1 is in force. Furthermore, assume that the initial data $v \in D(\mathbb{T}^d)\cap C^\infty(\mathbb{T}^d)$ and $u \in C([0,\infty), H^k(\mathbb{T}^d))$, for any $k \ge 1$, is a strong solution to*

$$\begin{aligned} du(t) =& \Big\{\tfrac{1}{2}\Delta u(t) + u(t)\int_{\mathbb{T}^d} u(t,x)R\star u(t,x)dx - u(t)R\star u(t)\Big\}dt \\ &+ u(t)dW(t) - u(t)\int_{\mathbb{T}^d} u(t,x)dW(t,x), \quad u(0) = v. \end{aligned} \tag{3.5}$$

Proof. Under Assumption 3.1 and by the fact that $v \in C^\infty(\mathbb{T}^d)$, we have

$$\mathcal{U}(t) = v + \int_0^t \tfrac{1}{2}\Delta\mathcal{U}(s)ds + \int_0^t \mathcal{U}(s)dW(s). \tag{3.6}$$

By (2.14), we have (which can also be obtained by integrating the above equation in x)

$$Z_t = \int_{\mathbb{T}^d}\mathcal{U}(t,x)dx = 1 + \int_0^t\int_{\mathbb{T}^d}\mathcal{U}(s,x)dW(s,x). \tag{3.7}$$

Using the above two equations, we have

$$\begin{aligned} \langle Z\rangle_t &= \int_0^t \mathcal{R}\big(\mathcal{U}(s)\big)ds, \\ \langle \mathcal{U}, Z\rangle_t &= \int_0^t \mathcal{U}(s)\mathcal{R}\star\mathcal{U}(s)ds. \end{aligned} \tag{3.8}$$

By (B.5), we know that $(Z_t)_{t\geq 0}$ is continuous and positive almost surely. Then by Itô's formula, we have

$$\begin{aligned} du(t) =& d[\mathcal{U}(t)Z_t^{-1}] \\ =& Z_t^{-1}d\mathcal{U}(t) - \mathcal{U}(t)Z_t^{-2}dZ_t + \mathcal{U}(t)Z_t^{-3}d\langle Z\rangle_t - Z_t^{-2}d\langle \mathcal{U}, Z\rangle_t. \end{aligned}$$

Invoking (3.6) - (3.8), the proof is complete. □

We introduce the following operators:

$$\begin{aligned} \mathscr{A}(f) &:= \tfrac{1}{2}\Delta f + f\langle f, R\star f\rangle_{L^2(\mathbb{T}^d)} - fR\star f, \qquad f\in H^2(\mathbb{T}^d), \\ \mathscr{B}_k(f) &:= fe_k - f\langle f, e_{-k}\rangle_{L^2(\mathbb{T}^d)}, \qquad f\in L^2(\mathbb{T}^d). \end{aligned} \tag{3.9}$$

Note that $e_k^* = e_{-k}$, so $\langle f, e_{-k}\rangle_{L^2(\mathbb{T}^d)} = \int_{\mathbb{T}^d} f(x)e_k(x)dx$, and for real-valued f, we have

$$\mathscr{B}_k(f)^* = \mathscr{B}_{-k}(f).$$

With the above notations, the equation (3.5) can be written in a more compact form:

$$du(t) = \mathscr{A}(u(t))dt + \sum_{k\in\mathbb{Z}^d}\mathscr{B}_k(u(t))dw_k(t),$$

and we have the following lemma:

Lemma 3.3. *Under Assumption 3.1 and further assume $v\in D(\mathbb{T}^d)\cap C^\infty(\mathbb{T}^d)$, then for any $T>0$ and $p\in[1,+\infty)$, there exists $C = C(T,p,R,d)$ such that*

$$\begin{aligned} &\sup_{t\in[0,T]} \mathbf{E}\big[\|\mathscr{A}(u(t))\|_{L^2(\mathbb{T}^d)}^p\big] \leq C, \\ &\sum_{k\in\mathbb{Z}^d}\sup_{t\in[0,T]} \mathbf{E}\big[\|\mathscr{B}_k(u(t))\|_{L^2(\mathbb{T}^d)}^p\big] \leq C. \end{aligned}$$

Proof. Since $u(t) = \mathcal{U}(t)Z_t^{-1}$, the result is standard and follows from properties of SHE combined with (B.5). □

4. Geometric ergodicity of directed polymer on torus

In this section, we study the endpoint distribution of the directed polymer, defined in (2.3), and prove Theorem 2.3.

4.1. **Factorization.** Recall that we consider the solution to SHE with the initial data $\nu\in\mathcal{M}_1(\mathbb{T}^d)$:

$$\partial_t\mathcal{U} = \tfrac{1}{2}\Delta\mathcal{U} + \mathcal{U}\xi, \qquad \mathcal{U}(0,dx) = \nu(dx). \tag{4.1}$$

For any $t>s$ and $x,y\in\mathbb{T}^d$, define $\mathcal{Z}_{t,s}^\omega(x,y)$ as the solution to

$$\begin{aligned} &\partial_t\mathcal{Z}_{t,s}^\omega(x,y) = \tfrac{1}{2}\Delta_x\mathcal{Z}_{t,s}^\omega(x,y) + \mathcal{Z}_{t,s}^\omega(x,y)\xi(t,x), \qquad t>s, \\ &\mathcal{Z}_{s,s}^\omega(x,y) = \delta(x-y). \end{aligned} \tag{4.2}$$

In other words, $\mathcal{Z}_{t,s}^\omega(x,y)$ is the propagator of the SHE from (s,y) to (t,x). We will keep its dependence on ω throughout this section. In light of (2.4), we have

$$\mathcal{Z}_{t,s}^{\theta_{r,z}(\omega)}(x,y) = \mathcal{Z}_{t+r,s+r}^\omega(x+z,y+z). \tag{4.3}$$

For any set $I \subset \mathbb{R}$, denote the σ-algebra

$$\mathcal{F}_I = \sigma\{\xi(t,\cdot) : t \in I\}. \tag{4.4}$$

We shall also write

$$\mathcal{F}_t := \mathcal{F}_{[0,t]}, \quad t \ge 0. \tag{4.5}$$

Note that the random variable $\mathcal{Z}^\omega_{t,s}$ is $\mathcal{F}_{[s,t]}$-measurable.

Fix $t > 1$. We have

$$\mathcal{U}(t,x;\nu) = \int_{\mathbb{T}^d} \mathcal{Z}^\omega_{t,t-1}(x,y_1)\mathcal{U}(t-1,y_1;\nu)dy_1. \tag{4.6}$$

Let

$$N(t) := [t] + 1. \tag{4.7}$$

Sometimes, when it leads to no confusion, for abbreviation we shall write $N = N(t)$. By iterating (4.6), we further derive that

$$\begin{aligned}\mathcal{U}(t,x;\nu) = \int_{\mathbb{T}^{Nd}} \mathcal{Z}^\omega_{t,t-1}(x,y_1)\left(\prod_{j=1}^{N-2} \mathcal{Z}^\omega_{t-j,t-j-1}(y_j,y_{j+1})\right)& \\ \times \mathcal{Z}^\omega_{t-N+1,0}(y_{N-1},y_N)dy_1\dots dy_{N-1}\nu(dy_N).&\end{aligned} \tag{4.8}$$

With the convention of $y_0 = x$, we can write

$$\mathcal{U}(t,x;\nu) = \int_{\mathbb{T}^{Nd}} \prod_{j=0}^{N-1} \mathcal{Z}^\omega_{t-j,(t-j-1)_+}(y_j,y_{j+1})dy_1\dots dy_{N-1}\nu(dy_N). \tag{4.9}$$

Here $(x)_+ := \max\{x,0\}$.

4.2. **Construction of a Markov chain.** In this section, we construct a time reversed Markov chain $\{Y_n\}_{n=1}^N$, which depends on the random environment, and allows to express the endpoint density $u(t,x;\nu)$, see (2.3), as an average with respect to Y_1, see (4.17) below. It turns out that, with some positive probability, the random realizations of the transition probability density kernel of the chain are uniformly positive, see (4.24) and (4.25). This, in turn, allows us to use a coupling argument that leads to the proof of the geometric ergodicity of the density process $\big(u(t;\nu)\big)_{t\ge 0}$.

The random Markov chain takes values in $\mathbb{T}^d$ and is constructed as follows. For a fixed $\omega \in \Omega$, we run the chain backward in time. Let $\pi^\omega_N(y_N)$ be the (random) density of Y_N and $\pi^\omega_k(y_k\,|\,y_{k+1})$ be the (random) transition density kernels:

$$\begin{aligned}\pi^\omega_1(y_1\,|\,y_2) &= \frac{\mathcal{Z}^\omega_{t-1,t-2}(y_1,y_2)}{\int_{\mathbb{T}^d} \mathcal{Z}^\omega_{t-1,t-2}(z_1,y_2)dz_1}\\ \pi^\omega_k(y_k\,|\,y_{k+1}) &= \mathcal{Z}^\omega_{t-k,(t-k-1)_+}(y_k,y_{k+1})\\ &\times \frac{\int_{\mathbb{T}^{(k-1)d}} \mathcal{Z}^\omega_{t-1,t-2}(z_1,z_2)\dots \mathcal{Z}^\omega_{t-k+1,t-k}(z_{k-1},y_k)dz_{1,k-1}}{\int_{\mathbb{T}^{kd}} \mathcal{Z}^\omega_{t-1,t-2}(z_1,z_2)\dots \mathcal{Z}^\omega_{t-k,(t-k-1)_+}(z_k,y_{k+1})dz_{1,k}}\end{aligned} \tag{4.10}$$

for $k = 1,\dots,N-1$. Here $dz_{1,k} := dz_1\dots dz_k$. Finally we let

$$\begin{aligned}\pi^\omega_N(dy_N) := A^\omega_{N,\nu} \int_{\mathbb{T}^{(N-1)d}} \left(\prod_{j=1}^{N-2} \mathcal{Z}^\omega_{t-j,t-j-1}(z_j,z_{j+1})\right)&\\ \times \mathcal{Z}^\omega_{t-N+1,0}(z_{N-1},y_N)\nu(dy_N)dz_{1,N-1},&\end{aligned} \tag{4.11}$$

with the (random) constant $A^\omega_{N,\nu}$ given by

$$A^\omega_{N,\nu} := \left(\int_{\mathbb{T}^{Nd}} \prod_{j=1}^{N-1} \mathcal{Z}^\omega_{t-j,(t-j-1)_+}(z_j,z_{j+1})\nu(dz_N)dz_{1,N-1}\right)^{-1}.$$

Sometimes, we shall write $\pi_N^\omega(dy_N;\nu)$ when we wish to highlight the dependence of the distribution on the initial data ν. Also, when $\nu(dx)=f(x)dx$ for $f\in D(\mathbb{T}^d)$ we shall write $\pi_N^\omega(y_N;f)$ as the density of $\pi_N^\omega(dy_N;\nu)$ w.r.t. the Lebesgue measure. Note that $\mathcal{Z}_{t,s}^\omega(x,y)\in\mathcal{F}_{[s,t]}$ for all $x,y\in\mathbb{T}^d$, cf (4.4). Let $\mathbb{P}_{\pi_N}^\omega$ be the path measure on $\mathbb{T}^{Nd}$ constructed from the above transition probabilities. It is the law of time reversed Markov chain $\{Y_n\}_{n=1}^N$, where

$$Y_k(y_1,\ldots,y_N):=y_k,\qquad (y_1,\ldots,y_N)\in\mathbb{T}^{Nd},\ k\in\{1,\ldots,N\}.$$

Let $\mathbb{E}_{\pi_N}^\omega$ denote the expectation with respect to this measure. Suppose that $\nu(dx)=f(x)dx$. Then, the joint density of $(Y_1,\ldots,Y_N)$ under $\mathbb{P}_{\pi_N}^\omega$ equals

$$\begin{aligned}\varphi_{1,N}^\omega(y_1,\ldots,y_N)&=\pi_1^\omega(y_1\,|\,y_2)\ldots\pi_{N-1}^\omega(y_{N-1}\,|\,y_N)\pi_N^\omega(y_N;f)\\&=A_{N,\nu}^\omega\left(\prod_{j=1}^{N-1}\mathcal{Z}_{t-j,(t-j-1)_+}^\omega(y_j,y_{j+1})\right)f(y_N).\end{aligned}\tag{4.12}$$

Here $A_{N,\nu}^\omega$ is the (random) normalizing constant.

For a given $2\le k\le N-1$ denote by $\mathbb{P}_k^\omega$ the path measure on $\mathbb{T}^{Nd}$, corresponding to a time reversed Markov chain obtained from the transition probability densities

$$p_j^\omega(y_j\,|\,y_{j+1})=\begin{cases}\pi_j^\omega(y_j\,|\,y_{j+1}), & 1\le j\le k-1,\\[2ex] 1, & k\le j\le N-1.\end{cases}\tag{4.13}$$

and $p_N^\omega(y_N)\equiv 1$. The respective expectation shall be denoted by $\mathbb{E}_k^\omega$. The joint density of $(Y_1,\ldots,Y_k)$ under $\mathbb{P}_k^\omega$ equals

$$\begin{aligned}\varphi_{1,k}^\omega(y_1,\ldots,y_k)&=p_1^\omega(y_1\,|\,y_2)\ldots p_{k-1}^\omega(y_{k-1}\,|\,y_k)\\&=A_{k,\mathbb{1}}^\omega\left(\prod_{j=1}^{k-1}\mathcal{Z}_{t-j,t-j-1}^\omega(y_j,y_{j+1})\right),\end{aligned}\tag{4.14}$$

with the (random) constant

$$A_{k,\mathbb{1}}^\omega:=\left(\int_{\mathbb{T}^{kd}}\prod_{j=1}^{k-1}\mathcal{Z}_{t-j,t-j-1}^\omega(z_j,z_{j+1})dz_{1,k}\right)^{-1}.\tag{4.15}$$

In other words, comparing the Markov chains sampled from the path measures $\mathbb{P}_k^\omega$ and $\mathbb{P}_{\pi_N}^\omega$, the only difference lies in the distribution of $(Y_k,\ldots,Y_N)$. For $\mathbb{P}_k^\omega$, we sample $Y_k,\ldots,Y_N$ independently from the uniform distribution on $\mathbb{T}^d$, while we follow (4.10) and (4.11) for $\mathbb{P}_{\pi_N}^\omega$.

By (4.8) and (4.12), one can write

$$\begin{aligned}\mathcal{U}(t,x;\nu)=&\big(A_{N,\nu}^\omega\big)^{-1}\int_{\mathbb{T}^{Nd}}\mathcal{Z}_{t,t-1}^\omega(x,y_1)\pi_1^\omega(y_1\,|\,y_2)\ldots\pi_{N-1}^\omega(y_{N-1}\,|\,y_N)\pi_N^\omega(dy_N)dy_{1,N-1}\\=&\big(A_{N,\nu}^\omega\big)^{-1}\mathbb{E}_{\pi_N}^\omega[\mathcal{Z}_{t,t-1}^\omega(x,Y_1)].\end{aligned}\tag{4.16}$$

This, in turn implies

$$u(t,x;\nu)=\frac{\mathcal{U}(t,x;\nu)}{\int_{\mathbb{T}^d}\mathcal{U}(t,x';\nu)dx'}=\frac{\mathbb{E}_{\pi_N}^\omega[\mathcal{Z}_{t,t-1}^\omega(x,Y_1)]}{\int_{\mathbb{T}^d}\mathbb{E}_{\pi_N}^\omega[\mathcal{Z}_{t,t-1}^\omega(x',Y_1)]dx'}.\tag{4.17}$$

We note that in the above formula, the dependence on ν is through the distribution π_N^ω. It is also clear that the dependence of $u(t,x;\nu)$ on the random environment $\{\xi(s,\cdot):s\le t-1\}$ is only through the random variable Y_1, hence it suffices to study the mixing property of the Markov chain.

4.3. **Properties of the Markov chain.**

Lemma 4.1. *For any $p \in [1, +\infty)$, there exists $C > 0$ only depending on p, d, R such that*

$$\mathbf{E}\Big[\big(\inf_{x,y\in\mathbb{T}^d} \mathcal{Z}^\omega_{t,t-1}(x,y)\big)^{-p}\Big] + \mathbf{E}\Big[\big(\sup_{x,y\in\mathbb{T}^d} \mathcal{Z}^\omega_{t,t-1}(x,y)\big)^{p}\Big] \le C, \qquad t > 1. \tag{4.18}$$

It is a classical result for the stochastic heat equation, and for the convenience of readers, we present its proof in Section B.3.

For any $t > 1$ and $\delta > 0$, define the event

$$B_t(\delta) := \left\{\omega \in \Omega : \frac{\inf_{x,y,z\in\mathbb{T}^d} \mathcal{Z}^\omega_{t+1,t}(x,y)\mathcal{Z}^\omega_{t,t-1}(y,z)}{\sup_{x,z\in\mathbb{T}^d} \int_{\mathbb{T}^d} \mathcal{Z}^\omega_{t+1,t}(x,y')\mathcal{Z}^\omega_{t,t-1}(y',z)dy'} > \delta\right\}, \tag{4.19}$$

which belongs to $\mathcal{F}_{[t-1,t+1]}$, cf (4.4), and, thanks to (2.4),

$$\theta^{-1}_{s,y}\Big(B_t(\delta)\Big) = B_{t+s}(\delta), \quad (s,y) \in \mathbb{R}_+ \times \mathbb{T}^d. \tag{4.20}$$

Lemma 4.1 implies

Lemma 4.2. *There exists a constant $C > 0$ only depending on d, R such that*

$$\mathbf{P}[B_t(\delta)] \ge 1 - C\delta, \quad \textit{for all } t > 1 \textit{ and } \delta > 0. \tag{4.21}$$

Proof. First it is clear that by the time homogeneity of the random environment, the probability $\mathbf{P}[B_t(\delta)]$ does not depend on t. To simplify the notation, denote

$$W_1 = \inf_{x,y,z\in\mathbb{T}^d} \mathcal{Z}^\omega_{t+1,t}(x,y)\mathcal{Z}^\omega_{t,t-1}(y,z), \qquad W_2 = \sup_{x,z\in\mathbb{T}^d} \int_{\mathbb{T}^d} \mathcal{Z}^\omega_{t+1,t}(x,y)\mathcal{Z}^\omega_{t,t-1}(y,z)dy,$$

then for any $\delta > 0$, we have

$$\begin{aligned}\mathbf{P}[B_t(\delta)] =& \mathbf{P}[W_1W_2^{-1} > \delta] = 1 - \mathbf{P}[W_1^{-1}W_2 \ge \delta^{-1}]\\ \ge& 1 - \delta\mathbf{E}[W_1^{-1}W_2] \ge 1 - \delta\sqrt{\mathbf{E}[W_1^{-2}]\mathbf{E}[W_2^2]}.\end{aligned}$$

By Lemma 4.1, there exists a constant $C > 0$ depending on d, R such that

$$\sqrt{\mathbf{E}[W_1^{-2}]\mathbf{E}[W_2^2]} \le C.$$

The proof is complete.□

The following lemma holds.

Lemma 4.3. *We have*

$$\inf_{y_k,y_{k+1}\in\mathbb{T}^d} \pi^\omega_k(y_k \,|\, y_{k+1}) > \delta, \qquad \omega \in B_{t-k}(\delta) \tag{4.22}$$

for all $k = 2, \dots, N-2$.

