

# FLUCTUATIONS OF THE KPZ EQUATION ON A LARGE TORUS

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ABSTRACT. We study the one-dimensional KPZ equation on a large torus, started at equilibrium. The main results are optimal variance bounds in the super-relaxation regime and part of the relaxation regime.

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## 1. INTRODUCTION

**1.1. Main result.** We consider the stochastic heat equation (SHE) on a one-dimensional torus of size  $L$ :

$$(1.1) \quad \partial_t \mathcal{U} = \frac{1}{2} \Delta \mathcal{U} + \mathcal{U} \eta(t, x), \quad x \in \mathbb{T}_L = [0, L],$$

where  $\eta$  is a spacetime white noise. We assume the initial data

$$(1.2) \quad \mathcal{U}(0, x) = e^{B(x)},$$

with  $B$  a standard Brownian bridge on  $[0, L]$  with  $B(0) = B(L) = 0$ . Thus, the Cole–Hopf solution  $h(t, x) = \log \mathcal{U}(t, x)$  to the corresponding Kardar–Parisi–Zhang (KPZ) equation starts from  $\log \mathcal{U}(0, x) = B(x)$ , and is stationary under the KPZ evolution

$$\partial_t h(t, x) = \frac{1}{2} \Delta h(t, x) + \frac{1}{2} (\partial_x h(t, x))^2 + \eta(t, x)$$

up to spatially constant height shifts. The goal of this paper is to study the asymptotics of the variance  $\text{Var}[\log \mathcal{U}(t, x)]$ , as  $t$  and  $L$  go to infinity together. We write  $a \lesssim b$  if  $a \leq Cb$  for some constant  $C > 0$  independent of  $t \geq 1$  and  $x \in [0, L]$ , and  $a \asymp b$  if  $a \lesssim b$  and  $b \lesssim a$ . Here is the main result:

**Theorem 1.1.** *Suppose that  $L = \lambda t^\alpha$  for some  $\lambda, \alpha > 0$ . Then, there is a universal constant  $\delta > 0$  so that the following holds:*

$$(1.3) \quad \text{Var}[\log \mathcal{U}(t, x)] \asymp \begin{cases} t^{1-\alpha/2}, & \text{if } \alpha \in [0, 2/3) \text{ and } \lambda \in (0, \infty); \\ t^{2/3}, & \text{if } \alpha = 2/3 \text{ and } \lambda \in (0, \delta). \end{cases}$$

*In the general case of  $\alpha = 2/3$  and  $\lambda \in (0, \infty)$ , we have the upper bound*

$$\text{Var}[\log \mathcal{U}(t, x)] \lesssim t^{2/3}.$$

**1.2. Context.** The study of the 1+1 KPZ universality class has witnessed important progress in recent years, with the construction of the fixed point and the proofs of various models converging to it under appropriate scalings. There is a huge body of literature on the subject, and we refer to the results e.g. in [2, 49, 50, 15, 16, 31, 41, 46, 52], the reviews [45, 29, 47] and the references therein. In particular, for the KPZ equation on the whole line, it was shown in [46, 52] that, under the 1 : 2 : 3 scaling, the centered height function converges to the KPZ fixed point. Nevertheless, most studies have been focusing on exact solvable models and relying on exploiting

certain integrable structures. So it still remains a major open problem to derive the correct scaling exponents for general models and to further prove the convergence of the rescaled random fluctuations.

In this paper, we study the KPZ equation posed on a torus of size  $L$ , started at equilibrium, and investigate the growth of the variance of the height function as  $t \rightarrow \infty$ . By choosing  $L \sim t^\alpha$  and tuning the parameter  $\alpha$ , the model interpolates two universality classes: the Gaussian class ( $\alpha = 0$ ,  $L \sim O(1)$ ) and the KPZ class ( $\alpha = \infty$ ,  $L = \infty$ ). The critical length scale is  $L \sim t^{2/3}$ , which is the so-called relaxation regime where the heights at all points on the torus happen to be correlated. The regimes of  $\alpha < 2/3$  and  $\alpha > 2/3$  are the so-called super- and sub-relaxation regimes, respectively. For  $\alpha = 0$ , which is the case of a fixed torus, the Gaussian fluctuations were proved in [35]; for  $\alpha = \infty$ , which is the case of the whole line, the  $t^{1/3}$  fluctuations were obtained in [9] and the weak convergence of the rescaled fluctuations were further derived in [16] in the stationary regime. Here we devise a new approach which does not rely on integrable tools, through which we obtain the correct fluctuation exponents in the super-relaxation regime and a part of the relaxation regime.

The idea of our approach is easy to describe: for a torus of fixed size  $L$ , the result in [35] shows the diffusive scaling of  $\text{Var}[\log \mathcal{U}(t, 0)] \approx \sigma_L^2 t$  for large  $t \gg 1$ , with a diffusion constant  $\sigma_L^2$ . To obtain sub-diffusive behavior of  $\text{Var}[\log \mathcal{U}(t, 0)] \sim t^{2/3} \ll t$  in the case of  $L = \infty$ , the diffusion constant  $\sigma_L^2$  must vanish as  $L \rightarrow \infty$ . While the expression of the diffusion constant is model-dependent, its decay rate is expected to be universal, and is related to the characteristic exponents of the universality class. The correct rate is expected to be  $\sigma_L^2 \sim L^{-1/2}$ , which was proved in [32] for TASEP on the torus, a different model in the universality class. In our setting, we will prove that  $\sigma_L^2 \sim L^{-1/2}$  in Theorem 4.1 below. Roughly speaking, the variance of the height function on the torus of size  $L$  is  $t\sigma_L^2$ , and the transversal roughness of the interface is at most of order  $L$ . In our case, the invariant measure for the height function modulo an additive constant is given by the law of the Brownian bridge, whose variance is of order  $L$  on the torus of size  $L$ . With the decay of  $\sigma_L^2 \sim L^{-1/2}$ , we need  $L \sim t^{2/3}$  to balance these two terms, which implies the variance of the height function is  $t\sigma_L^2 \sim t^{2/3}$ . The picture was sketched in [32] for TASEP, and the core of our argument is to make this heuristic rigorous for the KPZ equation.