Proof. Recall that

$$\begin{aligned}\pi^\omega_k(y_k \,|\, y_{k+1}) =& \frac{\int_{\mathbb{T}^{(k-1)d}} \mathcal{Z}^\omega_{t-1,t-2}(y_1,y_2)\dots\mathcal{Z}^\omega_{t-k,t-k-1}(y_k,y_{k+1})dy_1\dots dy_{k-1}}{\int_{\mathbb{T}^{kd}} \mathcal{Z}^\omega_{t-1,t-2}(y_1,y_2)\dots\mathcal{Z}^\omega_{t-k,t-k-1}(y_k,y_{k+1})dy_1\dots dy_{k-1}dy_k}\\ =& \frac{\int_{\mathbb{T}^d} g(y_{k-1})\mathcal{Z}^\omega_{t-k+1,t-k}(y_{k-1},y_k)\mathcal{Z}^\omega_{t-k,t-k-1}(y_k,y_{k+1})dy_{k-1}}{\int_{\mathbb{T}^{2d}} g(y_{k-1})\mathcal{Z}^\omega_{t-k+1,t-k}(y_{k-1},y_k)\mathcal{Z}^\omega_{t-k,t-k-1}(y_k,y_{k+1})dy_{k-1}dy_k},\end{aligned}$$

for some non-trivial $g \ge 0$. Then it is enough to use the fact that

$$\begin{aligned}&\inf_{y_k,y_{k+1}\in\mathbb{T}^d} \pi^\omega_k(y_k \,|\, y_{k+1})\\ \ge& \frac{\int_{\mathbb{T}^d} g(y)dy \cdot \inf_{y_{k-1},y_k,y_{k+1}\in\mathbb{T}^d} \mathcal{Z}^\omega_{t-k+1,t-k}(y_{k-1},y_k)\mathcal{Z}^\omega_{t-k,t-k-1}(y_k,y_{k+1})}{\int_{\mathbb{T}^d} g(y)dy \cdot \sup_{y_{k-1},y_{k+1}} \int_{\mathbb{T}^d} \mathcal{Z}^\omega_{t-k+1,t-k}(y_{k-1},y_k)\mathcal{Z}^\omega_{t-k,t-k-1}(y_k,y_{k+1})dy_k}\end{aligned}$$

and the definition of $B_t(\delta)$ in (4.19) to complete the proof. $\square$

Combining Lemmas 4.2 and 4.3, we conclude that there exists $\delta > 0$, depending only on d, R, and events

$$B_{t-k}(\delta) \in \mathcal{F}_{[t-k-1,t-k+1]}, \qquad k = 2, \dots, N-2, \tag{4.23}$$

such that

$$\mathbf{P}[B_{t-k}(\delta)] > \delta, \tag{4.24}$$

and the transition probability of the chain $\{Y_n\}_{n=1}^N$ satisfies

$$\inf_{y_k, y_{k+1} \in \mathbb{T}^d} \pi_k^\omega(y_k \,|\, y_{k+1}) > \delta, \qquad \text{if } \omega \in B_{t-k}(\delta), \quad k = 2, \dots, N-2. \tag{4.25}$$

4.4. **Coupling construction.** In this section, we use a standard couplingargument to study the dependence of $u(t,x;\nu)$ on the "remote" environment $\{\xi(r,\cdot) : r \le s\}$ with $t - s \gg 1$.

Let $t > 1$. Define

$$I_k := [t-k-1, t-k+1], \qquad k = 0, \dots, [t]-1.$$

We decompose the interval $[0,t]$ into subintervals

$$[0,t] = [0, t-2m_N - 1] \cup \bigcup_{m \in \Lambda_N} I_m \cup [t-1, t].$$

where, as we recall $N = [t]+1$ and

$$\Lambda_N := \{2, 4, \dots, 2m_N\}, \tag{4.26}$$

and $2m_N$ is the largest number so that $t - 2m_N - 1 \ge 0$:

$$2m_N = \begin{cases} N-2 & \text{if } N \text{ is even,} \\ N-3 & \text{if } N \text{ is odd.} \end{cases}$$

From (4.25) we know that there exists $\delta > 0$ that only depends on d, R such that $B_{t-k}(\delta) \in \mathcal{F}_{I_k}$, and

$$\inf_{y_k, y_{k+1} \in \mathbb{T}^d} \pi_k^\omega(y_k \,|\, y_{k+1}) > \delta, \qquad \text{if } \omega \in B_{t-k}(\delta), \quad k \in \Lambda_N.$$

We fix δ from now on. Let

$$q_k^\omega(y_k \,|\, y_{k+1}) := \frac{\pi_k^\omega(y_k \,|\, y_{k+1}) - \delta}{1-\delta}, \qquad \text{if } \omega \in B_{t-k}(\delta). \tag{4.27}$$

Let us introduce a sequence of i.i.d. Bernoulli random variables $\{\tau_k\}_{k\in\mathbb{Z}}$ that is defined on a probability space $\Big(\Sigma, \mathcal{A}, \mathbf{Q}\Big)$. We shall denote by $\mathbb{E}_{\mathbf{Q}}$ the corresponding expectation. We assume that

$$\mathbf{Q}[\tau_k = 1] = \delta, \qquad \mathbf{Q}[\tau_k = 0] = 1 - \delta.$$

Consider the following random transition kernels: for $k = 2, 3, \dots, N-2$, we have

$$\tilde{\pi}_k^{\omega,0}\Big(y_k \,|\, y_{k+1}\Big) = q_k^\omega(y_k | y_{k+1}), \quad \text{if } \omega \in B_{t-k}(\delta), \tag{4.28}$$

$$\tilde{\pi}_k^{\omega,1}\Big(y_k \,|\, y_{k+1}\Big) = 1, \qquad \text{if } \omega \in B_{t-k}(\delta), \tag{4.29}$$

$$\tilde{\pi}_k^{\omega,\tau}\Big(y_k \,|\, y_{k+1}\Big) = \pi_k^\omega(y_k \,|\, y_{k+1}), \qquad \text{if } \omega \in B_{t-k}^c(\delta), \tau \in \{0,1\}. \tag{4.30}$$

We also let

$$\tilde{\pi}_1^{\omega,\tau}\Big(y_1\,|\,y_2\Big) = \pi_1^{\omega}(y_1\,|\,y_2), \quad \tau \in \{0,1\}. \tag{4.31}$$

$$\tilde{\pi}_{N-1}^{\omega,\tau}\Big(y_{N-1}\,|\,y_N\Big) = \pi_{N-1}^{\omega}(y_{N-1}\,|\,y_N), \quad \tau \in \{0,1\}. \tag{4.32}$$

$$\tilde{\pi}_N^{\omega,\tau}\Big(dy_N\Big) = \pi_N^{\omega}(dy_N), \quad \tau \in \{0,1\}. \tag{4.33}$$

We shall write $\tilde{\pi}_N^{\omega,\tau}(y;\nu)$ when we wish to highlight the dependence of $\tilde{\pi}_N^{\omega,\tau}$ on the initial data ν, with the convention of writing $\tilde{\pi}_N^{\omega,\tau}(y;f)$, when $\nu(dx) = f(x)dx$. For a given realization of $\{\tau_k(\sigma)\}_{k=1}^N$, $\sigma \in \Sigma$, we consider the path measure $\mathbb{P}_{\tilde{\pi}_N}^{\omega,\sigma}$ on $\mathbb{T}^{Nd}$ corresponding to the time reversed Markov dynamics obtained by using $\tilde{\pi}_N^{\omega,\tau_N(\sigma)}$ as the final density and $\tilde{\pi}_k^{\omega,\tau_k(\sigma)}(y_k\,|\,y_{k+1})$ as transition probabilities between times $k+1$ and k, for $k = 1,\dots,N-1$. Let $\mathbb{E}_{\tilde{\pi}_N}^{\omega,\sigma}$ be the corresponding expectation, for each fixed realization of ω,σ. Note that

$$\int_{\Sigma} \mathbb{P}_{\tilde{\pi}_N}^{\omega,\sigma}(A)\mathbf{Q}(d\sigma) = \mathbb{P}_{\pi_N}^{\omega}(A), \quad A \in \mathcal{B}(\mathbb{T}^{Nd}). \tag{4.34}$$

Therefore, from (4.17) and (4.34),

$$u(t,x;\nu) = \frac{\int_{\Sigma}\mathbf{Q}(d\sigma)\mathbb{E}_{\tilde{\pi}_N}^{\omega,\sigma}[\mathcal{Z}_{t,t-1}^{\omega}(x,Y_1)]}{\int_{\mathbb{T}^d}dx'\int_{\Sigma}\mathbf{Q}(d\sigma)\mathbb{E}_{\tilde{\pi}_N}^{\omega,\sigma}[\mathcal{Z}_{t,t-1}^{\omega}(x',Y_1)]}. \tag{4.35}$$

Define the events

$$A_k^t = B_{t-k}(\delta)\times\{\tau_k = 1\} \in \mathcal{F}_{[t-k-1,t-k+1]}\otimes\mathcal{A}, \quad k = 2,3,\dots,N-2. \tag{4.36}$$

If some A_k^t occurs, the transition probability is given by the uniform distribution, as can be seen from (4.29), so the chain "renews". We emphasize here that N depends on t as well. If there is no danger of confusion we shall omit the superscript t from the notation of the events. Note that

$$\mathbf{P}\otimes\mathbf{Q}[A_k^t] = \mathbf{P}[B_{t-k}(\delta)]\mathbf{Q}[\tau_k = 1] > \delta^2, \quad k = 2,3,\dots,N-2. \tag{4.37}$$

It is also clear that the events A_m^t, $m \in \Lambda_N$ are independent under $\mathbf{P}\otimes\mathbf{Q}$ (recall that Λ_N was defined in (4.26)). In addition, thanks to (4.20), for any $1 < s < t$ and non-negative $\ell \in \mathbb{Z}$ we have

$$\theta_{t-s,y}^{-1}\Big(A_k^s\Big) = A_k^t, \quad k = 2,3,\dots,N(s)-2,\ y \in \mathbb{T}^d. \tag{4.38}$$

Here we have denoted $\theta_{r,y}(\omega,\sigma) \coloneqq \Big(\theta_{r,y}(\omega),\sigma\Big)$.

Note that for any $F \in B_b(\mathbb{T}^d)$ we have

$$\begin{aligned}\mathbb{E}_{\mathbf{Q}}\otimes\mathbb{E}_{\tilde{\pi}_N}^{\omega,\cdot}\Big[F(Y_1)\Big|\tau_k = 1\Big] &= \frac{1}{\delta}\int_{\{\tau_k=1\}}\mathbf{Q}(d\sigma)\mathbb{E}_{\tilde{\pi}_N}^{\omega,\sigma}[F(Y_1)]\\ &= \mathbb{E}_k^{\omega}\Big[F(Y_1)\Big], \qquad 2 \le k \le N-2,\ \omega \in B_{t-k}(\delta),\end{aligned} \tag{4.39}$$

where, as we recall, $\mathbb{E}_k^{\omega}$ is the expectation with respect to the Markov chain constructed in (4.13) and (4.14). Now it is clear that for any $t > 1$ and $2 \le k \le N-2$, the random variable

$$1_{B_{t-k}(\delta)}(\omega)\int_{\{\tau_k=1\}}\mathbf{Q}(d\sigma)\mathbb{E}_{\tilde{\pi}_N}^{\omega,\sigma}[\mathcal{Z}_{t,t-1}^{\omega}(x,Y_1)], \qquad \omega \in \Omega \tag{4.40}$$

is $\mathcal{F}_{[t-k-1,t]}$-measurable.

4.5. **Approximation.** For any $\nu \in \mathcal{M}_1(\mathbb{T}^d)$, define

$$X^\omega(t,x;\nu) := \mathbb{E}^\omega_{\pi_N(\nu)}[\mathcal{Z}^\omega_{t,t-1}(x,Y_1)] = \mathbb{E}_{\mathbf{Q}}\mathbb{E}^{\omega,\cdot}_{\tilde{\pi}_N(\nu)}[\mathcal{Z}^\omega_{t,t-1}(x,Y_1)]. \tag{4.41}$$

When $\nu(dx) = f(x)dx$ for $f \in D(\mathbb{T}^d)$ we shall write $X^\omega(t,x;f) := X^\omega(t,x;\nu)$. We omit writing the initial data ν from our notation when they are obvious from the context.

Fix $t > 10$ and $2 \le k \le N(t) - 2$. We shall approximate $X^\omega(t,x)$ by some random variable that is $\mathcal{F}_{[t-k,t]}$−measurable. Define

$$\Lambda_{N(t),k} = \{m : m \le k-1\} \cap \Lambda_{N(t)}, \tag{4.42}$$

$$\tilde{A}^t_k = \bigcup_{m\in\Lambda_{N(t),k}} A^t_m \in \mathcal{F}_{[t-k,t-1]} \otimes \mathcal{A}, \tag{4.43}$$

and

$$X^\omega_k(t,x) := \int_\Sigma \mathbf{Q}(d\sigma) 1_{\tilde{A}^t_k}(\omega,\sigma)\mathbb{E}^{\omega,\sigma}_{\tilde{\pi}_{N(t)}}[\mathcal{Z}^\omega_{t,t-1}(x,Y_1)], \tag{4.44}$$

and, thanks to (4.38),

$$\theta^{-1}_{t-s,0}\Big(\tilde{A}^s_k\Big) = \tilde{A}^t_k. \tag{4.45}$$

In particular, as can be seen from (4.39), the random field $\{X_k(t,x)\}$ does not depend on the initial data. Moreover, thanks to (4.3) and (4.45), we have

$$X^{\theta_{t-s,0}(\omega)}_k(s,x) = X^\omega_k(t,x), \quad k \le N(s) - 2. \tag{4.46}$$

As a direct consequence of the above and formula (4.39), we conclude:

Lemma 4.4. *Suppose that $\nu_1, \nu_2 \in D(\mathbb{T}^d)$. For any $t \ge s > k$ and $2 \le k \le N(s) - 2$, the laws of the fields $X_k(t,\cdot;\nu_1)$ and $X_k(s,\cdot;\nu_2)$ over the space $L^1(\mathbb{T}^d)$ are identical.*

Lemma 4.5. *For any $2 \le k \le N(t) - 2$ and $\nu \in \mathcal{M}_1(\mathbb{T}^d)$, the field $X_k(t,\cdot;\nu)$ is $\mathcal{F}_{[t-k,t]}$−measurable. Moreover, for any $p \in [1,+\infty)$, there exists $C, \lambda > 0$ only depending on p, d and R such that*

$$\mathbf{E}\Big[\sup_{\nu\in\mathcal{M}_1(\mathbb{T}^d)} \|X^\omega(t;\nu) - X^\omega_k(t;\nu)\|^p_{L^\infty(\mathbb{T}^d)}\Big] \le Ce^{-\lambda k} \quad \textit{for } 2 \le k \le N(t) - 2. \tag{4.47}$$

Proof. First we obviously have

$$\mathbb{E}^{\omega,\sigma}_{\tilde{\pi}_N}[\mathcal{Z}^\omega_{t,t-1}(x,Y_1)] \le \sup_{x,y\in\mathbb{T}^d} \mathcal{Z}^\omega_{t,t-1}(x,y), \tag{4.48}$$

which implies

$$\sup_{x\in\mathbb{T}^d} |X^\omega(t,x;\nu) - X^\omega_k(t,x;\nu)| \le \sup_{x,y\in\mathbb{T}^d} \mathcal{Z}^\omega_{t,t-1}(x,y) \int_\Sigma \mathbf{Q}(d\sigma)1_{(\tilde{A}^t_k)^c}(\omega,\sigma).$$

Thus, by Jensen's inequality

$$\begin{aligned}&\mathbf{E}\Big[\sup_{\nu\in\mathcal{M}_1(\mathbb{T}^d)} \|X^\omega(t;\nu) - X^\omega_k(t;\nu)\|^p_{L^\infty(\mathbb{T}^d)}\Big]\\ &\le \sqrt{\mathbf{E}\Big(\sup_{x,y\in\mathbb{T}^d} \mathcal{Z}^\omega_{t,t-1}(x,y)\Big)^{2p}}\sqrt{\mathbf{P}\otimes\mathbf{Q}[(\tilde{A}^t_k)^c]}.\end{aligned}$$

Since $(\tilde{A}^t_k)^c = \bigcap_{m\in\Lambda_{N,k}}(A^t_m)^c$ and the events A^t_m, $m \in \Lambda_{N,k}$ are independent, from (4.37) we have

$$\mathbf{P}\otimes\mathbf{Q}[(\tilde{A}^t_k)^c] = \prod_{m\in\Lambda_{N,k}} \mathbf{P}\otimes\mathbf{Q}[(A^t_m)^c] < (1-\delta^2)^{\#\Lambda_{N,k}}, \tag{4.49}$$

where δ depends only on R and d. Combining (4.49) with Lemma 4.1, we derive that

$$\mathbf{E}\Big[\sup_{\nu\in\mathcal{M}_1(\mathbb{T}^d)}\|X^\omega(t;\nu)-X_k^\omega(t;\nu)\|_{L^\infty(\mathbb{T}^d)}^p\Big]\le C(1-\delta^2)^{\#\Lambda_{N,k}/2},$$

where the constant $C>0$ does not depend on t. This proves (4.47).

We prove now that $X_k^\omega(t,x;\nu)$ is $\mathcal{F}_{[t-k,t]}$−measurable. Consider now the event $\tilde{A}_k$ given by (4.43). From this point on we shall drop t from the notation of the events and integers. Since N,k are both fixed here, to simplify the notations, let

$$M:=\max\{m:m\in\Lambda_{N,k}\}$$

and for any $m\in\Lambda_{N,k}$, define

$$D_m=\begin{cases} A_m\cap\Bigg(\displaystyle\bigcup_{m'>m,\,m'\in\Lambda_{N,k}}A_{m'}\Bigg)^c, & \text{if } m<M,\\ A_M, & \text{if } m=M.\end{cases}$$

Then,

$$1_{\tilde{A}_k}(\omega,\sigma)=\sum_{m\in\Lambda_{N,k}}1_{D_m}(\omega,\sigma), \tag{4.50}$$

and it suffices to show that for any $m\in\Lambda_{N,k}$,

$$\int_\Sigma\mathbf{Q}(d\sigma)1_{D_m}(\omega,\sigma)\mathbb{E}_{\tilde{\pi}_N}^{\omega,\sigma}[\mathcal{Z}_{t,t-1}^\omega(x,Y_1)]$$

is $\mathcal{F}_{[t-k,t]}$−measurable. We can write

$$\begin{aligned}&\int_\Sigma\mathbf{Q}(d\sigma)1_{D_m}(\omega,\sigma)\mathbb{E}_{\tilde{\pi}_N}^{\omega,\sigma}[\mathcal{Z}_{t,t-1}^\omega(x,Y_1)]\\ &=\int_\Sigma\mathbf{Q}(d\sigma)1_{A_m}(\omega,\sigma)\Bigg(\prod_{m'>m,m'\in\Lambda_{N,k}}1_{A_{m'}^c}(\omega,\sigma)\Bigg)\mathbb{E}_{\tilde{\pi}_N}^{\omega,\sigma}[\mathcal{Z}_{t,t-1}^\omega(x,Y_1)].\end{aligned}$$

By convention we understand a product over an empty set to be equal to 1. Invoking (4.39), the right hand side of the above equation can be written as

$$\begin{aligned}&\Big(\int_\Sigma\mathbf{Q}(d\sigma)1_{A_m}(\omega,\sigma)\mathbb{E}_m^\omega[\mathcal{Z}_{t,t-1}^\omega(x,Y_1)]\Big)\int_\Sigma\mathbf{Q}(d\sigma)\Bigg(\prod_{m'>m,m'\in\Lambda_{N,k}}1_{A_{m'}^c}(\omega,\sigma)\Bigg)\\ &=\mathbb{E}_m^\omega[\mathcal{Z}_{t,t-1}^\omega(x,Y_1)]\delta 1_{B_{t-m}(\delta)}(\omega)\prod_{m'>m,m'\in\Lambda_{N,k}}\Big(1-\delta 1_{B_{t-m'}(\delta)}(\omega)\Big),\end{aligned} \tag{4.51}$$

which is $\mathcal{F}_{[t-k,t]}$−measurable. This completes the proof. □

From the above proof, we can actually conclude that $X_k^\omega(t,x)$ only depends on t,x,k and the random environment $\{\xi(s,\cdot):s\in[t-k,t]\}$. Recall that $u(t,\cdot;\nu)$ denotes the endpoint density of the polymer path, assuming that the starting distribution is an arbitrary probability measure $\nu\in\mathcal{M}_1(\mathbb{T}^d)$. We have the following key proposition which underlies Theorem 2.3.

Proposition 4.6. *For any $2\le k\le N(t)-2$, there exists $u_k(t,x)$ that is $\mathcal{F}_{[t-k,t]}$−measurable and does not depend on the initial distribution ν, such that for any $p\in[1,+\infty)$,*

$$\sup_{\nu\in\mathcal{M}_1(\mathbb{T}^d)}\mathbf{E}\Big[\|u(t;\nu)-u_k(t)\|_{L^\infty(\mathbb{T}^d)}^p\Big]\le Ce^{-\lambda k}, \tag{4.52}$$

where the constants $C,\lambda>0$ depend only on p,d,R.