For TASEP on the torus, more precise results have been derived in [3, 4, 39, 5, 7, 6]. The rescaled random fluctuations are described explicitly in the relaxation regime, which has been further shown to interpolate the Gaussian statistics and the Tracy-Widom type statistics. Exact formulas of the transition probability for ASEP on the torus were derived in [40]. For models of first passage percolation [23] and longest increasing path [33], similar variance bounds as in (1.3) have been derived in the corresponding super-relaxation regimes and the Gaussian fluctuations were also proved.

A key part of our argument relies on the fact that the law of the Brownian bridge modulo an additive constant is the unique invariant measure for the KPZ equation on the torus; see [14, 34, 37, 38]. The fact that the invariant measure is explicit is crucial in our analysis, especially for the proof of the decay of  $\sigma_L^2 \sim L^{-1/2}$ . For the KPZ equation or the stochastic Burgers equation, on the torus with a noise that is white in time and smooth in space, the existence and uniqueness of the invariant measure is well-known (see e.g. [51, 48, 35]), but there is very little information regarding the quantitative properties of the invariant measure. Part of our proof applies to the case of a colored noise in all dimensions, and our analysis reduces the proof of results of the form (1.3) to the study of two explicit functionals of the invariant measure; see the discussion in Section 5.3. On this point, we mention the

recent works on deriving the explicit formulas for the invariant measure for the KPZ equation posed on an interval with Neumann boundary conditions [30, 20, 11].

Our analysis is restricted to the torus, in which case we can define the endpoint distribution of the directed polymer at stationarity and use it as the underlying driving Markov process. In terms of the solution to SHE, the endpoint distribution takes the form  $\rho(t, x) = \mathcal{U}(t, x) / \int_0^L \mathcal{U}(t, x') dx'$ , which is not well-defined on the whole line when we start the KPZ from a two-sided Brownian motion. In many studies of the models in the 1 + 1 KPZ universality class, the underlying process is chosen to be the solution of the Burgers equation, or some discretization of the Burgers equation, so that there is always a well-defined stationary measure. For the problem on the torus, we find the Markov process  $\{\rho(t, \cdot)\}_{t \geq 0}$  useful, since the centered height function can be written as a stochastic integral of an explicit functional of  $\rho$  with respect to the underlying noise; see Proposition 2.1 below. On the whole line, when  $\mathcal{U}$  starts from some  $L^1$  initial data, the Markov process  $\{\rho(t, \cdot)\}_{t \geq 0}$  is also well-defined, but it does not have an invariant probability measure, as the polymer endpoint spreads out as  $t \rightarrow \infty$ . Nevertheless, by embedding the endpoint distribution into an abstract space, which factors out the spatial shift, the localization properties of the endpoint distribution have been studied in [12, 18, 8].

**1.3. Ingredients of the proof.** For a torus of fixed size  $L$ , the following central limit theorem was proved in [35]: there exist  $\gamma_L, \sigma_L^2 > 0$  such that for any  $x \in \mathbb{T}_L$ ,

$$(1.4) \quad \frac{\log \mathcal{U}(t, x) + \gamma_L t}{\sqrt{t}} \Rightarrow N(0, \sigma_L^2), \quad \text{as } t \rightarrow \infty.$$

The variance  $\sigma_L^2$  was derived in [30] using a homogenization argument. The argument is based on a semimartingale decomposition of the free energy of the directed polymer, the construction of the corrector by solving a Poisson equation, and an application of the martingale central limit theorem. As a result, the effective diffusion constant  $\sigma_L^2$  depends on the corrector, and is expressed in terms of a Green–Kubo-type formula. This is not surprising, since the variance typically depends on the temporal correlations of the underlying process.

As mentioned previously, to study the size of the fluctuations for large  $L$ , the key is to derive the decay rate of the diffusion constant  $\sigma_L^2$  as  $L \rightarrow \infty$ . However, the formula in [35] for  $\sigma_L^2$  involves the corrector hence is not amenable to asymptotic analysis as  $L \rightarrow \infty$ . So the first task is to derive a more explicit expression for  $\sigma_L^2$ . Somewhat surprisingly, it can be written as the expectation of an explicit exponential functional of correlated Brownian bridges; see (2.10) below. Our approach is based on an application of the Clark–Ocone formula, which expresses the centered height function as an Itô integral with respect to the noise, with the integrand connected to the distribution of the directed polymer. At this point, it is worth stressing the crucial role of the Clark–Ocone formula: to understand how the height function depends on the underlying noise, one way is to compute some derivative of the height function with respect to the noise, which in our context is naturally the Malliavin derivative. While it is well-known that the Gaussian Poincaré inequality only yields the sub-optimal diffusive upper bound in this case, the Clark–Ocone formula provides an *identity* that we can further analyze, and in a sense it connects the vertical fluctuations of the height function and the transversal fluctuations of directed polymers. As a matter of fact, the way to improve the Gaussian Poincaré inequality to obtain so-called superconcentration has been explained, for example, in the monograph [22]. That approach relies on a similar variance representation formula.

The central limit theorem (1.4) holds in the regime when  $t \rightarrow \infty$  with fixed  $L$ , and so is not suited for our case where  $L$  and  $t$  go to infinity simultaneously. Thus, the second task is to create a type of “stationarity” and come up with an approximation of  $\log \mathcal{U}(t, x)$ , so that we can compute the variance of the approximation for all finite  $t$  and  $L$ . This is the reason we start the equation at equilibrium. The approximation we choose is of the form  $\log \int_0^L \mathcal{U}(t, x) e^{W(x)} dx$ , where  $W$  is another standard Brownian bridge on  $[0, L]$ , independent of the noise  $\eta$  and the initial data  $B$ . One should view the factor of  $e^{W(\cdot)}$ , after a normalization, as the prescribed endpoint distribution of the directed polymer of length  $t$ , so that the integral  $\int_0^L \mathcal{U}(t, x) e^{W(x)} dx$  is related to the partition function of the polymer which not only starts at equilibrium but also ends at an independent equilibrium. By further using the time-reversal symmetry of the Green’s function of the SHE in the Clark–Ocone representation, we obtain the aforementioned “stationarity.”

The final step is to prove the decay rate  $\sigma_L^2 \sim L^{-1/2}$ . Although the diffusion constant obtained in (2.10) below is the expectation of an explicit exponential functional of Brownian bridges, the analysis of this functional is nontrivial and constitutes the most technical part of the paper. The difficulty comes from the fact that there are two correlated Brownian bridges, the correlation ultimately coming from the conditional expectation taken in the Clark–Ocone formula. The decay rate of  $L^{-1/2}$  is intimately connected to the correlation. If the two Brownian bridges were independent or identical, then we could use Yor’s density formula in [53] to compute  $\sigma_L^2$  explicitly, yielding, respectively,  $L^{-1}$  decay or no decay at all. The nontrivial correlation interpolates between these scenarios and leads to the correct exponent  $-1/2$ .

Our analysis of  $\sigma_L^2$  proceeds by first writing the correlated Brownian bridges as a linear transformation of two independent Brownian bridges. It turns out that the contributions to the functional in (2.10) mainly come from the event that the correlated Brownian bridges do not travel too far above zero, which corresponds to the event that the independent Brownian bridges do not stray too far outside of a wedge in  $\mathbb{R}^2$ . The angle of the wedge ( $2\pi/3$ ) comes from the correlation, and is what leads to the exponent  $-1/2$ . While the natural approach to the notion of “too far outside of the wedge” would seem to involve Brownian motion with a “soft” cutoff aligned with the sides of the wedge, we do not at present know how to analyze this directly. (See the discussion in Section 5.2 below.) Instead, for the upper bound, we approximate the “soft” cutoff by a sequence of “hard” cutoffs and use a stopping time argument. For the lower bound, we approximate the “soft” cutoff by a single “hard” cutoff, and use an entropic repulsion argument similar to that of [17, 27]. These approximations allow us to use the well-known formula for the heat kernel of a standard planar Brownian motion killed upon exiting a wedge (that is, with a “hard” cutoff) to undergird the analysis. In particular, this formula is ultimately the source of the correct exponent.

**1.4. Organization of the paper.** The rest of the paper is organized as follows. In Section 2, we derive an explicit expression of the diffusion constant  $\sigma_L^2$ . In Section 3, we prove a decomposition of the centered height function into two parts, one corresponding to the diffusive fluctuation of order  $t\sigma_L^2$ , and the other corresponding to the transversal roughness of order  $L$ . Then the proof of Theorem 1.1 is completed by balancing these two terms. Section 4 is devoted to the proof of the decay  $\sigma_L^2 \sim L^{-1/2}$ . In Section 5, we discuss several possible extensions, including another proof of the central limit theorem in (1.4), a possible connection to Liouville quantum mechanics which could be related to computing  $\sigma_L^2$  exactly, and the case of the colored noise and the equation in high dimensions.

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## 2. A FORMULA FOR THE DIFFUSION CONSTANT

In this section, we derive a formula for the variance appearing in (1.4). The result is based on an application of the Clark–Ocone formula.

We first introduce some notations. Let  $C_+(\mathbb{T}_L)$  be the subset of  $C(\mathbb{T}_L)$  – the space of continuous functions on  $\mathbb{T}_L$  – consisting of all non-negative functions that are not identically equal to 0. For any  $f, g \in C_+(\mathbb{T}_L)$ , let  $\mathcal{U}(t, x; g)$  denote the solution to (1.1) starting from  $g$ , i.e. with  $\mathcal{U}(0, x; g) = g(x)$ , and let

$$(2.1) \quad X_{f,g}(t) := \log \int_0^L \mathcal{U}(t, x; g) f(x) dx.$$

Define  $\mathcal{Z}(t, x; s, y)$  to be the propagator of the SHE from  $(s, y)$  to  $(t, x)$ , i.e.,

$$(2.2) \quad \begin{aligned} \partial_t \mathcal{Z}(t, x; s, y) &= \frac{1}{2} \Delta_x \mathcal{Z}(t, x; s, y) + \mathcal{Z}(t, x; s, y) \xi(t, x), \quad t > s, \\ \mathcal{Z}(s, x; s, y) &= \delta(x - y). \end{aligned}$$

Then we can write

$$X_{f,g}(t) = \log \int_0^L \int_0^L \mathcal{Z}(t, x; 0, y) g(y) f(x) dx dy.$$

Let  $\mathbb{D}^\infty(\mathbb{T}_L)$  be the set of all Borel probability measures on  $\mathbb{T}_L$  with continuous densities with respect to the Lebesgue measure. In what follows we shall identify a measure from  $\mathbb{D}^\infty(\mathbb{T}_L)$  with its respective density. If  $f, g \in \mathbb{D}^\infty(\mathbb{T}_L)$  we can view  $\int_0^L \int_0^L \mathcal{Z}(t, x; 0, y) g(y) f(x) dx dy$  as the partition function of the point-to-point directed polymer of length  $t$ , with the two endpoints sampled from  $f, g$  respectively. Then  $X_{f,g}(t)$  is the corresponding free energy.