Proof. By (4.35) and (4.41), we have

$$u(t,x;\nu) = \frac{X^\omega(t,x;\nu)}{\int_{\mathbb{T}^d} X^\omega(t,x';\nu)dx'}.$$

Let

$$C_k(\delta) := \bigcup_{m\in\Lambda_{N,k}} B_{t-m}(\delta),$$

where we recall that $\Lambda_{N,k}$ was defined in (4.42). Assuming $\omega \in B_{t-m}(\delta)$ for some $m \in \Lambda_{N,k}$, then $1_{\tilde{A}_k}(\omega,\sigma) \ge 1_{\{\tau_m=1\}}(\sigma)$ ($\tilde{A}_k$ was defined in (4.43)), which implies

$$\begin{aligned} \inf_{x\in\mathbb{T}^d} X_k^\omega(t,x) &\ge \Big(\int_\Sigma \mathbf{Q}(d\sigma)1_{\{\tau_m=1\}}(\sigma)\Big) \inf_{x,y\in\mathbb{T}^d} \mathcal{Z}^\omega_{t,t-1}(x,y) \\ &= \delta \inf_{x,y\in\mathbb{T}^d} \mathcal{Z}^\omega_{t,t-1}(x,y), \qquad \text{if } \omega\in C_k(\delta), \end{aligned} \tag{4.53}$$

with X_k^ω defined in (4.44). Define

$$u_k(t,x) := \begin{cases} \frac{X_k^\omega(t,x)}{\int_{\mathbb{T}^d} X_k^\omega(t,x')dx'}, & \text{if } \omega\in C_k(\delta) \\ 0, & \text{if } \omega \in C_k^c(\delta). \end{cases} \tag{4.54}$$

From the definition (4.41), we conclude

$$\inf_{x,y\in\mathbb{T}^d} \mathcal{Z}^\omega_{t,t-1}(x,y) \le \inf_{x\in\mathbb{T}^d} X^\omega(t,x) \le \sup_{x\in\mathbb{T}^d} X^\omega(t,x) \le \sup_{x,y\in\mathbb{T}^d} \mathcal{Z}^\omega_{t,t-1}(x,y).$$

Concerning $X_k^\omega(t,x)$, we conclude from (4.53) that

$$\begin{aligned} \delta 1_{C_k(\delta)}(\omega) \inf_{x,y\in\mathbb{T}^d} \mathcal{Z}^\omega_{t,t-1}(x,y) &\le \inf_{x\in\mathbb{T}^d} X_k^\omega(t,x) \\ &\le \sup_{x\in\mathbb{T}^d} X_k^\omega(t,x) \le \sup_{x,y\in\mathbb{T}^d} \mathcal{Z}^\omega_{t,t-1}(x,y). \end{aligned}$$

Thus, for $\omega \in C_k(\delta)$, we have

$$\begin{aligned} \sup_{x\in\mathbb{T}^d} |u(t,x;\nu) - u_k(t,x)| &\le \sup_{x\in\mathbb{T}^d} |X^\omega(t,x) - X_k^\omega(t,x)| \\ &\times \Big(\inf_{x,y\in\mathbb{T}^d} \mathcal{Z}^\omega_{t,t-1}(x,y)\Big)^{-1} \Big(1 + \frac{1}{\delta}\frac{\sup_{x,y\in\mathbb{T}^d} \mathcal{Z}^\omega_{t,t-1}(x,y)}{\inf_{x,y\in\mathbb{T}^d} \mathcal{Z}^\omega_{t,t-1}(x,y)}\Big). \end{aligned} \tag{4.55}$$

We can therefore write

$$\mathbf{E}[\sup_{x\in\mathbb{T}^d} |u(t,x;\nu) - u_k(t,x)|^p] = I_1 + I_2, \tag{4.56}$$

where

$$\begin{aligned} I_1 &:= \mathbf{E}\Big[\sup_{x\in\mathbb{T}^d} |u(t,x;\nu) - u_k(t,x)|^p 1_{C_k(\delta)}\Big], \\ I_2 &:= \mathbf{E}\Big[\sup_{x\in\mathbb{T}^d} |u(t,x;\nu)|^p 1_{C_k^c(\delta)}\Big]. \end{aligned} \tag{4.57}$$

To estimate I_1, we use the bound (4.55) together with Lemmas 4.1 and 4.5, and obtain that there exist $C,\lambda > 0$ depend only on p,d,R such that $I_1 \le Ce^{-\lambda k}$. To deal with I_2, note that by the Hölder estimate for any $r > p$ we can write

$$\begin{aligned} I_2 &\le \Big\{\mathbf{E}\Big[\Big(\sup_{x\in\mathbb{T}^d} u(t,x;\nu)\Big)^r\Big]\Big\}^{p/r} \Big(\mathbf{P}[C_k^c(\delta)]\Big)^{1-p/r} \\ &= \Big\{\mathbf{E}\Big[\Big(\sup_{x\in\mathbb{T}^d} u(t,x;\nu)\Big)^r\Big]\Big\}^{p/r} \Big(\mathbf{P}[\bigcap_{m\in\Lambda_{N,k}} B^c_{t-m}(\delta)]\Big)^{1-p/r}. \end{aligned} \tag{4.58}$$

Since $B^c_{t-m}(\delta)$ are independent and satisfy (4.24), we conclude that the right hand side can be estimated by

$$\left\{\mathbf{E}\Big[\Big(\sup_{x\in\mathbb{T}^d} u(t,x;\nu)\Big)^r\Big]\right\}^{p/r}\Big(1-\delta\Big)^{(1-p/r)\#\Lambda_{N,k}}.$$

Using Lemma 4.1 we conclude that there exist $C,\lambda>0$ depend only on p,d,R such that $I_2\le Ce^{-\lambda k}$. The proof is complete. □

4.6. **Proof of Theorem 2.3.** Recall that given an initial distribution $\nu\in\mathcal{M}_1(\mathbb{T}^d)$ the polymer endpoint density $u(t;\nu)$ is given by (2.3). Fix $F\in\mathrm{Lip}(\mathcal{M}_1(\mathbb{T}^d))$. By virtue of Lemma 4.4 and (4.54), for $t>s>k>s/2$, we have

$$\mathbf{E}[F(u_k(t))]=\mathbf{E}[F(u_k(s))].$$

Therefore, for any $\nu_1,\nu_2\in\mathcal{M}_1(\mathbb{T}^d)$, we have

$$\begin{aligned}
&\Big|\mathcal{P}_tF(\nu_1)-\mathcal{P}_sF(\nu_2)\Big|=\Big|\mathbf{E}[F(u(t;\nu_1))]-\mathbf{E}[F(u(s;\nu_2))]\Big|\\
&=\Big|\mathbf{E}[F(u(t;\nu_1))]-\mathbf{E}[F(u_k(t))]-\mathbf{E}[F(u(s;\nu_2))]+\mathbf{E}[F(u_k(s))]\Big|\\
&\le\Big|\mathbf{E}[F(u(t;\nu_1))]-\mathbf{E}[F(u_k(t))]\Big|+\Big|\mathbf{E}[F(u(s;\nu_2))]-\mathbf{E}[F(u_k(s))]\Big|\\
&\le\|F\|_{\mathrm{Lip}}\Big(\sup_{\mu\in\mathcal{M}_1(\mathbb{T}^d)}\mathbf{E}\|u(t;\mu)-u_k(t)\|_{L^1(\mathbb{T}^d)}+\sup_{\mu\in\mathcal{M}_1(\mathbb{T}^d)}\mathbf{E}\|u(s;\mu)-u_k(s)\|_{L^1(\mathbb{T}^d)}\Big).
\end{aligned}$$

Using Proposition 4.6 we can further write that

$$\Big|\mathcal{P}_tF(\nu_1)-\mathcal{P}_sF(\nu_2)\Big|\le C\|F\|_{\mathrm{Lip}}e^{-\lambda k}\le C\|F\|_{\mathrm{Lip}}e^{-\lambda s/2},\tag{4.59}$$

which in turn implies that there exist $C,\lambda>0$ such that

$$\sup_{\nu_1,\nu_2\in\mathcal{M}_1(\mathbb{T}^d)}\Big\|\delta_{\nu_1}\mathcal{P}_t-\delta_{\nu_2}\mathcal{P}_s\Big\|_{\mathrm{FM}}\le Ce^{-\lambda s/2},\quad t>s>0.$$

For any $\Pi\in\mathrm{M}_1(\mathcal{M}_1(\mathbb{T}^d))$ and $F\in\mathrm{Lip}(\mathcal{M}_1(\mathbb{T}^d))$, we can write

$$\int_{\mathcal{M}_1(\mathbb{T}^d)}F(\nu)\Pi\mathcal{P}_t(d\nu)=\int_{\mathcal{M}_1(\mathbb{T}^d)}\Pi(d\nu)\Big(\int_{\mathcal{M}_1(\mathbb{T}^d)}\delta_\nu\mathcal{P}_t(d\mu)F(\mu)\Big),$$

so we also conclude that

$$\sup_{\Pi_1,\Pi_2\in\mathrm{M}_1(\mathcal{M}_1(\mathbb{T}^d))}\Big\|\Pi_1\mathcal{P}_t-\Pi_2\mathcal{P}_s\Big\|_{\mathrm{FM}}\le Ce^{-\lambda s/2},\quad t>s>0,\tag{4.60}$$

which implies that there exists a unique $\pi_\infty\in\mathrm{M}_1(\mathcal{M}_1(\mathbb{T}^d))$ such that

$$\lim_{t\to+\infty}\Pi\mathcal{P}_t=\pi_\infty\tag{4.61}$$

for any $\Pi\in\mathrm{M}_1(\mathcal{M}_1(\mathbb{T}^d))$, in the Fortet-Mourier metric on $\mathrm{M}_1(\mathcal{M}_1(\mathbb{T}^d))$. Hence

$$\pi_\infty=\pi_\infty\mathcal{P}_t\quad\text{for all }t>0.$$

Suppose that $p\in[1,+\infty]$. Since $\delta_\nu\mathcal{P}_t(D^p(\mathbb{T}^d))=1$ for all $\nu\in\mathcal{M}_1(\mathbb{T}^d)$ and any $t>0$, we have

$$\pi_\infty\Big(D^p(\mathbb{T}^d)\Big)=\pi_\infty\mathcal{P}_t\Big(D^p(\mathbb{T}^d)\Big)=\int_{\mathcal{M}_1(\mathbb{T}^d)}\pi_\infty(d\nu)\Big(\delta_\nu\mathcal{P}_t\Big(D^p(\mathbb{T}^d)\Big)\Big)=1,$$

which finishes the proof of part (i).

Part (ii) is a direct consequence of (4.60) and (4.61).

Suppose now that $p\in(1,\infty]$ and $F\in\mathrm{Lip}(D^p(\mathbb{T}^d))$. Proposition 4.6 also applies in this case with the $L^1(\mathbb{T}^d)$ norm, used above, replaced by the $L^p(\mathbb{T}^d)$ norm. In consequence we conclude (4.59) with the Lipschitz norm of F in the respective

$L^p(\mathbb{T}^d)$ metric. Hence, we infer the estimate (4.60) with the respective Fortet-Mourier norm on the l.h.s. The remaining part of the argument stays the same and, as a result, we conclude the statement of the theorem in the case of the space $D^p(\mathbb{T}^d)$. □

4.7. **Asymptotic stability of the transition probability operator.** A straightforward consequence of Theorem 2.3 is

Corollary 4.7. *Suppose that $F \in C_b(\mathcal{M}_1(\mathbb{T}^d))$ and $\Pi \in \mathrm{M}_1(\mathcal{M}_1(\mathbb{T}^d))$ then*

$$\lim_{t\to+\infty}\int_{\mathcal{M}_1(\mathbb{T}^d)} \mathcal{P}_t F d\Pi = \int_{\mathcal{M}_1(\mathbb{T}^d)} F(u)\pi_\infty(du). \tag{4.62}$$

Proposition 4.8. *Suppose that $F \in \mathrm{Lip}(\mathcal{M}_1(\mathbb{T}^d))$. Then,*

$$\lim_{t\to+\infty}\mathbf{E}_\Pi\left(\frac{1}{t}\int_0^t F(u(s))ds - \int_{\mathcal{M}_1(\mathbb{T}^d)} F(u)\pi_\infty(du)\right)^2 = 0. \tag{4.63}$$

where $\mathbf{E}_\Pi$ denotes the expectation w.r.t. the noise ξ realization and the initial data $u(0)$ distributed according to a Borel probability measure Π on $\mathcal{M}_1(\mathbb{T}^d)$. The same result also holds for $D^p(\mathbb{T}^d)$ in place of $\mathcal{M}_1(\mathbb{T}^d)$ for any $p \in [1,+\infty]$.

Proof. It suffices to show that as $t \to \infty$,

$$\mathbf{E}_\Pi\left(\frac{1}{t}\int_0^t F(u(s))ds\right) \to \int_{\mathcal{M}_1(\mathbb{T}^d)} F(u)\pi_\infty(du), \tag{4.64}$$

and

$$\mathbf{E}_\Pi\left(\frac{1}{t}\int_0^t F(u(s))ds\right)^2 \to \left(\int_{\mathcal{M}_1(\mathbb{T}^d)} F(u)\pi_\infty(du)\right)^2. \tag{4.65}$$

Equality (4.64) follows directly from (4.62). Concerning (4.65), we can write

$$\begin{aligned}
&\mathbf{E}_\Pi\left(\frac{1}{t}\int_0^t F(u(s))ds\right)^2 = \frac{2}{t^2}\mathbf{E}_\Pi\left(\int_0^t F(u(s))ds\int_0^s F(u(s'))ds'\right)\\
&= \frac{2}{t^2}\int_0^t ds\int_0^s ds'\left[\int_{\mathcal{M}_1(\mathbb{T}^d)} \mathcal{P}_{s'}\left(F\mathcal{P}_{s-s'}F\right)d\Pi\right]\\
&= \frac{2}{t^2}\int_0^t ds\int_0^s ds'\left[\int_{\mathcal{M}_1(\mathbb{T}^d)} \mathcal{P}_{s-s'}\left(F\mathcal{P}_{s'}F\right)d\Pi\right].
\end{aligned} \tag{4.66}$$

Using Theorem 2.3 part (ii), we conclude that

$$\left\|\mathcal{P}_{s-s'}\left(F\mathcal{P}_{s'}F\right) - \mathcal{P}_{s-s'}\left(F\int_{\mathcal{M}_1(\mathbb{T}^d)} F(u)\pi_\infty(du)\right)\right\|_\infty \le Ce^{-\lambda s'}.$$

Therefore,

$$\begin{aligned}
&\Bigg|\frac{2}{t^2}\int_0^t ds\int_0^s ds'\left[\int_{\mathcal{M}_1(\mathbb{T}^d)} \mathcal{P}_{s-s'}\left(F\mathcal{P}_{s'}F\right)d\Pi\right]\\
&-\frac{2}{t^2}\left(\int_{\mathcal{M}_1(\mathbb{T}^d)} Fd\pi_\infty\right)\int_0^t ds\int_0^s ds'\int_{\mathcal{M}_1(\mathbb{T}^d)} \mathcal{P}_{s-s'}Fd\Pi\Bigg|\\
&\le \frac{C}{t^2}\int_0^t ds\int_0^s e^{-\lambda s'}ds' \to 0, \qquad \text{as } t\to\infty.
\end{aligned} \tag{4.67}$$

On the other hand, upon another application of Theorem 2.3 part (ii), we conclude that

$$
\begin{aligned}
&\frac{2}{t^2}\int_0^t ds\int_0^s ds'\int_{\mathcal{M}_1(\mathbb{T}^d)}\mathcal{P}_{s-s'}Fd\Pi\\
&=2\int_0^1 ds\int_0^s ds'\int_{\mathcal{M}_1(\mathbb{T}^d)}\mathcal{P}_{t(s-s')}Fd\Pi \qquad (4.68)\\
&\to 2\int_{\mathcal{M}_1(\mathbb{T}^d)}F(u)\pi_\infty(du)\int_0^1 ds\int_0^s ds' = \int_{\mathcal{M}_1(\mathbb{T}^d)}F(u)\pi_\infty(du).
\end{aligned}
$$

Combining (4.67) with (4.68), we conclude (4.65), thus finishing the proof of the proposition. The proof for the case of $F\in \mathrm{Lip}(D^p(\mathbb{T}^d))$ is the same. □

4.8. **More on Theorem 2.3.** In the statement of Theorem 2.3, we have only considered the Lipschitz functional on $\mathcal{M}_1(\mathbb{T}^d)$, or $D^p(\mathbb{T}^d)$. In what follows we shall need the stability of the semigroup $\mathcal{P}_t$ on functionals that are only local Lipschitz. For that purpose we shall need the following lemma:

Lemma 4.9. *For any $p\in[1,+\infty)$, there exists $C=C(d,R,p)$ such that*

$$\sup_{t>1,\nu\in\mathcal{M}_1(\mathbb{T}^d)}\mathbf{E}\Big[\big(\sup_{x\in\mathbb{T}^d}u(t,x;\nu)\big)^p\Big]\le C.$$

Proof. By (4.17), we have when $t>1$ that

$$\sup_{x\in\mathbb{T}^d}u(t,x;\nu)\le\frac{\sup_{x,y\in\mathbb{T}^d}\mathcal{Z}^\omega_{t,t-1}(x,y)}{\inf_{x,y\in\mathbb{T}^d}\mathcal{Z}^\omega_{t,t-1}(x,y)},$$

then the result is a direct consequence of Lemma 4.1. □

Now consider the functional $\mathcal{R}:D^2(\mathbb{T}^d)\to\mathbb{R}$ defined in (2.16):

$$\mathcal{R}(v)=\int_{\mathbb{T}^{2d}}R(x-y)v(x)v(y)dxdy\quad\text{ and }\tilde{\mathcal{R}}=\mathcal{R}-2\gamma, \tag{4.69}$$

with γ given by (2.22). The functional is globally Lipschitz in the case when $R(\cdot)$ is bounded, but only locally Lipschitz when $R=\delta$, as then $\mathcal{R}(v)=\|v\|^2_{L^2(\mathbb{T}^d)}$.

Proposition 4.10. *There exist C,λ only depending on R,d such that*

$$\sup_{t>1,\nu\in\mathcal{M}_1(\mathbb{T}^d)}|\mathcal{P}_t\tilde{\mathcal{R}}(\nu)|\le Ce^{-\lambda t}.$$

Proof. By Lemma 4.9 and the Markov property, it suffices to prove the result for any $v\in D^2(\mathbb{T}^d)$ and $t>1$. First we note that $\mathcal{R}(\cdot)$ is continuous from $D^2(\mathbb{T}^d)$ to $\mathbb{R}$, therefore by Theorem 2.3, we have as $t\to\infty$ that

$$\mathcal{R}(u(t;v))\Rightarrow\mathcal{R}(\tilde{u})\qquad\text{in distribution,}$$

with $\tilde{u}$ sampled from π_∞. Lemma 4.9 ensures the uniform integrability so we have

$$\mathcal{P}_t\mathcal{R}(v)=\mathbf{E}[\mathcal{R}(u(t;v))]\to\mathbf{E}[\mathcal{R}(\tilde{u})]=2\gamma. \tag{4.70}$$

Next, for any $t_1>t_2>k>t_2/2$, we have

$$|\mathcal{P}_{t_1}\mathcal{R}(v)-\mathcal{P}_{t_2}\mathcal{R}(v)|\le\sum_{j=1}^2|\mathbf{E}[\mathcal{R}(u(t_j;v))-\mathcal{R}(u_k(t_j))]|,$$

with u_k defined in (4.54). By Proposition 4.6 and the fact that

$$
\begin{aligned}
&|\mathcal{R}(u(t_j;v))-\mathcal{R}(u_k(t_j))|\\
&\le\|u(t_j;v)-u_k(t_j)\|_{L^\infty(\mathbb{T}^d)}(\|u(t_j;v)\|_{L^1(\mathbb{T}^d)}+\|u_k(t_j)\|_{L^1(\mathbb{T}^d)})\\
&\le 2\|u(t_j;v)-u_k(t_j)\|_{L^\infty(\mathbb{T}^d)},
\end{aligned}
$$

we derive $|\mathcal{P}_{t_1}\mathcal{R}(v) - \mathcal{P}_{t_2}\mathcal{R}(v)| \le Ce^{-\lambda t_2}$. Sending $t_1 \to \infty$ and applying (4.70), we complete the proof. □

5. Proofs of Theorem 2.5 and Theorem 1.1

Recall from (2.24) that

$$\log Z_t + \gamma t = \int_0^t \int_{\mathbb{T}^d} u(s,y)\xi(s,y)dyds - \tfrac{1}{2}\int_0^t \tilde{\mathcal{R}}(u(s))ds$$

where $\tilde{\mathcal{R}} = \mathcal{R} - 2\gamma$,

$$u(s) = u(s;\nu) = Z_s^{-1}\mathcal{U}(s;\nu), \tag{5.1}$$

where ν is an arbitrary probability measure on $\mathbb{T}^d$. By the definition of γ in (2.22), we have

$$\int_{\mathcal{M}_1(\mathbb{T}^d)} \tilde{\mathcal{R}}(u)\pi_\infty(du) = 0.$$

The goal in this section is to prove Theorem 2.5: as $t \to \infty$,

$$\frac{\log Z_t + \gamma t}{\sqrt{t}} \Rightarrow N(0,\sigma^2). \tag{5.2}$$

With the above convergence, the proof of Theorem 1.1 goes as follows: for any $\nu \in \mathcal{M}_1(\mathbb{T}^d)$ and $x \in \mathbb{T}^d$, we write

$$\frac{\log\mathcal{U}(t,x;\nu) + \gamma t}{\sqrt{t}} = \frac{\log Z_t + \gamma t}{\sqrt{t}} + \frac{\log u(t,x;\nu)}{\sqrt{t}}. \tag{5.3}$$

For $t > 1$, by (4.17) we have

$$\frac{\inf_{x,y\in\mathbb{T}^d} \mathcal{Z}^\omega_{t,t-1}(x,y)}{\sup_{x,y\in\mathbb{T}^d} \mathcal{Z}^\omega_{t,t-1}(x,y)} \le u(t,x;\nu) \le \frac{\sup_{x,y\in\mathbb{T}^d} \mathcal{Z}^\omega_{t,t-1}(x,y)}{\inf_{x,y\in\mathbb{T}^d} \mathcal{Z}^\omega_{t,t-1}(x,y)}. \tag{5.4}$$

Thus, in light of Lemma 4.1, we conclude thatthere exists a constant $C > 0$ such that

$$\mathbf{E}|\log u(t,x;\nu)| \le C \quad \text{for all } t > 1, x \in \mathbb{T}^d.$$

As a result we obtain

$$\frac{\log u(t,x;\nu)}{\sqrt{t}} \to 0, \quad \text{as } t \to +\infty,$$

in probability and the proof of Theorem 1.1 is complete.

5.1. Construction of the corrector.

The purpose of the present section is to construct the solution of the equation $-\mathcal{L}\chi = \tilde{\mathcal{R}}$, where $\tilde{\mathcal{R}}$ is defined in (4.69) and $\mathcal{L}$ is a properly understood generator of the transition probability semigroup $(\mathcal{P}_t)_{t\ge0}$, see Remark 5.4 below. The field $\chi : D^\infty(\mathbb{T}^d) \to \mathbb{R}$ is called the corrector and is a crucial object in the proof of the CLT for $\log Z_t$. We start with the following lemma:

Lemma 5.1. *For each $p \in [1,+\infty)$ there exists a constant $C > 0$ such that*

$$\sup_{t\ge0} \mathbf{E}[\|u(t;v)\|^p_{L^\infty(\mathbb{T}^d)}] \le C(1 + \|v\|^p_{L^\infty(\mathbb{T}^d)}), \tag{5.5}$$

and

$$|\mathcal{P}_t\tilde{\mathcal{R}}(v)| \le C(e^{-\lambda t}1_{t>1} + (1 + \|v\|_{L^\infty(\mathbb{T}^d)})1_{t\in[0,1]}), \quad \textit{for any } v \in D^\infty(\mathbb{T}^d),\ t \ge 0. \tag{5.6}$$

Proof. The case of $t > 1$ is implied by Lemma 4.9 and Proposition 4.10. For $t \in [0,1]$, we have

$$\begin{aligned}\mathcal{P}_t\mathcal{R}(v) &= \mathbf{E}\int_{\mathbb{T}^{2d}} R(x-y)u(t,x;v)u(t,y;v)dxdy\\ &\le \mathbf{E}\|u(t;v)\|_{L^\infty(\mathbb{T}^d)}\int_{\mathbb{T}^{2d}} R(x-y)u(t,x;v)dxdy = \mathbf{E}\|u(t;v)\|_{L^\infty(\mathbb{T}^d)},\end{aligned}$$

and it suffices to apply (B.9) to conclude the proof. □

Lemma 5.2. *For any $v \in D^\infty(\mathbb{T}^d)$, the function $\mathcal{P}_t\tilde{\mathcal{R}}(v)$ is continuous in $t \ge 0$.*

Proof. It is a direct consequence of Lemma B.6 and (5.5). □

The time dependent corrector field is defined as

$$\chi(t,v) := \int_0^t \mathcal{P}_s\tilde{\mathcal{R}}(v)ds, \qquad t \ge 0,\ v \in D^\infty(\mathbb{T}^d). \tag{5.7}$$

We have the following result.

Proposition 5.3. *The function $\chi : [0,+\infty) \times D^\infty(\mathbb{T}^d) \to \mathbb{R}$ satisfies*

$$\frac{d\chi(t,v)}{dt} = \mathcal{P}_t\tilde{\mathcal{R}}(v), \quad t \ge 0. \tag{5.8}$$

In addition, for any $t \ge 0$ and $v \in D^\infty(\mathbb{T}^d)$, we have

$$\lim_{\delta\to 0}\delta^{-1}[\mathcal{P}_\delta\chi(t,v) - \chi(t,v)] = \mathcal{P}_t\tilde{\mathcal{R}}(v) - \tilde{\mathcal{R}}(v). \tag{5.9}$$

Proof. First, (5.8) is a consequence of Lemma 5.2. To show (5.9), for any $\delta > 0, t \ge 0$ and $v \in D^\infty(\mathbb{T}^d)$, we have

$$\begin{aligned}\delta^{-1}[\mathcal{P}_\delta\chi(t,v) - \chi(t,v)] &= \delta^{-1}\Big[\int_0^t \mathcal{P}_{s+\delta}\tilde{\mathcal{R}}(v)ds - \int_0^t \mathcal{P}_s\tilde{\mathcal{R}}(v)ds\Big]\\ &= \delta^{-1}\Big[\int_t^{t+\delta} \mathcal{P}_s\tilde{\mathcal{R}}(v)ds - \int_0^\delta \mathcal{P}_s\tilde{\mathcal{R}}(v)ds\Big].\end{aligned}$$

Sending $\delta \to 0$ and applying Lemma 5.2 again, the proof is complete. □

By virtue of Lemma 5.1, we can define $\chi : D^\infty(\mathbb{T}^d) \to \mathbb{R}$

$$\chi(v) := \int_0^\infty \mathcal{P}_t\tilde{\mathcal{R}}(v)dt. \tag{5.10}$$

Remark 5.4. Analogously to (5.9), we can show that

$$\lim_{\delta\to 0}\delta^{-1}[\mathcal{P}_\delta\chi(v) - \chi(v)] = -\tilde{\mathcal{R}}(v), \qquad v \in D^\infty(\mathbb{T}^d). \tag{5.11}$$

The left hand side of (5.11) can be treated as a pointwise definition of the generator $\mathcal{L}$ of the semigroup $(\mathcal{P}_t)_{t\ge 0}$.

As a consequence of Lemma 5.1, the following proposition holds:

Proposition 5.5. *We have for any $v \in D^\infty(\mathbb{T}^d)$,*

$$|\chi(v)| \le C(1 + \|v\|_{L^\infty(\mathbb{T}^d)}), \tag{5.12}$$

$$\sup_{v\in D^\infty(\mathbb{T}^d)} |\chi(v) - \chi(t,v)| \le \frac{C}{\lambda}e^{-\lambda t}, \qquad t \ge 1. \tag{5.13}$$

Proof. It is a direct consequence of (5.6). □

5.2. **Frechet gradient of corrector.** Using the corrector χ defined in (5.10) we shall be able to write $\log Z_t+\gamma t$ as a sum of a continuous square integrable martingale and a term of order 1, see (5.54) - (5.56) below. This fact is crucial in our proof of the CLT. The quadratic variation of the martingale can be expressed in terms of an appropriately defined Frechet gradient of the corrector. The present section is devoted to provide a definition of such an object. Note that the latter is not completely obvious, due to the fact that χ is defined on a set $D^\infty(\mathbb{T}^d)$ that has no interior points.

For any $t\geq 0$, $z\in\mathbb{T}^d$, $v\in D^\infty(\mathbb{T}^d)$ and $h\in L^2(\mathbb{T}^d)$, we let

$$U(t,x;z,v)=\frac{\mathcal{U}(t,x;z)}{\int_{\mathbb{T}^d}\mathcal{U}(t,x';v)dx'} \tag{5.14}$$

and

$$U(t,x;h,v)=\int_{\mathbb{T}^d}U(t,x;z,v)h(z)dz=\frac{\mathcal{U}(t,x;h)}{\int_{\mathbb{T}^d}\mathcal{U}(t,x';v)dx'},\quad x\in\mathbb{T}^d. \tag{5.15}$$

We extend the definition of $\mathcal{P}_t\mathcal{R}$ to an open subset of $L^2(\mathbb{T}^d)$ containing $D^\infty(\mathbb{T}^d)$. For any $t\geq 0$ and $v\in L^2(\mathbb{T}^d)$ such that $\int_{\mathbb{T}^d}\mathcal{U}(t,x;v)dx\neq 0$, define

$$\mathscr{P}_t(v):=\int_{\mathbb{T}^{2d}}R(x-y)\frac{\mathcal{U}(t,x;v)\mathcal{U}(t,y;v)}{\left(\int_{\mathbb{T}^d}\mathcal{U}(t,x';v)dx'\right)^2}dxdy. \tag{5.16}$$

Therefore, $\mathcal{P}_t\mathcal{R}(\cdot)$ coincides with $\mathbf{E}[\mathscr{P}_t(\cdot)]$ on $D^\infty(\mathbb{T}^d)$, but the latter is defined on a much bigger set. Define

$$L^2_+(\mathbb{T}^d):=\{f\in L^2(\mathbb{T}^d): f\geq 0,\ \|f\|_{L^2(\mathbb{T}^d)}\neq 0\}. \tag{5.17}$$

For any $v\in L^2_+(\mathbb{T}^d)$, we let

$$\tilde{v}(x)=\|v\|^{-1}_{L^1(\mathbb{T}^d)}v(x). \tag{5.18}$$

We have the following lemma for the Frechet derivative of $\mathscr{P}_t(v)$.

Lemma 5.6. *For any $t\geq 0, v\in D^\infty(\mathbb{T}^d)$ and almost every realization of ξ, $\mathscr{P}_t$ is twice Frechet differentiable at v and the respective first and second order Frechet derivatives equal*

$$\begin{aligned}\mathcal{D}\mathscr{P}_t(v)(z)=&2\int_{\mathbb{T}^{2d}}R(x-y)U(t,x;z,v)u(t,y;v)dxdy\\&-2\|U(t;z,v)\|_{L^1(\mathbb{T}^d)}\int_{\mathbb{T}^{2d}}R(x-y)u(t,x;v)u(t,y;v)dxdy,\end{aligned} \tag{5.19}$$

(5.20)
$$\begin{aligned}&\mathcal{D}^2\mathscr{P}_t(v)(z_1,z_2)\\&=2\int_{\mathbb{T}^{2d}}R(x-y)U(t,x;z_1,v)U(t,y;z_2,v)dxdy\\&-4\|U(t;z_2,v)\|_{L^1(\mathbb{T}^d)}\int_{\mathbb{T}^{2d}}R(x-y)U(t,x;z_1,v)u(t,y;v)dxdy\\&-4\|U(t;z_1,v)\|_{L^1(\mathbb{T}^d)}\int_{\mathbb{T}^{2d}}R(x-y)U(t,x;z_2,v)u(t,y;v)dxdy\\&+6\|U(t;z_1,v)\|_{L^1(\mathbb{T}^d)}\|U(t;z_2,v)\|_{L^1(\mathbb{T}^d)}\int_{\mathbb{T}^{2d}}R(x-y)u(t,x;v)u(t,y;v)dxdy.\end{aligned}$$

Proof. We only consider $t>0$ here (the case of $t=0$ is easy to analyze). One can write

$$\|\mathcal{U}(t;v)\|_{L^1(\mathbb{T}^d)}=\int_{\mathbb{T}^{2d}}\mathcal{U}(t,x;y)v(y)dxdy,$$

where $\mathcal{U}(t,x;y)$ is the Green's function of SHE, which, for almost all ξ, is a continuous and positive function in $(x,y)\in\mathbb{T}^{2d}$. Fix such a realization of ξ and $v\in D^\infty(\mathbb{T}^d)$, we have $\|\mathcal{U}(t;v)\|_{L^1(\mathbb{T}^d)} > 0$, thus, there exists $\delta > 0$ (which could depend on ξ) such that $\|\mathcal{U}(t;\tilde{v})\|_{L^1(\mathbb{T}^d)} > 0$ for any $\tilde{v}$ with $\|\tilde{v} - v\|_{L^2(\mathbb{T}^d)} < \delta$. Then $\mathscr{P}_t(\cdot)$ is well-defined for all such $\tilde{v}$, and it is a straightforward calculation to check that for any $h \in L^2(\mathbb{T}^d)$, as $\delta \to 0$

$$\delta^{-1}[\mathscr{P}_t(v + \delta h) - \mathscr{P}_t(v)] \to \langle \mathcal{D}\mathscr{P}_t(v), h\rangle_{L^2(\mathbb{T}^d)}.$$

The proof of (5.20) is similar. □

Using the functional $\mathcal{R}$, see (2.16), we can rewrite (5.19) in the form

$$\begin{aligned}\langle \mathcal{D}\mathscr{P}_t(v), h\rangle_{L^2(\mathbb{T}^d)} =& 2\mathcal{R}(U(t;h,v), u(t;v)) \\ &- 2\mathcal{R}(u(t;v)) \int_{\mathbb{T}^d} U(t,x;h,v)dx,\end{aligned} \tag{5.21}$$

for any $v \in D^\infty(\mathbb{T}^d), h \in L^2(\mathbb{T}^d)$. By further restricting to $h \in L^2_+(\mathbb{T}^d)$, with $\tilde{h} = h/\|h\|_{L^1(\mathbb{T}^d)}$, we can write

$$\begin{aligned}&\langle \mathcal{D}\mathscr{P}_t(v), h\rangle_{L^2(\mathbb{T}^d)} \\ &= 2\int_{\mathbb{T}^{3d}} R(x-y)[u(t,x;\tilde{h}) - u(t,x;v)]u(t,y;v)U(t,x';h,v)dxdydx' \\ &= 2\,\mathcal{R}(u(t;\tilde{h}) - u(t;v), u(t;v))\|U(t;h,v)\|_{L^1(\mathbb{T}^d)}.\end{aligned} \tag{5.22}$$

Next, we prove several technical lemmas on the Frechet derivatives, where we will only use the following estimate:

$$|\mathcal{R}(f,g)| \le \hat{R}_* \|f\|_{L^2(\mathbb{T}^d)} \|g\|_{L^2(\mathbb{T}^d)}, \quad f, g \in L^2(\mathbb{T}^d), \tag{5.23}$$

where

$$\hat{R}_* \coloneqq \sup_{k\in\mathbb{Z}^d} \hat{r}_k \tag{5.24}$$

and $(\hat{r}_k)_{k\in\mathbb{Z}^d}$ are the Fourier coefficients of R, cf. (3.1). By our assumption of $R \ge 0$ and $\int R = 1$, we actually have $\hat{R}_* = \hat{r}_0 = 1$.

Lemma 5.7. *For any $p \in [1, +\infty)$ and $T > 0$, there exists $C > 0$, depending only on p and T, such that*

$$\begin{aligned}&\sup_{t\in[0,T]} \left(\mathbf{E}\langle|\mathcal{D}\mathscr{P}_t(v)|, h\rangle^p_{L^2(\mathbb{T}^d)}\right)^{1/p} \le C\|h\|_{L^2(\mathbb{T}^d)}\|v\|_{L^2(\mathbb{T}^d)}(1 + \|v\|_{L^2(\mathbb{T}^d)}) \\ &\le C\|h\|_{L^2(\mathbb{T}^d)}\|v\|_{L^\infty(\mathbb{T}^d)}, \quad v \in D^\infty(\mathbb{T}^d),\, 0 \le h \in L^2(\mathbb{T}^d).\end{aligned}$$

Lemma 5.8. *For any $p \in [1, +\infty)$ and $T > 0$, there exists $C > 0$, depending only on p and T, such that*

$$\sup_{t\in[0,T]} \left(\mathbf{E}\langle h_1, |\mathcal{D}^2\mathscr{P}_t(v)|h_2\rangle^p_{L^2(\mathbb{T}^d)}\right)^{1/p} \le C\|h_1\|_{L^2(\mathbb{T}^d)}\|h_2\|_{L^2(\mathbb{T}^d)}(1 + \|v\|_{L^\infty(\mathbb{T}^d)})$$

for all $v \in D^\infty(\mathbb{T}^d)$, $0 \le h_1, h_2 \in L^2(\mathbb{T}^d)$.

Proof of Lemma 5.7. From (5.19), we have

$$\begin{aligned}\langle|\mathcal{D}\mathscr{P}_t(v)|, h\rangle_{L^2(\mathbb{T}^d)} \le& C\|U(t;h,v)\|_{L^2(\mathbb{T}^d)}\|u(t;v)\|_{L^2(\mathbb{T}^d)}(1 + \|u(t;v)\|_{L^2(\mathbb{T}^d)}) \\ \le& C\|U(t;h,v)\|_{L^2(\mathbb{T}^d)}\|u(t;v)\|_{L^\infty(\mathbb{T}^d)}.\end{aligned}$$

Here we have used the fact that $\|f\|^2_{L^2(\mathbb{T}^d)} \le \|f\|_{L^\infty(\mathbb{T}^d)}$ for any $f \in D(\mathbb{T}^d)$. The proof is complete by applying Lemma B.5 and the Hölder inequality. □

Proof of Lemma 5.8. From (5.20), we have

$$\begin{aligned}&\langle h_1, |\mathcal{D}^2\mathscr{P}_t(v)|h_2\rangle_{L^2(\mathbb{T}^d)} \\ &\le C\|U(t;h_1,v)\|_{L^2(\mathbb{T}^d)}\|U(t;h_2,v)\|_{L^2(\mathbb{T}^d)}(1 + \|u(t;v)\|_{L^\infty(\mathbb{T}^d)}).\end{aligned} \tag{5.25}$$

Similarly, the proof is complete by applying Lemma B.5 and Hölder inequality. □

Now we define the gradient of the time dependent corrector: for any $v \in D^\infty(\mathbb{T}^d)$, t, $T \geq 0$, we let

$$\mathcal{D}\mathcal{P}_t\mathcal{R}(v) := \mathbf{E}[\mathcal{D}\mathscr{P}_t(v)], \qquad \mathcal{D}^2\mathcal{P}_t\mathcal{R}(v) := \mathbf{E}[\mathcal{D}^2\mathscr{P}_t(v)], \tag{5.26}$$

where $\mathcal{D}\mathscr{P}_t$ and $\mathcal{D}^2\mathscr{P}_t$ are given by (5.19) and (5.20) respectively. Let

$$\mathcal{D}\chi(T,v) := \int_0^T \mathcal{D}\mathcal{P}_t\mathcal{R}(v)dt. \tag{5.27}$$

The following two lemmas concern the continuity of the Frechet derivatives, and their proofs also only rely on the property of $\mathcal{R}$ through (5.23).

Lemma 5.9. *There exists $C = C(T) > 0$ depending on $T > 0$ such that*

$$\begin{aligned}&\sup_{t\in[0,T]} |\langle \mathcal{D}\mathcal{P}_t\mathcal{R}(v_1) - \mathcal{D}\mathcal{P}_t\mathcal{R}(v_2), h\rangle_{L^2(\mathbb{T}^d)}| \\ &\leq C\|h\|_{L^2(\mathbb{T}^d)}\|v_1 - v_2\|_{L^2(\mathbb{T}^d)}P(\|v_1\|_{L^2(\mathbb{T}^d)}, \|v_2\|_{L^2(\mathbb{T}^d)}),\end{aligned} \tag{5.28}$$

for all $v_1, v_2 \in D^\infty(\mathbb{T}^d)$ and $h \in L^2(\mathbb{T}^d)$. Here $P(\cdot,\cdot)$ is some polynomial function.

Moreover, we have also

$$\begin{aligned}&\sup_{t\in[0,T]} |\langle h_1, \big(\mathcal{D}^2\mathcal{P}_t\mathcal{R}(v_1) - \mathcal{D}^2\mathcal{P}_t\mathcal{R}(v_2)\big)h_2\rangle_{L^2(\mathbb{T}^d)}| \\ &\leq C\|h_1\|_{L^2(\mathbb{T}^d)}\|h_2\|_{L^2(\mathbb{T}^d)}\|v_1 - v_2\|_{L^2(\mathbb{T}^d)}P(\|v_1\|_{L^2(\mathbb{T}^d)}, \|v_2\|_{L^2(\mathbb{T}^d)}),\end{aligned} \tag{5.29}$$

for all $v_1, v_2 \in D^\infty(\mathbb{T}^d)$ and $h_1, h_2 \in L^2(\mathbb{T}^d)$. Here, again, $P(\cdot,\cdot)$ is some polynomial.