For any  $g \in C_+(\mathbb{T}_L)$ , define

$$(2.3) \quad \rho(t, x; g) = \frac{\mathcal{U}(t, x; g)}{\int_0^L \mathcal{U}(t, x'; g) dx'},$$

which is the endpoint distribution of the directed polymer with the initial point sampled from the density  $g(\cdot) / \int_0^L g(x') dx'$  on  $[0, L]$ . In particular, (2.3) with  $t = 0$  means that

$$\rho(0, x; g) = \frac{g(x)}{\int_0^L g(x') dx'}.$$

From [35, Theorem 2.3], we know that as a Markov process,  $\{\rho(t)\}_{t \geq 0}$  has a unique invariant probability measure  $\pi_\infty$  on  $\mathbb{D}^\infty(\mathbb{T}_L)$ . Thanks to [34, Theorems 2.1 (part (1)) and 1.1], the invariant measure  $\pi_\infty$  is the law of the random element

$$\frac{e^{B(\cdot)}}{\int_0^L e^{B(x')} dx'},$$

where  $B$  is a Brownian bridge on  $[0, L]$  with  $B(0) = B(L) = 0$ . Note that  $\pi_\infty$  is invariant in law under periodic rotations of the torus  $\mathbb{T}_L$ , even though the law of the Brownian bridge  $B$  is not. We also define the probability measure  $\mu$  on  $C_+(\mathbb{T}_L)$  to be the distribution of  $e^{B(\cdot)}$ , and later we will sample  $f, g$ , which appeared in (2.1), independently from  $\mu$ .

Since there are several sources of randomness in our setup, to avoid confusion, in Sections 2 and 3, we use  $\mathbf{E}$  to denote the expectation *only* with respect to the noise  $\eta$ . The expectations with respect to  $f$ ,  $g$ , and the Brownian bridges  $B$  and  $W$  will be denoted by  $\mathbb{E}_f$ ,  $\mathbb{E}_g$ ,  $\mathbb{E}_B$ , and  $\mathbb{E}_W$ , respectively. When there is no possibility of confusion, we use  $\mathbb{E}$  as the expectation with respect to all possible randomnesses.

For any  $t > s \geq 0$ ,  $f, h \in C_+(\mathbb{T}_L)$  and  $y \in \mathbb{T}_L$ , define the deterministic functional

$$(2.4) \quad \mathcal{G}_{t,s}(f, h, y) := \mathbf{E} \left[ \frac{\int_0^L f(x) \mathcal{Z}(t, x; s, y) dx}{\int_0^L \int_0^L f(x) \mathcal{Z}(t, x; s, y') h(y') dx dy'} \right].$$

We have the following proposition which comes from a direct application of the Clark–Ocone formula.

**Proposition 2.1.** *Fix any  $t, L > 0$  and  $f, g \in C_+(\mathbb{T})$ , we have*

$$(2.5) \quad X_{f,g}(t) - \mathbf{E} X_{f,g}(t) = \int_0^t \int_0^L \mathcal{G}_{t,s}(f, \rho(s; g), y) \rho(s, y; g) \eta(s, y) dy ds.$$

We postpone the proof of Proposition 2.1 until Section 2.1 and present first some corollaries of the result.

As mentioned already, one can view  $X_{f,g}(t)$  as the free energy of the directed polymer of length  $t$ , with the two endpoints sampled from  $f(\cdot)/\int_0^L f(x') dx'$  and  $g(\cdot)/\int_0^L g(x') dx'$ , respectively. The idea is to sample  $f, g$  from  $\mu$  so that the start- and end-point distributions of the polymer path are both stationary but independent of each other and the noise.

First, when we sample  $f$  from  $\mu$  (in symbols, when  $f \sim \mu$ ), independent of the noise  $\eta$ , the integrand on the r.h.s. of (2.5) can be simplified. We use  $\mathbb{E}_f$  to denote the expectation with respect to  $f \sim \mu$ . In this case it is straightforward to check, for any fixed  $t > s$ , that if we define the random density

$$(2.6) \quad \tilde{\rho}_{t,f}(s, y) := \frac{\int_0^L f(x) \mathcal{Z}(t, x; s, y) dx}{\int_0^L \int_0^L f(x) \mathcal{Z}(t, x; s, y') dx dy'},$$

then  $\tilde{\rho}_{t,f}(s, \cdot) \sim \pi_\infty$ . Indeed,  $\tilde{\rho}_{t,f}(s, \cdot)$  can be viewed as the endpoint distribution of the “backward” polymer of length  $t - s$  with its starting point distributed according to  $f(\cdot)/\int_0^L f(x) dx$ . In particular, this means that

$$\tilde{\rho}_{t,f}(s, \cdot) \stackrel{\text{law}}{=} \frac{e^{B(\cdot)}}{\int_0^L e^{B(x')} dx'}.$$

So for any  $h \in C_+(\mathbb{T}_L)$  and  $y \in \mathbb{T}_L$ , we have

$$(2.7) \quad \begin{aligned} \mathbb{E}_f \mathcal{G}_{t,s}(f, h, y) &= \mathbb{E}_f \mathbf{E} \left[ \frac{\int_0^L f(x) \mathcal{Z}(t, x; s, y) dx}{\int_0^L \int_0^L f(x) \mathcal{Z}(t, x; s, y') h(y') dx dy'} \right] \\ &= \mathbb{E}_f \mathbf{E} \left[ \frac{\tilde{\rho}_{t,f}(s, y)}{\int_0^L \tilde{\rho}_{t,f}(s, y') h(y') dy'} \right] \\ &= \mathbb{E}_B \left[ \frac{e^{B(y)}}{\int_0^L e^{B(y')} h(y') dy'} \right] =: \mathcal{A}(h, y). \end{aligned}$$

Note that the functional  $\mathcal{A}$  is deterministic.