Proof of Lemma 5.9. We only prove (5.28), as the argument for (5.29) follows the same lines. From Lemma B.5 and the fact that $\mathcal{U}$ solves a linear equation, we know that for any $t \in [0,T]$, $h \in L^2(\mathbb{T}^d)$, $v_1, v_2 \in D^\infty(\mathbb{T}^d)$, we have

(i) $\big(\mathbf{E}[\|u(t;v_1) - u(t;v_2)\|^p_{L^2(\mathbb{T}^d)}]\big)^{1/p} \leq C\|v_1 - v_2\|_{L^2(\mathbb{T}^d)}(1 + \|v_1\|_{L^2(\mathbb{T}^d)})$,

(ii) $\big(\mathbf{E}[\|U(t;h,v_1) - U(t;h,v_2)\|^p_{L^2(\mathbb{T}^d)}]\big)^{1/p} \leq C\|h\|_{L^2(\mathbb{T}^d)}\|v_1 - v_2\|_{L^2(\mathbb{T}^d)}$.

To see (i) note that

$$\|u(t;v_1) - u(t;v_2)\|_{L^2(\mathbb{T}^d)} \leq \frac{\|\mathcal{U}(t;v_1 - v_2)\|_{L^2(\mathbb{T}^d)}}{\|\mathcal{U}(t;v_2)\|_{L^1(\mathbb{T}^d)}} + \frac{\|u(t;v_1)\|_{L^2(\mathbb{T}^d)}\|\mathcal{U}(t;v_1 - v_2)\|_{L^1(\mathbb{T}^d)}}{\|\mathcal{U}(t;v_2)\|_{L^1(\mathbb{T}^d)}},$$

The estimate follows then directly from Lemma B.5 and estimate (B.5) (recall that $Z_t = \|\mathcal{U}(t;v)\|_{L^1(\mathbb{T}^d)}$). Formula (ii) follows by similar considerations.

Using (5.19), the above estimates, (5.23) and the Hölder inequality, it is a straightforward calculation to derive that

$$\begin{aligned}&|\langle \mathbf{E}\mathcal{D}\mathscr{P}_t(v_1), h\rangle_{L^2(\mathbb{T}^d)} - \langle \mathbf{E}\mathcal{D}\mathscr{P}_t(v_2), h\rangle_{L^2(\mathbb{T}^d)}| \\ &\leq C\|h\|_{L^2(\mathbb{T}^d)}\|v_1 - v_2\|_{L^2(\mathbb{T}^d)}P(\|v_1\|_{L^2(\mathbb{T}^d)}, \|v_2\|_{L^2(\mathbb{T}^d)})\end{aligned}$$

for some polynomial function P. The proof is complete. □

5.2.1. *Estimate on $\mathcal{D}\mathcal{P}_t\mathcal{R}(v)$.* For the gradient of the corrector $\mathcal{D}\chi(T,v)$ to be well-defined with T approaching to infinity, one needs to refine the estimates on $\mathcal{D}\mathcal{P}_t\mathcal{R}(v)$ for large t. Here is the main result of this section.

Proposition 5.10. *There exists $C, \lambda > 0$ depending only on R, d such that*

$$\Big|\langle \mathcal{D}\mathcal{P}_t\mathcal{R}(v), h\rangle_{L^2(\mathbb{T}^d)}\Big| \leq C\|h\|_{L^1(\mathbb{T}^d)}\big(1_{[1,+\infty)}(t)e^{-\lambda t} + 1_{[0,1]}(t)\|v\|_{L^\infty(\mathbb{T}^d)}\big) \tag{5.30}$$

for all $t \geq 0$, $v \in D^\infty(\mathbb{T}^d)$ and $h \in L^2(\mathbb{T}^d)$.

Directly from Proposition 5.10 we conclude the following.

Corollary 5.11. *For $C, \lambda > 0$ as in Proposition 5.10 we have*

$$\begin{aligned} &\|\mathcal{D}\mathcal{P}_t\mathcal{R}(v)\|_{L^\infty(\mathbb{T}^d)} \le C\big(1_{t>1}e^{-\lambda t} + 1_{t\in[0,1]}\|v\|_{L^\infty(\mathbb{T}^d)}\big), \\ &\|\mathcal{D}\chi(t,v)\|_{L^\infty(\mathbb{T}^d)} \le C(1 + \|v\|_{L^\infty(\mathbb{T}^d)}) \end{aligned} \tag{5.31}$$

for all $t \ge 0$, $v \in D^\infty(\mathbb{T}^d)$.

The proof of Proposition 5.10 relies on the following lemmas.

Lemma 5.12. *For any $p \in [1, +\infty)$ there exist $C, \lambda > 0$ depending only on R, d, p such that*

$$\mathbf{E}\|u(t;\nu_1) - u(t;\nu_2)\|^p_{L^\infty(\mathbb{T}^d)} \le Ce^{-\lambda t}, \qquad t > 1,\ \nu_1, \nu_2 \in \mathcal{M}_1(\mathbb{T}^d).$$

Proof. By Proposition 4.6, for any $\frac{t}{2} < k < N(t) - 2$, we have

$$\begin{aligned} &\mathbf{E}\|u(t;\nu_1) - u(t;\nu_2)\|^p_{L^\infty(\mathbb{T}^d)} \\ &\le C\big(\mathbf{E}\|u(t;\nu_1) - u_k(t)\|^p_{L^\infty(\mathbb{T}^d)} + \mathbf{E}\|u(t;\nu_2) - u_k(t)\|^p_{L^\infty(\mathbb{T}^d)}\big) \le Ce^{-\lambda k} \end{aligned}$$

for some $C, \lambda > 0$ only depending on R, d, p. The proof is complete. □

We have the following estimate on U:

Lemma 5.13. *If $p \in [1, +\infty)$, then there exists C depending on R, d, p such that*

$$\mathbf{E}\Big[\|U(t;h,v)\|^p_{L^1(\mathbb{T}^d)}\Big] \le C\|h\|^p_{L^1(\mathbb{T}^d)}, \qquad v \in D(\mathbb{T}^d),\ 0 \le h \in L^1(\mathbb{T}^d),\ t \ge 0.$$

Proof. By definition, we have

$$\int_{\mathbb{T}^d} U(t,x;h,v)dx = \frac{\int_{\mathbb{T}^d}\mathcal{U}(t,x;h)dx}{\int_{\mathbb{T}^d}\mathcal{U}(t,x;v)dx}.$$

Through a time reversal of the noise $\xi(\cdot,\cdot) \mapsto \xi(t-\cdot,\cdot)$, we have, for each $t \ge 0$,

$$\frac{\int_{\mathbb{T}^d}\mathcal{U}(t,x;h)dx}{\int_{\mathbb{T}^d}\mathcal{U}(t,x;v)dx} \overset{\text{law}}{=} \frac{\int_{\mathbb{T}^d}\mathcal{U}(t,x;\mathbb{1})h(x)dx}{\int_{\mathbb{T}^d}\mathcal{U}(t,x;\mathbb{1})v(x)dx} = \frac{\int_{\mathbb{T}^d}u(t,x;\mathbb{1})h(x)dx}{\int_{\mathbb{T}^d}u(t,x;\mathbb{1})v(x)dx},$$

where $\mathbb{1}(x) \equiv 1$. Thus, it suffices to estimate the moments of the r.h.s. We have

$$\frac{\int_{\mathbb{T}^d}u(t,x;\mathbb{1})h(x)dx}{\int_{\mathbb{T}^d}u(t,x;\mathbb{1})v(x)dx} \le \|h\|_{L^1(\mathbb{T}^d)}\sup_{x\in\mathbb{T}^d}u(t,x;\mathbb{1})\big(\inf_{y\in\mathbb{T}^d}u(t,x;\mathbb{1})\big)^{-1}.$$

For $t > 1$, we use (5.4) and apply Lemma 4.1 to complete the proof. For $t \le 1$, we use the estimate

$$\begin{aligned} &\sup_{x\in\mathbb{T}^d}u(t,x;\mathbb{1})\big(\inf_{x\in\mathbb{T}^d}u(t,x;\mathbb{1})\big)^{-1} = \sup_{x\in\mathbb{T}^d}\mathcal{U}(t,x;\mathbb{1})\sup_{x\in\mathbb{T}^d}\mathcal{U}(t,x;\mathbb{1})^{-1} \\ &\le \tfrac{1}{2}\sup_{x\in\mathbb{T}^d}\mathcal{U}(t,x;\mathbb{1})^2 + \tfrac{1}{2}\sup_{x\in\mathbb{T}^d}\mathcal{U}(t,x;\mathbb{1})^{-2} \end{aligned}$$

and Lemma B.7 to complete the proof. □

Proof of Proposition 5.10. Assuming first that $h \in L^2_+(\mathbb{T}^d)$, recall from (5.22), we have

$$\langle\mathcal{D}\mathcal{P}_t\mathcal{R}(v), h\rangle_{L^2(\mathbb{T}^d)} = 2\mathbf{E}[\mathcal{R}(u(t;\tilde{h}) - u(t;v), u(t;v))\|U(t;h,v)\|_{L^1(\mathbb{T}^d)}].$$

Here, as we recall, $\tilde{h} = \|h\|^{-1}_{L^1(\mathbb{T}^d)}h$. For $t > 1$, we use the fact that

$$|\langle\mathcal{D}\mathcal{P}_t\mathcal{R}(v), h\rangle_{L^2(\mathbb{T}^d)}| \le 2\mathbf{E}[\|u(t;\tilde{h}) - u(t;v)\|_{L^\infty(\mathbb{T}^d)}\|u(t;v)\|_{L^1(\mathbb{T}^d)}\|U(t;h,v)\|_{L^1(\mathbb{T}^d)}]$$

and apply Lemmas 5.12, 5.13 and the Hölder inequality to derive that

$$|\langle\mathcal{D}\mathcal{P}_t\mathcal{R}(v), h\rangle_{L^2(\mathbb{T}^d)}| \le Ce^{-\lambda t}\|h\|_{L^1(\mathbb{T}^d)}.$$

For $t \le 1$, we use a different expression: in (5.19), we use the fact that

$$\int_{\mathbb{T}^{2d}} R(x-y)U(t,x;h,v)u(t,y;v)dxdy \le \|R\|_{L^1(\mathbb{T}^d)}\|u(t;v)\|_{L^\infty(\mathbb{T}^d)}\|U(t;h,v)\|_{L^1(\mathbb{T}^d)}$$

to derive that

$$|\langle \mathcal{D}\mathscr{P}_t(v), h\rangle_{L^2(\mathbb{T}^d)}| \le 4\|R\|_{L^1(\mathbb{T}^d)}\|u(t;v)\|_{L^\infty(\mathbb{T}^d)}\|U(t;h,v)\|_{L^1(\mathbb{T}^d)},$$

Taking the expectation, applying Lemma 5.13, estimate (B.9) and the Hölder inequality, we obtain

$$|\langle \mathcal{D}\mathcal{P}_t\mathcal{R}(v), h\rangle_{L^2(\mathbb{T}^d)}| \le C\|v\|_{L^\infty(\mathbb{T}^d)}\|h\|_{L^1(\mathbb{T}^d)}.$$

To generalize the proof to $h \in L^2(\mathbb{T}^d)$, it suffices to write $h = h_+ - h_-$ and apply the above discussion to h_+, h_- separately. The proof is complete. □

Define

$$\mathcal{D}\chi(v) = \int_0^\infty \mathcal{D}\mathcal{P}_t\mathcal{R}(v)dt, \qquad v \in D^\infty(\mathbb{T}^d). \tag{5.32}$$

As a direct consequence of (5.22) and Corollary 5.11, we have

Proposition 5.14. *For any $v \in D^\infty(\mathbb{T}^d)$, we have*

$$\langle v, \mathcal{D}\chi(v)\rangle_{L^2(\mathbb{T}^d)} = 0. \tag{5.33}$$

In addition there exists $C > 0$ such that

$$\|\mathcal{D}\chi(v)\|_{L^\infty(\mathbb{T}^d)} \le C(1 + \|v\|_{L^\infty(\mathbb{T}^d)}), \quad v \in D^\infty \tag{5.34}$$

and

$$\sup_{v\in D^\infty(\mathbb{T}^d)} \|\mathcal{D}\chi(t,v) - \mathcal{D}\chi(v)\|_{L^\infty(\mathbb{T}^d)} \le \frac{C}{\lambda}e^{-\lambda t}, \quad t > 1. \tag{5.35}$$

5.3. **A semimartingale decomposition.** The goal in this section is to obtain a semimartingale decomposition of the process $\{\mathcal{P}_s\mathcal{R}(u(t;v))\}_{t\ge 0}$, for any fixed $s \ge 0$ and $v \in D^\infty(\mathbb{T}^d)$. Recall that, for any (fixed) $v \in D^\infty(\mathbb{T}^d)$, we have

$$\mathcal{P}_s\mathcal{R}(v) = \mathbf{E}[\mathcal{R}(u(s;v))] = \mathbf{E}[\mathscr{P}_s(v)].$$

From now on, assume that, in (5.16), we use an independent copy of noise, denoted by $\tilde{\xi}$, to generate the random variable $\mathscr{P}_s(v)$. To emphasize the dependence, we will write $\mathscr{P}_s(v) = \mathscr{P}_s(v;\tilde{\xi})$, so for any $v \in D^\infty(\mathbb{T}^d)$,

$$\mathcal{P}_s\mathcal{R}(v) = \mathbf{E}_{\tilde{\xi}}[\mathscr{P}_s(v;\tilde{\xi})],$$

and

$$\mathcal{P}_s\mathcal{R}(u(t;v,\xi)) = \mathbf{E}_{\tilde{\xi}}[\mathscr{P}_s(u(t;v,\xi);\tilde{\xi})],$$

where $\mathbf{E}_{\tilde{\xi}}$ denotes the expectation only with respect to $\tilde{\xi}$, and the $u(t;v,\xi)$ in the above expression is based on the original noise ξ.

By Lemma 5.6, we know that for almost every realization of $\tilde{\xi}$, $\mathscr{P}_s(\cdot;\tilde{\xi})$ is twice Frechet differentiable at any $v \in D^\infty(\mathbb{T}^d)$. The same proof actually shows that $\mathscr{P}_s(\cdot;\tilde{\xi})$ is infinitely Frechet differentiable. The idea is to fix the realization of $\tilde{\xi}$, apply the Itô formula to $\mathscr{P}_s(u(t;v);\tilde{\xi})$, then take the expectation with respect to $\tilde{\xi}$ to obtain a semimartingale decompoisition of $\mathcal{P}_s\mathcal{R}(u(t;v))$.

5.3.1. *The case of smooth noise and initial data.* In this section, we assume that $v \in D(\mathbb{T}^d) \cap C^\infty(\mathbb{T}^d)$ (so it is automatically in $D^\infty(\mathbb{T}^d)$) and the noise ξ satisfies Assumption 3.1, i.e., it contains only finitely many Fourier modes. Since the initial data is fixed, we omit its dependence to write $u(t) = u(t; v)$. By Proposition 3.2, we know that $u(t)$ is a strong solution to the following SPDE

$$du(t) = \mathscr{A}(u(t))dt + \sum_{k\in\mathbb{Z}^d} \mathscr{B}_k(u(t))dw_k(t), \tag{5.36}$$

with the operators $\mathscr{A}, \mathscr{B}_k$ defined in (3.9).

Fix $s \geq 0$ and a realization of $\tilde{\xi}$, applying Itô formula, see e.g. [26, Theorem 4.32, p. 106], we have

(5.37)

$$\begin{aligned}\mathscr{P}_s(u(t);\tilde{\xi}) - \mathscr{P}_s(v;\tilde{\xi}) = &\int_0^t \langle \mathcal{D}\mathscr{P}_s(u(r);\tilde{\xi}), u(r)dW(r)\rangle_{L^2(\mathbb{T}^d)} \\ &+ \int_0^t \langle \mathcal{D}\mathscr{P}_s(u(r);\tilde{\xi}), \mathscr{A}(u(r))\rangle_{L^2(\mathbb{T}^d)}dr \\ &+ \tfrac{1}{2}\sum_{k\in\mathbb{Z}^d}\int_0^t \langle \mathscr{B}_k(u(r)), \mathcal{D}^2\mathscr{P}_s(u(r);\tilde{\xi})\mathscr{B}_k(u(r))\rangle_{L^2(\mathbb{T}^d)}dr.\end{aligned}$$

Here we have used the fact that

$$\langle \mathcal{D}\mathscr{P}_s(v;\tilde{\xi}), v\rangle_{L^2(\mathbb{T}^d)} = 0, \qquad \text{if } v \in D^\infty(\mathbb{T}^d),$$

which comes from (5.22).

Freeze $(u(r;\xi))_{r\in[0,t]}$ and take the expectation with respect to $\tilde{\xi}$ in (5.37). By Lemmas 5.7 and 5.8, we interchange the expectation and the integral to obtain

$$\begin{aligned}\mathcal{P}_s\mathcal{R}(u(t)) - \mathcal{P}_s\mathcal{R}(v) = &\int_0^t \langle \mathcal{D}\mathcal{P}_s\mathcal{R}(u(r)), u(r)dW(r)\rangle_{L^2(\mathbb{T}^d)} \\ &+ \int_0^t \mathcal{L}\mathcal{P}_s\mathcal{R}(u(r))dr,\end{aligned} \tag{5.38}$$

with

$$\begin{aligned}\mathcal{L}\mathcal{P}_s\mathcal{R}(v) :=&\langle \mathcal{D}\mathcal{P}_s\mathcal{R}(v), \mathscr{A}(v)\rangle_{L^2(\mathbb{T}^d)} \\ &+ \frac{1}{2}\sum_{k\in\mathbb{Z}^d}\langle \mathscr{B}_k(v), \mathcal{D}^2\mathcal{P}_s\mathcal{R}(v)\mathscr{B}_k(v)\rangle_{L^2(\mathbb{T}^d)}.\end{aligned} \tag{5.39}$$

for any $s \geq 0$ and $v \in D(\mathbb{T}^d) \cap C^\infty(\mathbb{T}^d)$. In addition $\mathcal{D}\mathcal{P}_s\mathcal{R}$ and $\mathcal{D}^2\mathcal{P}_s\mathcal{R}$ are defined in (5.26).

The following lemma holds:

Lemma 5.15. *For any $T > 0$, $v \in D(\mathbb{T}^d) \cap C^\infty(\mathbb{T}^d)$ and $p \in [1, +\infty)$, we have*

$$\sup_{s,r\in[0,T]} \mathbf{E}|\mathcal{L}\mathcal{P}_s\mathcal{R}(u(r))|^p \leq C(T,p,v). \tag{5.40}$$

For any $s \geq 0$ and v as above, we have

$$\mathcal{L}\mathcal{P}_s\mathcal{R}(v) = \lim_{\delta\to 0}\delta^{-1}[\mathcal{P}_{s+\delta}\mathcal{R}(v) - \mathcal{P}_s\mathcal{R}(v)]. \tag{5.41}$$

Proof. First, by Lemmas 5.7 and 5.8, we have

$$\begin{aligned}|\mathcal{L}\mathcal{P}_s\mathcal{R}(u(r))| \leq &Ce^{CT}\|\mathscr{A}(u(r))\|_{L^2(\mathbb{T}^d)}\|u(r)\|_{L^\infty(\mathbb{T}^d)} \\ &+ Ce^{CT}\sum_{k\in\mathbb{Z}^d}\|\mathscr{B}_k(u(r))\|^2_{L^2(\mathbb{T}^d)}(1 + \|u(r)\|_{L^\infty(\mathbb{T}^d)}).\end{aligned}$$

Then we apply Lemmas 3.3 and 5.1 to conclude (5.40). To show (5.41), fix $s \geq 0$. Define a continuous local martingale

$$\mathbf{M}_s(t) := \int_0^t \langle \mathcal{D}\mathcal{P}_s\mathcal{R}(u(r)), u(r)dW(r)\rangle_{L^2(\mathbb{T}^d)},$$

By the equality (5.26) and Lemma 5.7, its quadratic variation satisfies

$$\langle \mathbf{M}_s\rangle_t = \int_0^t \mathcal{R}\Big(\mathcal{D}\mathcal{P}_s\mathcal{R}(u(r))u(r)\Big)dr \leq C\int_0^t \|u(r)\|^4_{L^\infty(\mathbb{T}^d)}dr.$$

Next we apply Lemma 5.1 to conclude that $\mathbf{E}[\langle \mathbf{M}_s\rangle_t] \leq Ct$, so $\mathbf{M}_s$ is a bona fide, square integrable, continuous trajectory martingale. Therefore from (5.38) we have

$$\begin{aligned}\delta^{-1}[\mathcal{P}_{s+\delta}\mathcal{R}(v) - \mathcal{P}_s\mathcal{R}(v)] =& \delta^{-1}\big(\mathbf{E}[\mathcal{P}_s\mathcal{R}(u(\delta))] - \mathcal{P}_s\mathcal{R}(v)\big)\\ =& \delta^{-1}\int_0^\delta \mathbf{E}[\mathcal{L}\mathcal{P}_s\mathcal{R}(u(r))]dr.\end{aligned}$$

By (5.40), Lemma 5.9, and the fact that the sample path of $u(\cdot)$ lies in $C([0,\infty), H^k(\mathbb{T}^d))$ for any $k \geq 1$, we conclude the proof. □

Remark 5.16. Formula (5.38) yields the semimartingale decomposition of the process $(\mathcal{P}_s\mathcal{R}(u(t)))_{t\geq 0}$ for any $s \geq 0$.

The drift term in (5.38) can be simplified after a time integration:

Lemma 5.17. *For any $T, t \geq 0$,*

$$\int_0^T \Big(\int_0^t \mathcal{L}\mathcal{P}_s\mathcal{R}(u(r))dr\Big)ds = \int_0^t \mathcal{P}_T\mathcal{R}(u(r))dr - \int_0^t \mathcal{R}(u(r))dr.$$

Proof. First, by (5.40), we have

$$\int_0^T \Big(\int_0^t \mathcal{L}\mathcal{P}_s\mathcal{R}(u(r))dr\Big)ds = \int_0^t \Big(\int_0^T \mathcal{L}\mathcal{P}_s\mathcal{R}(u(r))ds\Big)dr.$$

It remains to use (5.41) to derive that for any $v \in D(\mathbb{T}^d) \cap C^\infty(\mathbb{T}^d)$,

$$\mathcal{P}_T\mathcal{R}(v) - \mathcal{R}(v) = \int_0^T \mathcal{L}\mathcal{P}_s\mathcal{R}(v)ds.$$

The proof is complete. □

Using the semimartingale decomposition in (5.38) and Lemma 5.17, we have

Proposition 5.18. *Suppose that Assumption 3.1 is in force and further assume that the initial condition $v \in D(\mathbb{T}^d) \cap C^\infty(\mathbb{T}^d)$. Then, for any $T, t \geq 0$ we have*

$$\begin{aligned}\int_0^t \mathcal{R}(u(r;v))dr =& \int_0^t \langle \mathcal{D}\chi(T, u(r;v)), u(r;v)dW(r)\rangle_{L^2(\mathbb{T}^d)}\\ &- \chi(T, u(t;v)) + \chi(T, v) + \int_0^t \mathcal{P}_T\mathcal{R}(u(r;v))dr.\end{aligned} \tag{5.42}$$

Proof. In (5.38), we integrate s from 0 to T and apply Lemma 5.17 to derive

$$\begin{aligned}\chi(T, u(t)) - \chi(T, v) =& \int_0^T \Big(\int_0^t \langle \mathcal{D}\mathcal{P}_s\mathcal{R}(u(r)), u(r)dW(r)\rangle_{L^2(\mathbb{T}^d)}\Big)ds\\ &+ \int_0^t \mathcal{P}_T\mathcal{R}(u(r))dr - \int_0^t \mathcal{R}(u(r))dr.\end{aligned}$$

For the first term in the right hand side, we use the stochastic Fubini theorem, see e.g. [26, Theorem 4.33, p. 110], and obtain

$$\int_0^T \Big(\int_0^t \langle \mathcal{D}\mathcal{P}_s\mathcal{R}(u(r)), u(r)dW(r)\rangle_{L^2(\mathbb{T}^d)}\Big)ds$$
$$= \int_0^t \langle \int_0^T \mathcal{D}\mathcal{P}_s\mathcal{R}(u(r))ds, u(r)dW(r)\rangle_{L^2(\mathbb{T}^d)} = \int_0^t \langle \mathcal{D}\chi(T, u(r)), u(r)dW(r)\rangle_{L^2(\mathbb{T}^d)},$$

where $\mathcal{D}\chi$ was defined in (5.27). The proof is complete. □

5.3.2. *The general case.* The following is the main result of this section, which removes the smoothness assumptions on the noise and the initial data in Proposition 5.18.

Proposition 5.19. *For any $T, t \geq 0$ and $v \in D^\infty(\mathbb{T}^d)$, the decomposition (5.42) holds.*

A direct consequence of the above proposition is the decomposition of the additive functional appearing in the formula for $\log Z_t$, see (2.21), into a stochastic integral with respect to $dW(t)$ and boundary terms.

Corollary 5.20. *For any $t \geq 0$ and $v \in D^\infty(\mathbb{T}^d)$, we have*

$$\int_0^t \tilde{\mathcal{R}}(u(r;v))dr = \int_0^t \langle \mathcal{D}\chi(u(r;v)), u(r;v)dW(r)\rangle_{L^2(\mathbb{T}^d)} - \chi(u(t;v)) + \chi(v). \tag{5.43}$$

Proof. First, in both sides of (5.42), we subtract $2\gamma t$ to obtain

$$\int_0^t \tilde{\mathcal{R}}(u(r;v))dr = \int_0^t \langle \mathcal{D}\chi(T, u(r;v)), u(r;v)dW(r)\rangle_{L^2(\mathbb{T}^d)} - \chi(T, u(t;v)) + \chi(T, v) + \int_0^t \mathcal{P}_T\tilde{\mathcal{R}}(u(r;v))dr. \tag{5.44}$$

Sending $T \to \infty$ in the above equation, and applying Lemma 5.1 together with estimates (5.13) and (5.35), we conclude the proof. □

The rest of the section is devoted to proving Proposition 5.19. Fix $v \in D^\infty(\mathbb{T}^d)$ from now on. Let us introduce some notations. Recall that $(\hat{r}_k)_{k\in\mathbb{Z}^d}$ are the Fourier coefficients of $R(\cdot)$, see (3.1). For any $N > 0$, define

$$R_N(x) = \sum_{|k|\leq N} \hat{r}_k e^{i2\pi k\cdot x}. \tag{5.45}$$

Let ξ_N be a finite dimensional noise that is white in time and smooth in space with the spatial covariance function R_N, and

$$dW_N(t,x) = \xi_N(t,x)dt = \sum_{|k|\leq N} \sqrt{\hat{r}_k} e^{i2\pi k\cdot x} dw_k(t).$$

Given $v \in D^\infty(\mathbb{T}^d)$, we choose and fix a sequence of $v_N \in D(\mathbb{T}^d) \cap C^\infty(\mathbb{T}^d)$ such that

$$\|v_N - v\|_{L^\infty(\mathbb{T}^d)} \to 0, \qquad N \to \infty. \tag{5.46}$$

Let $\mathcal{U}_N(t;v)$ be the solution to SHE with noise ξ_N and initial data v. Let furthermore

$$u_N(t,x;v) = \frac{\mathcal{U}_N(t,x;v)}{\int_{\mathbb{T}^d} \mathcal{U}_N(t,x';v)dx'},$$
$$U_N(t,x;h,v) = \frac{\mathcal{U}_N(t,x;h)}{\int_{\mathbb{T}^d} \mathcal{U}_N(t,x';v)dx'}.$$

Define

$$\mathcal{R}_N(v_1,v_2) = \int_{\mathbb{T}^{2d}} R_N(x-y)v_1(x)v_2(y)dxdy, \qquad v_1,v_2 \in L^2(\mathbb{T}^d),$$
$$\mathcal{R}_N(v) = \mathcal{R}_N(v,v), \qquad v \in L^2(\mathbb{T}^d),$$

and denote $\mathcal{P}_t^N$ the semigroup associated with u_N, so

$$\mathcal{P}_t^N \mathcal{R}_N(v) = \mathbf{E}[\mathcal{R}_N(u_N(t;v))].$$

Define

$$\chi_N(T,v) = \int_0^T \mathcal{P}_t^N \mathcal{R}_N(v)dt - 2\gamma T,$$
$$\mathscr{P}_t^N(v) := \int_{\mathbb{T}^{2d}} R_N(x-y)\frac{\mathcal{U}_N(t,x;v)\mathcal{U}_N(t,y;v)}{\left(\int_{\mathbb{T}^d}\mathcal{U}_N(t,x';v)dx'\right)^2}dxdy,$$

and

$$\mathcal{D}\chi_N(T,v) = \int_0^T \mathbf{E}[\mathcal{D}\mathscr{P}_t^N(v)]dt.$$

Then by Proposition 5.18, we have

$$\begin{aligned}\int_0^t \mathcal{R}_N(u_N(r;v_N))dr &= \int_0^t \langle \mathcal{D}\chi_N(T,u_N(r;v_N)), u_N(r;v_N)dW_N(r)\rangle_{L^2(\mathbb{T}^d)} \\ &\quad - \chi_N(T,u_N(t;v_N)) + \chi_N(T,v_N) + \int_0^t \mathcal{P}_T^N\mathcal{R}_N(u_N(r;v_N))dr.\end{aligned} \tag{5.47}$$

The goal is to pass to the limit in (5.47) as $N\to\infty$. We first state the following elementary lemmas.

Lemma 5.21. *For any $T\ge 0$, $p\in[1,+\infty)$, $h\in L^2(\mathbb{T}^d)$, the following convergences take place*

$$\sup_{t\in[0,T]} \mathbf{E}\|u_N(t;v)-u(t;v)\|^p_{L^2(\mathbb{T}^d)} \to 0,$$
$$\sup_{t\in[0,T]} \mathbf{E}\|u_N(t;v_N)-u(t;v)\|^p_{L^2(\mathbb{T}^d)} \to 0,$$
$$\sup_{t\in[0,T]} \mathbf{E}\|U_N(t;h,v)-U(t;h,v)\|^p_{L^2(\mathbb{T}^d)} \to 0, \quad \text{as } N\to\infty.$$

Furthermore, for any $t>0$, as $N\to\infty$,

$$\mathbf{E}\Big(\int_{\mathbb{T}^{2d}}[\mathcal{U}_N(t,x;y)-\mathcal{U}(t,x;y)]^2dxdy\Big)^p \to 0. \tag{5.48}$$

The proof is standard and we present it in Section B.3 of the appendix.

Lemma 5.22. *(i) For any $v_1,v_2\in L^2(\mathbb{T}^d)$, we have*

$$|\mathcal{R}_N(v_1,v_2)-\mathcal{R}(v_1,v_2)| \le \|v_1\|_{L^2(\mathbb{T}^d)}\|v_2\|_{L^2(\mathbb{T}^d)},$$
$$\mathcal{R}_N(v_1,v_2)\to\mathcal{R}(v_1,v_2), \qquad N\to\infty.$$

(ii) For any $v_j\in L^2(\mathbb{T}^d), j=1,\ldots,4$, we have

$$|\mathcal{R}_N(v_1,v_2)-\mathcal{R}_N(v_3,v_4)| \le \|v_1-v_3\|_{L^2(\mathbb{T}^d)}\|v_2\|_{L^2(\mathbb{T}^d)} + \|v_2-v_4\|_{L^2(\mathbb{T}^d)}\|v_3\|_{L^2(\mathbb{T}^d)}.$$

Proof. It suffices to write the integrals in the Fourier domain

$$\mathcal{R}_N(v_1,v_2) = \sum_{|k|\le N}\hat r_k \hat v_1(k)\hat v_2^*(k), \qquad \mathcal{R}(v_1,v_2) = \sum_{k\in\mathbb{Z}^d}\hat r_k \hat v_1(k)\hat v_2^*(k),$$

and use the fact that $0\le \hat r_k\le \hat R_* = 1$ to conclude the proof of (i). Similarly, for (ii), by virtue of the Cauchy-Schwarz inequality, we have

$$\begin{aligned}|\mathcal{R}_N(v_1,v_2)-\mathcal{R}_N(v_3,v_4)| &\le \sum_{k\in\mathbb{Z}^d}\left|\hat v_1(k)\hat v_2^*(k)-\hat v_3(k)\hat v_4^*(k)\right| \\ &\le \|v_1-v_3\|_{L^2(\mathbb{T}^d)}\|v_2\|_{L^2(\mathbb{T}^d)} + \|v_2-v_4\|_{L^2(\mathbb{T}^d)}\|v_3\|_{L^2(\mathbb{T}^d)}.\end{aligned}$$

The proof is complete. □

With the above two lemmas, we have

Proposition 5.23. *For any $t, T \geq 0$, we have*

$$
\begin{aligned}
&\chi_N(T, v_N) \to \chi(T, v),\\
&\chi_N(T, u_N(t; v_N)) \to \chi(T, u(t; v)),\\
&\int_0^t \mathcal{R}_N(u_N(r; v_N))dr \to \int_0^t \mathcal{R}(u(r; v))dr,\\
&\int_0^t \mathcal{P}_T^N \mathcal{R}_N(u_N(r; v_N))dr \to \int_0^t \mathcal{P}_T \mathcal{R}(u(r; v))dr, \quad \textit{as } N \to \infty.
\end{aligned} \tag{5.49}
$$

The convergence holds in $L^p(\Omega)$ for any $p \in [1, +\infty)$.

Proof. The proofs for all cases are similar, so we take the last one as an example. One can write

$$\int_0^t \mathcal{P}_T^N \mathcal{R}_N(u_N(r; v_N))dr = \int_0^t \mathbf{E}[\mathcal{R}_N(u_N(r+T; v_N))|\mathcal{F}_r]dr,$$

and

$$\int_0^t \mathcal{P}_T \mathcal{R}(u(r; v))dr = \int_0^t \mathbf{E}[\mathcal{R}(u(r+T; v))|\mathcal{F}_r]dr.$$

So the difference can be decomposed into two terms:

$$
\begin{aligned}
I_1 &= \int_0^t \mathbf{E}[\mathcal{R}_N(u_N(r+T; v_N))|\mathcal{F}_r]dr - \int_0^t \mathbf{E}[\mathcal{R}_N(u(r+T; v))|\mathcal{F}_r]dr,\\
I_2 &= \int_0^t \mathbf{E}[\mathcal{R}_N(u(r+T; v))|\mathcal{F}_r]dr - \int_0^t \mathbf{E}[\mathcal{R}(u(r+T; v))|\mathcal{F}_r]dr.
\end{aligned}
$$

We apply Lemmas 5.21 and 5.22 to conclude that as $N \to \infty$, $I_1, I_2 \to 0$ in $L^p(\Omega)$ for any $p \in [1, +\infty)$. □

To deal with the martingale term in (5.47), we need the following lemmas:

Lemma 5.24. *For any $f \in D^\infty(\mathbb{T}^d)$ and $t > 0$, we have as $N \to \infty$,*

$$\mathcal{D}\mathcal{P}_t^N \mathcal{R}_N(f) \to \mathcal{D}\mathcal{P}_t \mathcal{R}(f) \quad \textit{in} \quad L^2(\mathbb{T}^d).$$

Proof. To simplify the notation we shall assume that $\hat{R}_* = 1$. Fix $t > 0$. Recall from (5.19) that

$$
\begin{aligned}
\mathcal{D}\mathscr{P}_t^N(f)(z) =& 2\int_{\mathbb{T}^{2d}} R_N(x-y) U_N(t, x; z, f) u_N(t, y; f) dxdy\\
&- 2\|U_N(t; z, f)\|_{L^1(\mathbb{T}^d)} \int_{\mathbb{T}^{2d}} R_N(x-y) u_N(t, x; f) u_N(t, y; f) dxdy
\end{aligned}
$$

and $\mathcal{D}\mathcal{P}_t^N \mathcal{R}_N(f) = \mathbf{E}\mathcal{D}\mathscr{P}_t^N(f)$.

To prove the result, it suffices to show as $N \to \infty$,

$$\mathbf{E}\int_{\mathbb{T}^d} |\mathcal{D}\mathscr{P}_t^N(f)(z) - \mathcal{D}\mathscr{P}_t(f)(z)|^2 dz \to 0. \tag{5.50}$$

One can write the error as $\mathcal{D}\mathscr{P}_t^N(f)(z) - \mathcal{D}\mathscr{P}_t(f)(z) = 2(I_1^{(N)}(z) + I_2^{(N)}(z))$ with

$$I_1^{(N)}(z) = \int_{\mathbb{T}^{2d}} R_N(x-y) U_N(t, x; z, f) u_N(t, y; f) dxdy - \int_{\mathbb{T}^{2d}} R(x-y) U(t, x; z, f) u(t, y; f) dxdy$$

and

$$
\begin{aligned}
I_2^{(N)}(z) =& \|U(t; z, f)\|_{L^1(\mathbb{T}^d)} \int_{\mathbb{T}^{2d}} R(x-y) u(t, x; f) u(t, y; f) dxdy\\
&- \|U_N(t; z, f)\|_{L^1(\mathbb{T}^d)} \int_{\mathbb{T}^{2d}} R_N(x-y) u_N(t, x; f) u_N(t, y; f) dxdy.
\end{aligned}
$$

By Lemma 5.22, we have

$$\begin{aligned}|I_1^{(N)}(z)| \le & \|U_N(t;z,f)-U(t;z,f)\|_{L^2(\mathbb{T}^d)}\|u_N(t;f)\|_{L^2(\mathbb{T}^d)}\\ & + \|U(t;z,f)\|_{L^2(\mathbb{T}^d)}\|u_N(t;f)-u(t;f)\|_{L^2(\mathbb{T}^d)}\\ & + \Big|\int_{\mathbb{T}^{2d}}(R_N(x-y)-R(x-y))U(t,x;z,f)u(t,y;f)dxdy\Big|\\ & =: J_1^{(N)}(z)+J_2^{(N)}(z)+J_3^{(N)}(z)\end{aligned}$$

To deal with J_1, we first note that

$$\begin{aligned}&|U_N(t,x;z,f)-U(t,x;z,f)| \le |\mathcal{U}_N(t,x;z)-\mathcal{U}(t,x;z)|\cdot\|\mathcal{U}_N(t;f)\|^{-1}_{L^1(\mathbb{T}^d)}\\ &+U(t,x;z,f)\|\mathcal{U}_N(t;f)\|^{-1}_{L^1(\mathbb{T}^d)}\|\mathcal{U}_N(t;f)-\mathcal{U}(t;f)\|_{L^1(\mathbb{T}^d)}.\end{aligned}$$

Using Lemma 5.21, (B.2) and Lemma B.2, together with the Hölder inequality, we conclude that for any $p\in[1,+\infty)$

$$\lim_{N\to+\infty}\mathbf{E}\Big(\int_{\mathbb{T}^{2d}}|U_N(t,x;z,f)-U(t,x;z,f)|^2dxdz\Big)^p=0.$$

Invoking Lemma 5.21 and the Hölder inequality we conclude that

$$\mathbf{E}\|J_1^{(N)}\|^2_{L^2(\mathbb{T}^d)}=\mathbf{E}\Big[\|u_N(t;f)\|^2_{L^2(\mathbb{T}^d)}\int_{\mathbb{T}^{2d}}|U_N(t,x;z,f)-U(t,x;z,f)|^2dxdz\Big]\to 0.$$

Similarly, we have

$$\mathbf{E}\|J_2^{(N)}\|^2_{L^2(\mathbb{T}^d)}=\mathbf{E}\Big[\|u_N(t;f)-u(t;f)\|^2_{L^2(\mathbb{T}^d)}\int_{\mathbb{T}^{2d}}U(t,x;z,f)^2dxdz\Big]\to 0.$$

Concerning $J_3^{(N)}$, denote by $\hat{U}_k(t;z,f),\hat{u}_k(t;f)$ the Fourier coefficients of $U(t;z,f)$ and $u(t;f)$, respectively. We have

$$\mathbf{E}\|J_3^{(N)}\|^2_{L^2(\mathbb{T}^d)}\le\mathbf{E}\Bigg[\sum_{|k|>N}\int_{\mathbb{T}^d}|\hat{U}_k(t;z,f)|^2dz\sum_{|k|>N}|\hat{u}_k(t;f)|^2\Bigg]\to 0,$$

by virtue of the monotone convergence theorem. This shows that $\mathbf{E}\|I_1^{(N)}\|^2_{L^2(\mathbb{T}^d)}\to 0$. The proof that $\mathbf{E}\|I_2^{(N)}\|^2_{L^2(\mathbb{T}^d)}\to 0$ can be carried out along similar lines. □