A direct corollary of (2.5) and (2.7) is the following.

**Corollary 2.2.** *For any  $g \in C_+(\mathbb{T}_L)$  and  $t, L > 0$ , we have*

$$(2.8) \quad \mathbb{E}_f X_{f,g}(t) - \mathbb{E}_f \mathbf{E} X_{f,g}(t) = \int_0^t \int_0^L \mathcal{A}(\rho(s; g), y) \rho(s, y; g) \eta(s, y) dy ds.$$

Furthermore, when we sample  $g \sim \mu$  independently,  $\rho(s, \cdot; g)$  is the endpoint distribution of the “forward” polymer of length  $s$ , starting from the stationary distribution, so the variance of (2.8) can be computed explicitly. We use  $\mathbb{E}_g$  to denote the expectation with respect to  $g \sim \mu$  which is independent of  $f$  and  $\eta$ .

**Corollary 2.3.** *For any  $t, L > 0$ , we have*

$$(2.9) \quad \begin{aligned} & \mathbb{E}_g \mathbf{E} \left[ \left( \mathbb{E}_f X_{f,g}(t) - \mathbb{E}_f \mathbf{E} X_{f,g}(t) \right)^2 \right] \\ &= \int_0^t \int_0^L \mathbb{E}_g \mathbf{E} \left[ \mathcal{A}(\rho(s; g), y) \rho(s, y; g) \right]^2 dy ds = t \sigma_L^2, \end{aligned}$$

with

$$(2.10) \quad \sigma_L^2 = \mathbb{E}_B \frac{\int_0^L e^{B_1(x)+B_2(x)+2B_3(x)} dx}{\int_0^L e^{B_1(x)+B_3(x)} dx \int_0^L e^{B_2(x)+B_3(x)} dx},$$

where  $B_1, B_2, B_3$  are independent Brownian bridges on  $[0, L]$  with  $B_j(0) = B_j(L) = 0$ ,  $j = 1, 2, 3$ .

*Proof.* For any deterministic  $g$ , applying the Itô isometry to (2.8), we obtain

$$\mathbf{E} \left[ \left( \mathbb{E}_f X_{f,g}(t) - \mathbb{E}_f \mathbf{E} X_{f,g}(t) \right)^2 \right] = \int_0^t \int_0^L \mathbf{E} \left[ \left( \mathcal{A}(\rho(s; g), y) \rho(s, y; g) \right)^2 \right] dy ds.$$

Further taking the expectation with respect to  $g \sim \mu$  we conclude the first equality in (2.9). To prove the second one, we use the fact that  $\rho(s, \cdot; g) \sim \pi_\infty$  for any  $s$ . Then, using (2.7), we obtain

$$\begin{aligned} & \mathbb{E}_g \mathbf{E} \left[ \left( \mathcal{A}(\rho(s; g), y) \rho(s, y; g) \right)^2 \right] \\ &= \mathbb{E}_B \left[ \mathcal{A} \left( e^{B_3(\cdot)} / \int_0^L e^{B_3(x)} dx, y \right)^2 \left( e^{B_3(y)} / \int_0^L e^{B_3(y')} dy' \right)^2 \right] \\ &= \mathbb{E}_B \left[ \frac{e^{B_1(y)+B_2(y)+2B_3(y)}}{\prod_{j=1}^2 \int_0^L \left( e^{B_j(y')+B_3(y')} / \int_0^L e^{B_3(x)} dx \right) dy'} \left( \int_0^L e^{B_3(y')} dy' \right)^{-2} \right], \end{aligned}$$

and thus (2.10).  $\square$

**2.1. Proof of Proposition 2.1.** By the Clark–Ocone formula (see e.g. [24, Proposition 6.3]), we have

$$(2.11) \quad X_{f,g}(t) - \mathbf{E} X_{f,g}(t) = \int_0^t \int_0^L \mathbf{E} [\mathcal{D}_{s,y} X_{f,g}(t) | \mathcal{F}_s] \eta(s, y) dy ds,$$

where  $\mathcal{D}$  is the Malliavin derivative with respect to  $\eta$  and  $\{\mathcal{F}_s\}_{s \geq 0}$  is the filtration generated by  $\eta$ . Using the chain rule, we derive

$$(2.12) \quad \begin{aligned} \mathcal{D}_{s,y} X_{f,g}(t) &= \mathcal{D}_{s,y} \log \int_0^L \mathcal{U}(t, x; g) f(x) dx \\ &= \frac{\int_0^L \mathcal{D}_{s,y} \mathcal{U}(t, x; g) f(x) dx}{\int_0^L \mathcal{U}(t, x; g) f(x) dx}. \end{aligned}$$

The Malliavin derivative of the SHE can be found, for example, in the proof of [25, Theorem 3.2]. It is given by

$$\mathcal{D}_{s,y} \mathcal{U}(t, x; g) = \mathcal{Z}(t, x; s, y) \mathcal{U}(s, y; g),$$

where we recall that  $\mathcal{Z}$  is the propagator of the SHE, defined in (2.2). Then from the fact that

$$\mathcal{U}(t, x; g) = \int_0^L \mathcal{Z}(t, x; s, y') \mathcal{U}(s, y'; g) dy', \quad t > s,$$



we obtain, after substituting into (2.12),

$$\begin{aligned} \mathcal{D}_{s,y}X_{f,g}(t) &= \frac{\int_0^L f(x)\mathcal{Z}(t,x;s,y)dx}{\int_0^L \int_0^L f(x)\mathcal{Z}(t,x;s,y')\mathcal{U}(s,y';g)dxdy'} \times \mathcal{U}(s,y;g) \\ &= \frac{\int_0^L f(x)\mathcal{Z}(t,x;s,y)dx}{\int_0^L \int_0^L f(x)\mathcal{Z}(t,x;s,y')\rho(s,y';g)dxdy'} \times \rho(s,y;g). \end{aligned}$$

The conditional expectation in (2.11) is then equal to

$$\mathbf{E}[\mathcal{D}_{s,y}X_{f,g}(t) | \mathcal{F}_s] = \mathbf{E}\left[\frac{\int_0^L f(x)\mathcal{Z}(t,x;s,y)dx}{\int_0^L \int_0^L f(x)\mathcal{Z}(t,x;s,y')\rho(s,y';g)dxdy'} \middle| \mathcal{F}_s\right] \rho(s,y;g).$$

Since (for  $t \geq s$ )  $\mathcal{Z}(t, \cdot; s, \cdot)$  is independent of  $\mathcal{F}_s$ , the conditional expectation on the r.h.s. of the above display is equal to  $\mathcal{G}(f, \rho(s; g), y)$ , and this completes the proof.  $\square$

### 3. A DECOMPOSITION OF THE CENTERED HEIGHT FUNCTION

In the present section, we prove a key decomposition that relates  $\log \mathcal{U}(t, 0; g)$  to  $X_{f,g}$  defined in (2.1). Here  $g(x) = e^{B(x)}$ . For any  $L > 0$ , define the stationary process

$$(3.1) \quad Y_L(t) := \mathbb{E}_W \log \int_0^L \frac{\mathcal{U}(t,x;g)}{\mathcal{U}(t,0;g)} e^{W(x)} dx, \quad t \geq 0,$$

where  $W$  is a Brownian bridge on  $[0, L]$  with  $W(0) = W(L) = 0$ , and is independent of  $\eta$  and  $B$ . The above expectation  $\mathbb{E}_W$  is only on  $W$ , so  $Y_L(t)$  retains the randomnesses from  $\eta$  and  $B$ . Here is the main result:

**Proposition 3.1.** *For any  $t, L > 0$ , we have*

$$(3.2) \quad \log \mathcal{U}(t, 0; g) - \mathbb{E}_g \mathbf{E} \log \mathcal{U}(t, 0; g) = I_L(t) + J_L(t),$$

with

$$(3.3) \quad I_L(t) := \mathbb{E}_f X_{f,g}(t) - \mathbb{E}_f \mathbf{E} X_{f,g}(t).$$

and

$$(3.4) \quad J_L(t) := Y_L(0) - Y_L(t).$$

*Proof.* We start from the following algebraic identity, which follows from the definitions (2.1) and (3.3):

$$(3.5) \quad \log \mathcal{U}(t, 0; g) + \mathbb{E}_f \log \int_0^L \frac{\mathcal{U}(t,x;g)}{\mathcal{U}(t,0;g)} f(x) dx = I_L(t) + \mathbb{E}_f \mathbf{E} X_{f,g}(t).$$

Now we consider each of the expectations in (3.5) (the one on the left and the one on the right). First, since  $\mathbb{E}_f$  is the expectation on  $f \sim \mu$ , we have

$$\mathbb{E}_f \log \int_0^L \frac{\mathcal{U}(t,x;g)}{\mathcal{U}(t,0;g)} f(x) dx = Y_L(t).$$

Secondly, we claim that

$$(3.6) \quad \mathbb{E}_f \mathbf{E} X_{f,g}(t) = \mathbb{E}_f \mathbf{E} \log \int_0^L \mathcal{U}(t,x;g) f(x) dx = \mathbb{E}_f \mathbf{E} \log \int_0^L \mathcal{U}(t,x;f) g(x) dx.$$

We will show (3.6) momentarily. First, we use it to finish the proof of the proposition. From (3.6), we further derive

$$(3.7) \quad \mathbb{E}_f \mathbf{E} X_{f,g}(t) = \mathbb{E}_f \mathbf{E} \log \int_0^L \frac{\mathcal{U}(t,x;f)}{\mathcal{U}(t,0;f)} g(x) dx + \mathbb{E}_f \mathbf{E} \log \mathcal{U}(t, 0; f).$$



Note that with  $f \sim \mu$ , we have

$$\frac{\mathcal{U}(t, \cdot; f)}{\mathcal{U}(t, 0; f)} \stackrel{\text{law}}{=} e^{W(\cdot)},$$

which implies

$$(3.8) \quad \mathbb{E}_f \mathbf{E} \log \int_0^L \frac{\mathcal{U}(t, x; f)}{\mathcal{U}(t, 0; f)} g(x) dx = \mathbb{E}_W \log \int_0^L e^{W(x)} g(x) dx, \\ = Y_L(0) + \log g(0) = Y_L(0)$$

since  $g(0) = 1$ ,  $\mu$ -a.s. Since, obviously,

$$(3.9) \quad \mathbb{E}_f \mathbf{E} \log \mathcal{U}(t, 0; f) = \mathbb{E}_g \mathbf{E} \log \mathcal{U}(t, 0; g),$$

combining (3.8) and (3.9) in (3.7), we obtain

$$\mathbb{E}_f \mathbf{E} X_{f,g}(t) = Y_L(0) + \mathbb{E}_g \mathbf{E} \log \mathcal{U}(t, 0; g), \quad \mu\text{-a.s. in } g.$$

Substituting for the respective expressions into (3.5) we get

$$\log \mathcal{U}(t, 0; g) + Y_L(t) = I_L(t) + Y_L(0) + \mathbb{E}_g \mathbf{E} \log \mathcal{U}(t, 0; g),$$

which completes the proof.