Lemma 5.25. *For any $T,t\ge 0$, $p\in[1,+\infty)$, we have*

$$\sup_{r\in[0,t],N>1}\mathbf{E}\big(\|\mathcal{D}\chi_N(T,u_N(r;v_N))\|^p_{L^2(\mathbb{T}^d)}+\|\mathcal{D}\chi(T,u(r;v))\|^p_{L^2(\mathbb{T}^d)}\big)\le C. \tag{5.51}$$

In addition, for any $r\in[0,t]$,

$$\mathbf{E}\|\mathcal{D}\chi_N(T,u_N(r;v_N))-\mathcal{D}\chi(T,u(r;v))\|^p_{L^2(\mathbb{T}^d)}\to 0,\ \textit{as } N\to\infty. \tag{5.52}$$

Proof. To simplify the notations, we let $f_N=u_N(r;v_N)$ and $f=u(r;v)$. Concerning (5.51), note that by Lemma 5.7 for a fixed $T>0$ we can find $C>0$ such that

$$\|\mathcal{D}\chi_N(T,f_N)\|_{L^2(\mathbb{T}^d)}+\|\mathcal{D}\chi(T,f)\|_{L^2(\mathbb{T}^d)}\le C\Big(\|f_N\|_{L^\infty(\mathbb{T}^d)}+\|f\|_{L^\infty(\mathbb{T}^d)}\Big).$$

Applying Lemma 5.1, we conclude estimate (5.51). For (5.52), we write

$$\begin{aligned}&\mathcal{D}\chi_N(T,f_N)-\mathcal{D}\chi(T,f)\\ &=\int_0^T\big[\mathcal{D}\mathcal{P}_t^N\mathcal{R}_N(f_N)-\mathcal{D}\mathcal{P}_t\mathcal{R}(f)\big]dt\\ &=\int_0^T\big[\mathcal{D}\mathcal{P}_t^N\mathcal{R}_N(f_N)-\mathcal{D}\mathcal{P}_t^N\mathcal{R}_N(f)\big]dt+\int_0^T\big[\mathcal{D}\mathcal{P}_t^N\mathcal{R}_N(f)-\mathcal{D}\mathcal{P}_t\mathcal{R}(f)\big]dt.\end{aligned}$$

For the first term, by Lemma 5.9, we have

$$\|\mathcal{D}\mathcal{P}_t^N\mathcal{R}_N(f_N)-\mathcal{D}\mathcal{P}_t^N\mathcal{R}_N(f)\|_{L^2(\mathbb{T}^d)}\le C\|f_N-f\|_{L^2(\mathbb{T}^d)}P(\|f_N\|_{L^2(\mathbb{T}^d)},\|f\|_{L^2(\mathbb{T}^d)})$$

where P is some given polynomial function. Then we apply Lemmas 5.1 and 5.21 to derive that

$$\mathbf{E}\int_0^T \|\mathcal{D}\mathcal{P}_t^N\mathcal{R}_N(f_N)-\mathcal{D}\mathcal{P}_t^N\mathcal{R}_N(f)\|_{L^2(\mathbb{T}^d)}^p dt \to 0, \text{ as } N\to\infty.$$

For the second term, we first note that, by Lemma 5.7, we have

$$\|\mathcal{D}\mathcal{P}_t^N\mathcal{R}_N(f)-\mathcal{D}\mathcal{P}_t\mathcal{R}(f)\|_{L^2(\mathbb{T}^d)}\le C\|f\|_{L^\infty(\mathbb{T}^d)}.$$

Then we apply Lemmas 5.1 and 5.24 to derive that

$$\mathbf{E}\int_0^T \|\mathcal{D}\mathcal{P}_t^N\mathcal{R}_N(f)-\mathcal{D}\mathcal{P}_t\mathcal{R}(f)\|_{L^2(\mathbb{T}^d)}^p dt \to 0, \text{ as } N\to\infty.$$

The proof is complete. □

Finally, we can prove the convergence of the martingale term in (5.47):

Proposition 5.26. *For any $t,T\ge 0$, we have*

$$\begin{aligned}&\int_0^t \langle \mathcal{D}\chi_N(T,u_N(r;v_N)),u_N(r;v_N)dW_N(r)\rangle_{L^2(\mathbb{T}^d)}\\ &\quad\to \int_0^t\langle\mathcal{D}\chi(T,u(r;v)),u(r;v)dW(r)\rangle_{L^2(\mathbb{T}^d)},\ \textit{as } N\to\infty,\ \textit{in } L^2(\Omega).\end{aligned}$$

Proof. To simplify the notation, we denote

$$\begin{aligned}f_N(r,x)&=\mathcal{D}\chi_N(T,u_N(r;v_N))(x)u_N(r,x;v_N),\\ f(r,x)&=\mathcal{D}\chi(T,u(r;v))(x)u(r,x;v),\end{aligned}$$

then the goal is to show $\int_0^t\int_{\mathbb{T}^d} f_N(r,x)dW_N(r,x)\to\int_0^t\int_{\mathbb{T}^d} f(r,x)dW(r,x)$ in $L^2(\Omega)$. The error can be written as a sum of two terms:

$$\begin{aligned}I_{1,N}&=\int_0^t\int_{\mathbb{T}^d}[f_N(r,x)-f(r,x)]dW_N(t,x),\\ I_{2,N}&=\int_0^t\int_{\mathbb{T}^d} f(r,x)dW_N(t,x)-\int_0^t\int_{\mathbb{T}^d} f(r,x)dW(t,x).\end{aligned}$$

For the first term, we have

$$\begin{aligned}\mathbf{E}[I_{1,N}^2]=&\mathbf{E}\int_0^t\int_{\mathbb{T}^d}[f_N(r,x)-f(r,x)][f_N(r,y)-f(r,y)]R_N(x-y)dxdy\\ \le&\int_0^t\mathbf{E}\|f_N(r)-f(r)\|_{L^2(\mathbb{T}^d)}^2dr.\end{aligned}$$

We bound the L^2 norm by

$$\begin{aligned}\|f_N(r)-f(r)\|_{L^2(\mathbb{T}^d)}\le&\|\mathcal{D}\chi_N(T,u_N(r;v_N))-\mathcal{D}\chi(T,u(r;v))\|_{L^2(\mathbb{T}^d)}\|u_N(r;v_N)\|_{L^\infty(\mathbb{T}^d)}\\ &+\|\mathcal{D}\chi(T,u(r;v))\|_{L^\infty(\mathbb{T}^d)}\|u_N(r;v_N)-u(r;v)\|_{L^2(\mathbb{T}^d)}\end{aligned}$$

Then it suffices to apply Lemmas 5.1, 5.21, 5.25 and (5.31) to conclude that $\mathbf{E}[I_{1,N}^2]\to 0$.

For the second term, with $\hat f_k(r)=\int_{\mathbb{T}^d} f(r,x)e^{-i2\pi k\cdot x}dx$, we have

$$\mathbf{E}[I_{2,N}^2]=\sum_{|k|>N}\hat r_k\int_0^t\mathbf{E}|\hat f_k(r)|^2dr\le\sum_{|k|>N}\int_0^t\mathbf{E}|\hat f_k(r)|^2dr,$$

which goes to zero as $N\to\infty$, by invoking the fact that

$$\begin{aligned}\mathbf{E}\|f(r)\|_{L^2(\mathbb{T}^d)}^p\le&\mathbf{E}\Big[\|\mathcal{D}\chi(T,u(r;v))\|_{L^2(\mathbb{T}^d)}^p\|u(r;v)\|_{L^\infty}^p\Big]\\ \le& C\mathbf{E}\Big[(1+\|u(r;v)\|_{L^\infty(\mathbb{T}^d)})^p\|u(r;v)\|_{L^\infty}^p\Big]\le C,\end{aligned}$$

where we have used (5.31) in the second inequality and (5.5) in the third one. Here C is a generic constant, independent of N. □

Now we can combine Propositions 5.23 and 5.26 to conclude Proposition 5.19.

5.4. **Martingale CLT.** Define

$$\mathfrak{z}_t := \log Z_t + \gamma t = \int_0^t \int_{\mathbb{T}^d} u(s,y;\nu)\xi(s,y)dyds - \tfrac{1}{2}\int_0^t \tilde{\mathcal{R}}(u(s;\nu))ds, \tag{5.53}$$

where $\nu \in \mathcal{M}_1(\mathbb{T}^d)$. Recall that the goal is to show

$$\frac{\mathfrak{z}_t}{\sqrt{t}} \Rightarrow N(0,\sigma^2), \qquad \text{as } t \to \infty.$$

It is clear that as $t \to \infty$, the random variables $\mathfrak{z}_2/\sqrt{t}$ tend to 0 in probability, thus, it suffices to consider

$$\mathfrak{z}_t - \mathfrak{z}_2 = \int_2^t \int_{\mathbb{T}^d} u(s,y;\nu)\xi(s,y)dyds - \tfrac{1}{2}\int_2^t \tilde{\mathcal{R}}(u(s;\nu))ds.$$

Since $u(2;\nu)$ is a continuous function almost surely, by the Markov property, we can assume that in (5.53), we start from $\nu(dx) = v(x)dx$ with some fixed $v \in D^\infty(\mathbb{T}^d)$.

By (5.43), we write

$$\mathfrak{z}_t = \mathcal{N}_t + \mathfrak{r}_t, \tag{5.54}$$

with the martingale term

$$\mathcal{N}_t = \int_0^t \big\langle u(r;v)(1 - \tfrac{1}{2}\mathcal{D}\chi(u(r;v))), dW(r)\big\rangle_{L^2(\mathbb{T}^d)}, \tag{5.55}$$

and the remainder term

$$\mathfrak{r}_t = \tfrac{1}{2}\chi(u(t;v)) - \tfrac{1}{2}\chi(v). \tag{5.56}$$

The quadratic variation of $\mathcal{N}_t$ is

$$\langle \mathcal{N}\rangle_t = \int_0^t \mathcal{R}\big(u(r;v)(1 - \tfrac{1}{2}\mathcal{D}\chi(u(r;v)))\big)dr,$$

which satisfies

$$\langle \mathcal{N}\rangle_t \le \int_0^t \|u(r;v)\|^2_{L^\infty(\mathbb{T}^d)}(1 + \tfrac{1}{2}\|\mathcal{D}\chi(u(r;v))\|_{L^\infty(\mathbb{T}^d)})^2 dr.$$

Applying Lemma 5.1 and (5.34), we conclude that for any $p \in [1,+\infty)$,

$$\mathbb{E}[\langle \mathcal{N}\rangle_t^p] \le Ct^p. \tag{5.57}$$

Define

$$\sigma^2 = \int_{\mathcal{M}_1(\mathbb{T}^d)} \mathcal{R}\big(u(1 - \tfrac{1}{2}\mathcal{D}\chi(u))\big)\pi_\infty(du). \tag{5.58}$$

The following proposition shows the convergence of the (rescaled) quadratic variation:

Proposition 5.27. *As $t \to \infty$,*

$$\mathbf{E}[|t^{-1}\langle \mathcal{N}\rangle_t - \sigma^2|^2] \to 0.$$

Proof. For any $T > 1$, define

$$\langle \mathcal{N}^T\rangle_t = \int_0^t \mathcal{R}\big(u(r;v)(1 - \tfrac{1}{2}\mathcal{D}\chi(T, u(r;v)))\big)dr,$$

and

$$\sigma_T^2 = \int_{\mathcal{M}_1(\mathbb{T}^d)} \mathcal{R}\big(u(1 - \tfrac{1}{2}\mathcal{D}\chi(T,u))\big)\pi_\infty(du).$$

By Lemma 5.22, we have

$$|\langle \mathcal{N}^T\rangle_t - \langle \mathcal{N}\rangle_t| \le \int_0^t \|u(r;v)\|^2_{L^\infty(\mathbb{T}^d)} \|\mathcal{D}\chi(T,u(r;v)) - \mathcal{D}\chi(u(r;v))\|_{L^\infty(\mathbb{T}^d)} \\ \times (1 + \|\mathcal{D}\chi(T,u(r;v))\|_{L^\infty(\mathbb{T}^d)} + \|\mathcal{D}\chi(u(r;v))\|_{L^\infty(\mathbb{T}^d)})dr.$$

Further applying (5.34) and (5.35), we derive

$$|\langle \mathcal{N}^T\rangle_t - \langle \mathcal{N}\rangle_t| \le Ce^{-\lambda T}\int_0^t \|u(r;v)\|^2_{L^\infty(\mathbb{T}^d)}(1 + \|u(r;v)\|_{L^\infty(\mathbb{T}^d)})dr.$$

Similarly, we have

$$|\sigma_T^2 - \sigma^2| \le Ce^{-\lambda T}\int_{\mathcal{M}_1(\mathbb{T}^d)} \|u\|^2_{L^\infty}(1 + \|u\|_{L^\infty(\mathbb{T}^d)})\pi_\infty(du).$$

Thus, applying Lemma 5.1, we derive

$$\sup_{t>0} \mathbf{E}[t^{-2}|\langle \mathcal{N}^T\rangle_t - \langle \mathcal{N}\rangle_t|^2] + |\sigma_T^2 - \sigma^2| \le Ce^{-\lambda T},$$

so we only need to show that for each T, as $t \to \infty$,

$$\mathbf{E}[|t^{-1}\langle \mathcal{N}^T\rangle_t - \sigma_T^2|^2] \to 0. \tag{5.59}$$

Fix $T > 0$. By Lemmas 5.9, 5.22 and Corollary 5.11, the functional $\mathsf{F} : D^\infty(\mathbb{T}^d) \to \mathbb{R}$

$$\mathsf{F}(u) := \mathcal{R}\big(u(1 - \tfrac{1}{2}\mathcal{D}\chi(T,u))\big)$$

satisfies

$$|\mathsf{F}(u_1) - \mathsf{F}(u_2)| \le \|u_1 - u_2\|_{L^\infty(\mathbb{T}^d)} P(\|u_1\|_{L^\infty(\mathbb{T}^d)}, \|u_2\|_{L^\infty(\mathbb{T}^d)}),$$

where P is some polynomial function. In other words, F is locally Lipschitz. For any $M > 0$, let $\mathsf{F}_M : D^\infty(\mathbb{T}^d) \to \mathbb{R}$ be a global Lipschitz function that is a cutoff of F in the sense that $\mathsf{F}_M(u) = \mathsf{F}(u)$ if $\|u\|_{L^\infty(\mathbb{T}^d)} \le M$, and $|\mathsf{F}_M| \le |\mathsf{F}|$. Applying Proposition 4.8, we have

$$\mathbf{E}[|t^{-1}\int_0^t \mathsf{F}_M(u(r;v))dr - \sigma_{T,M}^2|^2] \to 0,$$

where $\sigma_{T,M}^2 = \int_{\mathcal{M}_1(\mathbb{T}^d)} \mathsf{F}_M(u)\pi_\infty(du)$. For the error induced by M, we have

$$t^{-1}\int_0^t \mathbf{E}|\mathsf{F}_M(u(r;v)) - \mathsf{F}(u(r;v))|^2 dr \\ \le 4t^{-1}\int_0^t \mathbf{E}\big[|\mathsf{F}(u(r;v))|^2 1_{\|u(r;v)\|_{L^\infty(\mathbb{T}^d)}>M}\big]dr \le \frac{C}{\sqrt{M}},$$

where for the last estimate we have applied the Hölder inequality and Lemma 5.1. Similarly, we can show $\sigma_{T,M}^2 \to \sigma_T^2$ as $M \to \infty$. The proof is complete. ☐

The following two lemmas combine to complete the proof of Theorem 2.5.

Lemma 5.28. *As $t \to \infty$, $\frac{1}{\sqrt{t}}\mathfrak{r}_t \to 0$ in probability.*

Lemma 5.29. *As $t \to \infty$, $\frac{1}{\sqrt{t}}\mathcal{N}_t \Rightarrow N(0,\sigma^2)$ in distribution.*

Proof of Lemma 5.28. By (5.12), we have

$$|\mathfrak{r}_t| \le C(1 + \|v\|_{L^\infty(\mathbb{T}^d)} + \|u(t;v)\|_{L^\infty(\mathbb{T}^d)}).$$

Taking expectation and applying Lemma 5.1, we derive that $\mathbf{E}|\mathfrak{r}_t| \le C(1+\|v\|_{L^\infty(\mathbb{T}^d)})$. The proof is complete. ☐

Proof of Lemma 5.29. By the same proof for (5.57), we have for any $t \ge s$

$$\mathbf{E}(\langle \mathcal{N}\rangle_t - \langle \mathcal{N}\rangle_s)^p \le C(t-s)^p.$$

Together with Proposition 5.27, we conclude that the process $(\varepsilon^2\langle\mathcal{N}\rangle_{t/\varepsilon^2})_{t\ge 0}$ converges in $C[0,\infty)$ to $(\sigma^2 t)_{t\ge 0}$, as $\varepsilon\to 0$. Then by the martingale central limit theorem, we derive that

$$\varepsilon\mathcal{N}_{t/\varepsilon^2}\Rightarrow \sigma B_t$$

in $C[0,\infty)$ in distribution, where B is a standard Brownian motion. The proof is complete. □

The following lemma guarantees the non-degeneracy of γ,σ.

Lemma 5.30. *We have $\gamma,\sigma\in(0,+\infty)$.*

Proof. Recall that

$$\gamma=\tfrac12\int_{\mathcal{M}_1(\mathbb{T}^d)}\mathcal{R}(u)\pi_\infty(du),\qquad \sigma^2=\int_{\mathcal{M}_1(\mathbb{T}^d)}\mathcal{R}\big(u(1-\tfrac12\mathcal{D}\chi(u))\big)\pi_\infty(du).$$

By (5.34), Lemma 5.1, Theorem 2.3 and the fact that

$$\mathcal{R}(f)\le\|f\|^2_{L^2(\mathbb{T}^d)}\le\|f\|_{L^\infty(\mathbb{T}^d)}\qquad\text{for any } f\in D^\infty(\mathbb{T}^d),$$

we conclude that $\gamma,\sigma^2<+\infty$. On the other hand, by the fact that for any $t>1$,

$$\mathcal{R}(u(t))\ge\inf_{x\in\mathbb{T}^d}u(t,x)^2\ge\left(\frac{\inf_{x,y\in\mathbb{T}^d}\mathcal{Z}^\omega_{t,t-1}(x,y)}{\sup_{x,y\in\mathbb{T}^d}\mathcal{Z}^\omega_{t,t-1}(x,y)}\right)^2$$

we have $\mathbf{E}[\mathcal{R}(u(t))]\ge c$ for some constant $c>0$, independent of $t>1$. This implies that $\gamma>0$.

Next, for any $u\in D^\infty(\mathbb{T}^d)$, we let

$$g=u(1-\tfrac12\mathcal{D}\chi(u))\in L^\infty(\mathbb{T}^d).$$

Then (recall that $\hat r_0=1$, see (1.2))

$$\mathcal{R}(g)=\sum_{k\in\mathbb{Z}^d}\hat r_k|\hat g_k|^2\ge|\hat g_0|^2.$$

Since

$$\hat g_0=\int_{\mathbb{T}^d}g(x)dx=1-\tfrac12\langle u,\mathcal{D}\chi(u)\rangle_{L^2(\mathbb{T}^d)}=1,$$

where the last equality comes from (5.33), we conclude that $\sigma^2\ge 1$. □

Appendix A. Fortet-Mourier metric on the space of Borel probability measures

Suppose that $(\mathbb{X},\mathrm{d})$ is a metric space. Given a set $A\subset\mathbb{X}$ and a function $F:A\to\mathbb{R}$ we denote

$$\|F\|_{\infty,A}:=\sup_{x\in A}|F(x)|.$$

If $A=\mathbb{X}$, we omit writing the subsript A in the notation of the norm.

Let $\mathrm{M}(\mathbb{X})$ be the space of all Borel signed measures on $\mathbb{X}$ with a finite total variation. By $\mathrm{M}_1(\mathbb{X})\subset\mathrm{M}(\mathbb{X})$ we denote the subset consisting of all probability measures.

For any $F:\mathbb{X}\to\mathbb{R}$ we let

$$\|F\|_{\mathrm{Lip}}:=\|F\|_\infty+\sup_{f\ne g,f,g\in\mathbb{X}}\frac{|F(f)-F(g)|}{\mathrm{d}(f,g)}.\tag{A.1}$$

Denote by $\mathrm{Lip}(\mathbb{X})$ the space of all functions F for which $\|F\|_{\mathrm{Lip}}<+\infty$.

Given $\Pi \in \mathrm{M}(\mathbb{X})$, we define

$$\|\Pi\|_{\mathrm{FM}} := \sup\left\{\left|\int_{\mathbb{X}} F d\Pi\right| : \|F\|_{\mathrm{Lip}} \le 1\right\}. \tag{A.2}$$

It is a norm on $\mathrm{M}(\mathbb{X})$, see e.g. [27, Lemma 6, p. 254]. In fact, when restricted to $\mathrm{M}_1(\mathbb{X})$ it defines a metric, called the Fortet-Mourier metric:

$$\mathrm{d}_{\mathrm{FM}}\Big(\Pi_1, \Pi_2\Big) := \|\Pi_1 - \Pi_2\|_{\mathrm{FM}}, \quad \Pi_1, \Pi_2 \in \mathrm{M}_1(\mathbb{X}).$$

The topology of $(\mathrm{M}_1(\mathbb{X}), \mathrm{d}_{\mathrm{FM}})$ topology coincides with the topology of the weak convergence of probability measures, see e.g. [10, Theorem 3.2.2, p. 111]. Suppose that $(\mathbb{X}, \mathrm{d})$ is complete metric space, then the metric space $(\mathrm{M}_1(\mathbb{X}), \mathrm{d}_{\mathrm{FM}})$ is complete. If in addition $(\mathbb{X}, \mathrm{d})$ is separable then $(\mathrm{M}_1(\mathbb{X}), \mathrm{d}_{\mathrm{FM}})$ is also separable, see ibid. In this way we can define the norms $\|\cdot\|_{\mathrm{FM},\mathcal{M}_1}$ and $\|\cdot\|_{\mathrm{FM},p}$ on the spaces $\mathrm{M}_1(\mathcal{M}_1(\mathbb{T}^d))$ and $\mathrm{M}_1(D^p(\mathbb{T}^d))$ for any $p \in [1, +\infty)$.

Appendix B. Technical lemmas on SHE

Recall that $\mathcal{U}(t,x;y)$ is Green's function of the SHE, while $\mathcal{U}(t,x;\nu)$ is the solution to the SHE with initial data ν belonging to $\mathcal{M}_1(\mathbb{T}^d)$. Let

$$Z_t = \int_{\mathbb{T}^d} \mathcal{U}(t,x;\nu)dx \tag{B.1}$$

be the partition function of the directed polymer.

B.1. Markov property of the endpoint distribution. Proof of Lemma 2.2.

In the proof we will write $\mathcal{U}(t;\nu,\omega)$ and $u(t;\nu,\omega)$ to emphasize the dependence of the solution on $\omega \in \Omega$. Since $\mathcal{U}$ is the solution of a linear equation, we conclude that

$$\begin{aligned}\mathcal{U}(t+s;\nu,\omega) &= \mathcal{U}(s;\mathcal{U}(t;\nu,\omega), \theta_{t,0}(\omega)) \\ &= \|\mathcal{U}(t;\nu,\omega)\|_{L^1(\mathbb{T}^d)} \mathcal{U}(s;u(t;\nu,\omega), \theta_{t,0}(\omega)).\end{aligned}$$

Therefore

$$\begin{aligned}u(t+s;\nu,\omega) &= \frac{\mathcal{U}(t+s;\nu,\omega)}{\|\mathcal{U}(t+s;\nu,\omega)\|_{L^1(\mathbb{T}^d)}} \\ &= \frac{\mathcal{U}(s;u(t;v,\omega), \theta_{t,0}(\omega))}{\|\mathcal{U}(s;u(t;\nu,\omega), \theta_{t,0}(\omega))\|_{L^1(\mathbb{T}^d)}} = u\Big(s;u(t;\nu,\omega), \theta_{t,0}(\omega)\Big),\end{aligned}$$

which completes the proof. □

B.2. Some properties of SHE.

In this section, we recall and state several results on the positive and negative moments of the solution to SHE. We consider two cases: either (i) $R(\cdot) \in C(\mathbb{T}^d)$ and $d \ge 1$, or (ii) $R(\cdot) = \delta(\cdot)$ and $d = 1$. All results are rather standard here, so we only sketch their proofs.

Lemma B.1. *For any $T > 0, p \in [1,\infty)$, there exists $C = C(T,p) > 0$ such that for $t \in (0,T], x \in \mathbb{T}^d$, we have*

$$\mathbf{E}[\mathcal{U}(t,x;0)^p] \le \begin{cases} Cp_t(x)^p, & \text{in case (i)}, \\ C[1+p_t(0)p_t(x)]^{p/2}, & \text{in case (ii)}. \end{cases} \tag{B.2}$$

Recall that $p_t(x)$ is the heat kernel on the torus, see (2.13). Futhermore, for each $\varepsilon > 0$ and $\delta \in (0, \frac{1}{2})$, we have

$$\sup_{t\in(\varepsilon,\varepsilon^{-1}), x\ne y\in\mathbb{T}^d} \frac{\mathbf{E}[|\mathcal{U}(t,x;0) - \mathcal{U}(t,y;0)|^p]}{|x-y|^{\delta p}} \le C(\varepsilon,\delta,p), \tag{B.3}$$

and

$$\sup_{t\in(\varepsilon,\varepsilon^{-1}),x\in\mathbb{T}^d} \mathbf{E}[\mathcal{U}(t,x;0)^{-p}] \le C(\varepsilon,p). \tag{B.4}$$

Proof. For (B.2), case (i) comes from a direct application of the Feynman-Kac formula. To show it in case (ii), one can invoke [22, Theorem 3.3], which was proved for the equation posed on the whole space, however the proof applies verbatim to our setting. The estimate (B.3) can be found e.g. in [54, Chapter 3]. For (B.4), one can follow the same proof for [37, Corollary 4.9]. Since the proofs are almost unchanged, we do not reproduce them here. □

Lemma B.2. *For any $T>0, p\in[1,\infty)$, there exists $C=C(T,p)$ such that*

$$\mathbf{E}[Z_t^p]+\mathbf{E}[Z_t^{-p}] \le C, \qquad t\in[0,T]. \tag{B.5}$$

Proof. Using (B.1), for the positive moments, we conclude by the Jensen inequality that

$$\mathbf{E}[Z_t^p] \le \int_{\mathbb{T}^d}\mathbf{E}\left(\int_{\mathbb{T}^d}\mathcal{U}(t,x;y)dx\right)^p \nu(dy).$$

For each fixed $y\in\mathbb{T}^d$ and $t>0$, we know $\int_{\mathbb{T}^d}\mathcal{U}(t,x;y)dx \overset{\text{law}}{=} \mathcal{U}(t,0;\mathbb{1})$, which implies

$$\mathbf{E}[Z_t^p] \le \mathbf{E}[\mathcal{U}(t,0;\mathbb{1})^p].$$

Similarly, for negative moments we have

$$\mathbf{E}[Z_t^{-p}] \le \mathbf{E}[\mathcal{U}(t,0;\mathbb{1})^{-p}].$$

The positive and negative moments bound on $\mathcal{U}(t,0;\mathbb{1})$ is a standard result on SHE, see e.g. [40, Proposition 4.1] and [45, Theorem 2]. □

Lemma B.3. *For any $T>0, p\in[1,\infty)$, there exists $C=C(T,p)$ such that for all $t\in(0,T]$, we have*

$$\sup_{\nu\in\mathcal{M}_1(\mathbb{T}^d)}\left(\mathbf{E}[\|\mathcal{U}(t;\nu)\|^{2p}_{L^{2p}(\mathbb{T}^d)}]+\mathbf{E}[\|u(t;\nu)\|^{2p}_{L^{2p}(\mathbb{T}^d)}]\right) \le C(1+t^{-(p-1/4)}), \qquad \textit{in case (ii).}$$

Proof. First, we have

$$\|u(t;\nu)\|^{2p}_{L^{2p}(\mathbb{T}^d)} = Z_t^{-2p}\int_{\mathbb{T}^d}\mathcal{U}(t,x;\nu)^{2p}dx.$$

An application of the Hölder inequality leads to

$$\begin{aligned}\mathbf{E}[\|u(t;\nu)\|^{2p}_{L^{2p}(\mathbb{T}^d)}] \le& \sqrt{\mathbf{E}[Z_t^{-4p}]\int_{\mathbb{T}^d}\mathbf{E}[\mathcal{U}(t,x;\nu)^{4p}]dx}\\ \le&\sqrt{\mathbf{E}[Z_t^{-4p}]\int_{\mathbb{T}^{2d}}\mathbf{E}[\mathcal{U}(t,x;y)^{4p}]dx\,\nu(dy)}\\ =&\sqrt{\mathbf{E}[Z_t^{-4p}]\int_{\mathbb{T}^d}\mathbf{E}[\mathcal{U}(t,x;0)^{4p}]dx}.\end{aligned}$$

Applying Lemmas B.1 and B.2 we complete the proof for u. The proof for $\mathcal{U}$ is similar. □

Lemma B.4. *For any $p\in[1,\infty)$, we have $\mathbf{E}\langle Z\rangle_t^p \le C(t,p)$.*

Proof. Recall that

$$\langle Z\rangle_t = \int_0^t\int_{\mathbb{T}^{2d}}\mathcal{U}(s,y)\mathcal{U}(s,y')R(y-y')dydy'ds.$$

In case (i), we have

$$\langle Z\rangle_t \le \|R\|_{L^\infty(\mathbb{T}^d)}\int_0^t Z_s^2 ds,$$

then it suffices to apply Lemma B.2 to complete the proof. In case (ii), we have $\langle Z\rangle_t = \int_0^t \|\mathcal{U}(s;\nu)\|^2_{L^2(\mathbb{T}^d)}ds$. Applying Lemma B.3, we have

$$\left(\mathbb{E}\langle Z\rangle_t^p\right)^{1/p} \le \int_0^t \left(\mathbb{E}\|\mathcal{U}(s;\nu)\|^{2p}_{L^2(\mathbb{T}^d)}\right)^{1/p} ds \le C\int_0^t (1+s^{-(1-\frac{1}{4p})})ds,$$

which concludes the proof. □

Lemma B.5. *For any $T>0, p\in[1,\infty)$, there exists a constant $C=C(T,p)$ such that*

(i) for any $0\le v\in L^2(\mathbb{T}^d)$, we have

$$\sup_{t\in[0,T]} \mathbf{E}[\|\mathcal{U}(t;v)\|^{2p}_{L^2(\mathbb{T}^d)}] \le C\|v\|^{2p}_{L^2(\mathbb{T}^d)}, \tag{B.6}$$

(ii) for any $0\le v\in L^\infty(\mathbb{T}^d)$, we have

$$\sup_{t\in[0,T]} \mathbf{E}[\|\mathcal{U}(t;v)\|^{2p}_{L^\infty(\mathbb{T}^d)}] \le C\|v\|^{2p}_{L^\infty(\mathbb{T}^d)}, \tag{B.7}$$

(iii) for any $v\in D^2(\mathbb{T}^d)$, we have

$$\sup_{t\in[0,T]} \mathbf{E}[\|u(t;v)\|^{2p}_{L^2(\mathbb{T}^d)}] \le C\|v\|^{2p}_{L^2(\mathbb{T}^d)}, \tag{B.8}$$

(iv) for any $v\in D^\infty(\mathbb{T}^d)$, we have

$$\sup_{t\in[0,T]} \mathbf{E}[\|u(t;v)\|^{2p}_{L^\infty(\mathbb{T}^d)}] \le C\|v\|^{2p}_{L^\infty(\mathbb{T}^d)}, \tag{B.9}$$

(v) for any $h\in L^2(\mathbb{T}^d), v\in D(\mathbb{T}^d)$, we have

$$\sup_{t\in[0,T]} \mathbf{E}[\|U(t;h,v)\|^{2p}_{L^2(\mathbb{T}^d)}] \le C\|h\|^{2p}_{L^2(\mathbb{T}^d)}. \tag{B.10}$$

Proof. By Lemma B.2 and the Hölder inequality, (B.8), (B.9) and (B.10) are consequences of (B.6) and (B.7). So we focus on (B.6) and (B.7).

By comparison principle [19, Theorem 1.3], we have

$$\|\mathcal{U}(t;v)\|_{L^\infty(\mathbb{T}^d)} \le \|v\|_{L^\infty(\mathbb{T}^d)}\|\mathcal{U}(t;\mathbb{1})\|_{L^\infty(\mathbb{T}^d)}.$$

By applying Lemma B.7 formulated below, we conclude (B.7).

To prove (B.6), the case of $R(\cdot)\in C(\mathbb{T}^d)$ can be studied by the Feynman-Kac formula in a straightforward way, so we focus on the $1+1$ spacetime white noise. By the mild formulation, we have

$$\mathcal{U}(t,x;v) = p_t\star v(x) + \int_0^t\int_{\mathbb{T}^d} p_{t-s}(x-y)\mathcal{U}(s,y;v)\xi(s,y)dyds =: p_t\star v(x) + M_t(x),$$

so

$$\int_{\mathbb{T}^d}\mathcal{U}(t,x;v)^2dx \le 2\int_{\mathbb{T}^d}|p_t\star v(x)|^2dx + 2\int_{\mathbb{T}^d}M_t(x)^2dx.$$

The first term on the r.h.s. is bounded from above by $2\|v\|^2_{L^2(\mathbb{T}^d)}$. For the second term, applying [39, Proposition 4.4], we have

$$\|M_t(x)\|^2_{L^{2p}(\Omega)} \le C\int_0^t\int_{\mathbb{T}^d} p_{t-s}(x-y)^2\|\mathcal{U}(s,y;v)\|^2_{L^{2p}(\Omega)}dyds.$$

By Lemma B.1, we have

$$\begin{aligned}\|\mathcal{U}(s,y;v)\|_{L^{2p}(\Omega)} =& \left\|\int_{\mathbb{T}^d}\mathcal{U}(s,y;z)v(z)dz\right\|_{L^{2p}(\Omega)} \le \int_{\mathbb{T}^d}\|\mathcal{U}(s,y;z)\|_{L^{2p}(\Omega)}v(z)dz\\ \le& C\|v\|_{L^2(\mathbb{T}^d)}\sqrt{\int_{\mathbb{T}^d}(1+p_s(0)p_s(y-z))dz}\\ \le& C\|v\|_{L^2(\mathbb{T}^d)}(1+s^{-1/4}),\end{aligned}$$

which implies

$$\|M_t(x)\|^2_{L^{2p}(\Omega)} \le C\|v\|^2_{L^2(\mathbb{T}^d)}\int_0^t \frac{1}{\sqrt{t-s}}(1+\frac{1}{\sqrt{s}})ds \le C\|v\|^2_{L^2(\mathbb{T}^d)},$$

and

$$\mathbf{E}|\int_{\mathbb{T}^d} M_t(x)^2dx|^p \le \mathbf{E}\int_{\mathbb{T}^d} M_t(x)^{2p}dx \le C\|v\|^{2p}_{L^2(\mathbb{T}^d)},$$

which completes the proof. □

Lemma B.6. *Assume $v \in D^2(\mathbb{T}^d)$, then for any $s \ge 0$ and $p \in [1,\infty)$*

$$\lim_{t\to s}\mathbf{E}[\|u(t;v)-u(s;v)\|^p_{L^2(\mathbb{T}^d)}] = 0.$$

Proof. We have

$$\begin{aligned}&\|u(t;v)-u(s;v)\|_{L^2(\mathbb{T}^d)}\\ &\le \|\mathcal{U}(t;v)-\mathcal{U}(s;v)\|_{L^2(\mathbb{T}^d)}\big(Z_t^{-1}+\|\mathcal{U}(s;v)\|_{L^2(\mathbb{T}^d)}Z_t^{-1}Z_s^{-1}\big).\end{aligned}$$

To complete the proof it suffices to use the fact that

$$\mathbf{E}[\|\mathcal{U}(t;v)-\mathcal{U}(s;v)\|^p_{L^2(\mathbb{T}^d)}] \to 0, \quad \text{as } t\to s$$

for any $p \ge 1$, together with Lemma B.2 and estimate (B.6). □

Lemma B.7. *For any $T \ge 0, p \in [1,\infty)$, we have*

$$\mathbf{E}\Big[\sup_{t\in[0,T],x\in\mathbb{T}^d}\mathcal{U}(t,x;\mathbb{1})^p\Big] + \mathbf{E}\Big[\sup_{t\in[0,T],x\in\mathbb{T}^d}\mathcal{U}(t,x;\mathbb{1})^{-p}\Big] \le C(T,p).$$

Proof. The proof is similar to Lemma 4.1, so we do not repeat it here.□

B.3. **Proof of Lemmas 4.1, 5.21.** *Proof of Lemma 4.1.* From (4.2), we have $\mathcal{Z}_{t,0}(x,y) = \mathcal{U}(t,x;y)$, so the goal reduces to prove that there exists $C > 0$ only depending on d,p,R such that

$$\mathbf{E}[\sup_{x,y\in\mathbb{T}^d}\mathcal{U}(1,x;y)^p] \le C, \qquad \mathbf{E}[(\inf_{x,y\in\mathbb{T}^d}\mathcal{U}(1,x;y))^{-p}] \le C. \tag{B.11}$$

For the second inequality, by writing

$$(\inf_{x,y\in\mathbb{T}^d}\mathcal{U}(1,x;y))^{-p} = \sup_{x,y\in\mathbb{T}^d}\mathcal{U}(1,x;y)^{-p},$$

it suffices to prove

$$\mathbb{E}\big[\sup_{w\in I}X(w)^p\big] \le C \tag{B.12}$$

with $w = (x,y)$, $I = \mathbb{T}^{2d}$, and $X(w) = \mathcal{U}(1,x;y)$ or $\mathcal{U}(1,x;y)^{-1}$. By a well-known chaining argument, see e.g. [40, Proposition 5.8], we need to show that there exists $\alpha > 0$ such that for any $p \ge 2$,

$$\begin{aligned}&\mathbf{E}[X(w)^p] < \infty \qquad \text{for some } w \in I,\\ &\sup_{w_1,w_2\in I,w_1\ne w_2}\mathbf{E}\left[\frac{|X(w_1)-X(w_2)|^p}{|w_1-w_2|^{p\alpha}}\right] < \infty.\end{aligned} \tag{B.13}$$

This comes from the following estimates:

$$
\begin{aligned}
&\sup_{x,y\in\mathbb{T}^d}\mathbf{E}[\mathcal{U}(1,x;y)^p]+\sup_{x,y\in\mathbb{T}^d}\mathbf{E}[\mathcal{U}(1,x;y)^{-p}]\le C,\\
&\sup_{x\in\mathbb{T}^d}\mathbf{E}[|\mathcal{U}(1,x;y_1)-\mathcal{U}(1,x;y_2)|^p]\le C|y_1-y_2|^{\delta p},\\
&\sup_{y\in\mathbb{T}^d}\mathbf{E}[|\mathcal{U}(1,x_1;y)-\mathcal{U}(1,x_2;y)|^p]\le C|x_1-x_2|^{\delta p},
\end{aligned}
\tag{B.14}
$$

which are direct consequences of Lemma B.1 and the fact that for fixed $x,y\in\mathbb{T}^d$, we have

$$\mathcal{U}(1,x;y)\overset{\text{law}}{=}\mathcal{U}(1,x-y;0).$$

□

Proof of Lemma 5.21. We only sketch the proof for the case of $1+1$ spacetime white noise. First, we have

$$\sup_{t\in[0,T]}\mathbf{E}[\|\mathcal{U}_N(t;v)-\mathcal{U}(t;v)\|^p_{L^2(\mathbb{T}^d)}]\to 0, \tag{B.15}$$

which is a consequence of [8, Theorem 2.2]. While the result of [8] is formulated for the equation posed on the whole space, the proof applies verbatim to our setting. Denote $Z_N(t)=\|\mathcal{U}_N(t;v)\|_{L^1(\mathbb{T}^d)}$, then as in Lemma B.2, we also have

$$\mathbf{E}[Z_N(t)^p]+\mathbf{E}[Z_N(t)^{-p}]\le C(T,p),\qquad t\in[0,T].$$

The proofs for all cases are similar, so we take $\mathbf{E}[\|u_N(t;v)-u(t;v)\|^p_{L^2(\mathbb{T}^d)}]$ as an example. We have

$$
\begin{aligned}
\|u_N(t;v)-u(t;v)\|_{L^2(\mathbb{T}^d)}\le& Z_N(t)^{-1}\|\mathcal{U}_N(t;v)-\mathcal{U}(t;v)\|_{L^2(\mathbb{T}^d)}\\
&+(Z_N(t)Z_t)^{-1}\|\mathcal{U}(t;v)\|_{L^2(\mathbb{T}^d)}\|\mathcal{U}_N(t;v)-\mathcal{U}(t;v)\|_{L^2(\mathbb{T}^d)},
\end{aligned}
$$

then it suffices to apply Hölder inequality and (B.15) to conclude the proof.□

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