It remains to prove (3.6). The first equality is just the definition of  $X_{f,g}(t)$ . For the second equality, we use the propagator of SHE and write

$$\int_0^L \mathcal{U}(t, x; g) f(x) dx = \int_0^L \int_0^L \mathcal{Z}(t, x; 0, y) g(y) f(x) dx dy.$$

Then (3.6) is a consequence of the fact that for each fixed  $t > 0$ , we have

$$\{\mathcal{Z}(t, x; 0, y)\}_{x, y \in \mathbb{T}_L} \stackrel{\text{law}}{=} \{\mathcal{Z}(t, y; 0, x)\}_{x, y \in \mathbb{T}_L},$$

and the proof is complete.  $\square$

It turns out that the marginal distribution of  $Y_L(t)$  is rather explicit, and we have the following lemma:

**Lemma 3.2.** *For any  $t \geq 0$ , we have as  $L \rightarrow \infty$ ,*

$$(3.10) \quad \frac{Y_L(t) - \log L}{\sqrt{L}} \Rightarrow \mathbb{E}_{\mathcal{W}} \max_{x \in [0, 1]} \{\mathcal{B}(x) + \mathcal{W}(x)\}$$

*in distribution, and*

$$(3.11) \quad \frac{1}{L} \text{Var}[Y_L(t)] \rightarrow \text{Var}[\mathbb{E}_{\mathcal{W}} \max_{x \in [0, 1]} \{\mathcal{B}(x) + \mathcal{W}(x)\}] > 0.$$

*Here  $\mathcal{B}, \mathcal{W}$  are independent Brownian bridges on  $[0, 1]$  satisfying  $\mathcal{B}(0) = \mathcal{B}(1) = \mathcal{W}(0) = \mathcal{W}(1) = 0$ .*

*Proof.* With  $g(x) = e^{B(x)}$ , we know that for each  $t > 0$ ,

$$\log \mathcal{U}(t, \cdot; g) - \log \mathcal{U}(t, 0; g) \stackrel{\text{law}}{=} B(\cdot),$$

so

$$Y_L(t) \stackrel{\text{law}}{=} \mathbb{E}_{\mathcal{W}} \log \int_0^L e^{B(x) + \mathcal{W}(x)} dx \\ \stackrel{\text{law}}{=} \log L + \mathbb{E}_{\mathcal{W}} \log \int_0^1 e^{\sqrt{L}(\mathcal{B}(x) + \mathcal{W}(x))} dx,$$

where  $\mathcal{B}, \mathcal{W}$  are Brownian bridges on  $[0, 1]$  and we used the scaling property of the Brownian bridges in the second equality in law. By the Laplace principle, we have

$$\frac{1}{\sqrt{L}} \log \int_0^1 e^{\sqrt{L}(\mathcal{B}(x) + \mathcal{W}(x))} dx \rightarrow \max_{x \in [0, 1]} \{\mathcal{B}(x) + \mathcal{W}(x)\}$$

almost surely, as  $L \rightarrow \infty$ . Since

$$\min_{x \in [0,1]} \{\mathcal{B}(x) + \mathcal{W}(x)\} \leq \frac{1}{\sqrt{L}} \log \int_0^1 e^{\sqrt{L}(\mathcal{B}(x) + \mathcal{W}(x))} dx \leq \max_{x \in [0,1]} \{\mathcal{B}(x) + \mathcal{W}(x)\},$$

the laws of the random variable  $\frac{1}{\sqrt{L}} \log \int_0^1 e^{\sqrt{L}(\mathcal{B}(x) + \mathcal{W}(x))} dx$  and its square are tight as  $L \rightarrow \infty$ , so we can pass to the limit to complete the proof.  $\square$

To summarize, we have decomposed the centered height function into two parts

$$(3.12) \quad \log \mathcal{U}(t, 0; g) - \mathbb{E}_g \mathbf{E} \log \mathcal{U}(t, 0; g) = I_L(t) + J_L(t),$$

both parts of mean zero. (See (3.3) and (3.4).) For any  $t, L > 0$ , by Corollary 2.3, we have

$$(3.13) \quad \text{Var}[I_L(t)] = t\sigma_L^2,$$

with  $\sigma_L^2$  given in (2.10). By Lemma 3.2, we have

$$(3.14) \quad \text{Var}[J_L(t)] \leq CL$$

for some universal constant  $C > 0$ . In Section 4 below, we will show (Theorem 4.1) that

$$(3.15) \quad C^{-1}L^{-1/2} \leq \sigma_L^2 \leq CL^{-1/2}, \quad L \geq 1,$$

with some universal constant  $C > 0$ . That will be the last ingredient in the following proof of Theorem 1.1.

*Proof of Theorem 1.1.* Recall that we choose  $L = \lambda t^\alpha$ , where  $\lambda > 0$  is a given constant. Using (3.12), for any  $x \in \mathbb{T}_L$ , we write

$$\log \mathcal{U}(t, x; g) - \mathbb{E}_g \mathbf{E} \log \mathcal{U}(t, x; g) = I_L(t) + J_L(t) + K_L(t, x),$$

with

$$K_L(t, x) = \log \mathcal{U}(t, x; g) - \log \mathcal{U}(t, 0; g).$$

Note that  $\mathbb{E}_g \mathbf{E} K_L(t, x) = \mathbb{E}_B B(x) = 0$  and

$$\text{Var} K_L(t, x) = \text{Var} B(x) = x(L-x)/L \leq L/4.$$

The rest of the proof is based on applying (3.13), (3.14), (3.15) and the triangle inequality

$$\begin{aligned} \sqrt{\text{Var} I_L(t)} - \sqrt{\text{Var}[J_L(t) + K_L(t, x)]} &\leq \sqrt{\text{Var} \log \mathcal{U}(t, x; g)} \\ &\leq \sqrt{\text{Var} I_L(t)} + \sqrt{\text{Var}[J_L(t) + K_L(t, x)]}. \end{aligned}$$

If  $\alpha \in [0, 2/3)$ , then we have  $\text{Var}[J_L(t) + K_L(t, x)] \ll \text{Var} I_L(t) \asymp t^{1-\frac{\alpha}{2}}$ , which implies that

$$\text{Var} \log \mathcal{U}(t, x; g) \asymp t^{1-\frac{\alpha}{2}}.$$

If  $\alpha = 2/3$ , there is a constant  $C > 0$  so that

$$C^{-1}\lambda^{-1/2}t^{2/3} \leq \text{Var} I_L(t) \leq C\lambda^{-1/2}t^{2/3}$$

and

$$\text{Var}[J_L(t) + K_L(t, x)] \leq C\lambda t^{2/3}.$$

We clearly have the upper bound  $\text{Var} \log \mathcal{U}(t, x; g) \lesssim t^{2/3}$ . By choosing  $\lambda$  small so that

$$C^{-1}\lambda^{-1/2} > C\lambda,$$

we also obtain a lower bound. The proof is complete.  $\square$

*Remark 3.3.* We note that (1.3) covers the super-relaxation regime and only a part of the relaxation regime. The natural prediction in the relaxation and sub-relaxation regime is  $\text{Var} \log \mathcal{U}(t, 0) \asymp t^{2/3}$ . To further refine our result and to possibly cover the sub-relaxation regime, one needs to study the term  $J_L(t)$  on the r.h.s. of (3.12). For fixed  $t > 0$ , sending  $L \rightarrow \infty$  in (3.12), the l.h.s. should converge to the corresponding centered solution of the KPZ on the whole line, starting from a two-sided Brownian motion. For the first term on the r.h.s., we have  $I_L(t) \rightarrow 0$  as  $L \rightarrow \infty$ , which comes from the fact that  $\text{Var}[I_L(t)] = t\sigma_L^2 \rightarrow 0$  by (3.15). This implies that the second term on the r.h.s.,  $J_L(t) = Y_L(0) - Y_L(t)$ , must converge as  $L \rightarrow \infty$ . Therefore, to study the fluctuations in the sub-relaxation regime, it is necessary to understand the cancellation in  $Y_L(0) - Y_L(t)$ , because the process  $\{Y_L(t)\}_{t \geq 0}$  is stationary, and as shown in Lemma 3.2, its marginal distribution does not converge as  $L \rightarrow \infty$ .

#### 4. ASYMPTOTICS OF $\sigma_L^2$

Our goal in this section is to estimate the quantity  $\sigma_L^2$  appearing in (2.10). For  $j = 1, 2$ , we define the Brownian bridge  $U_j = B_j + B_3$ , so we can rewrite (2.10) as

$$(4.1) \quad \sigma_L^2 = \mathbb{E} \left[ \frac{\int_0^L e^{(U_1+U_2)(x)} dx}{\int_0^L e^{U_1(x)} dx \int_0^L e^{U_2(x)} dx} \right].$$

**Theorem 4.1.** *There exists a constant  $C > 0$  so that, for all  $L \geq 1$ ,*

$$C^{-1}L^{-1/2} \leq \sigma_L^2 \leq CL^{-1/2}.$$

The upper bound will be proved as Proposition 4.7, and the lower bound will be proved as Proposition 4.13. Throughout this section, we will let  $C$  represent a positive, finite constant that is always independent of  $L$  but may change from line to line in a computation. In the next two subsections, we introduce some preliminary notions.

**4.1. Translation-invariance.** We first make a simple reduction using the circular translation-invariance of the modified Brownian bridge with mean zero rather than endpoints equal to zero.

**Lemma 4.2.** *We have*

$$(4.2) \quad \sigma_L^2 = L \mathbb{E} \left[ \frac{1}{\int_0^L e^{U_1(x)} dx \int_0^L e^{U_2(x)} dx} \right].$$

*Proof.* Let

$$D_j = \frac{1}{L} \int_0^L U_j(x) dx.$$

Then  $(\tilde{U}_1, \tilde{U}_2) := (U_1 - D_1, U_2 - D_2)$  is invariant under circular shifts of the interval  $[0, L]$ , i.e. under translations considered on  $\mathbb{R}/(L\mathbb{Z})$ . Therefore, we have

$$\begin{aligned} \sigma_L^2 &= \mathbb{E} \left[ \frac{\int_0^L e^{(\tilde{U}_1+\tilde{U}_2)(x)} dx}{\int_0^L e^{\tilde{U}_1(x)} dx \int_0^L e^{\tilde{U}_2(x)} dx} \right] = L \mathbb{E} \left[ \frac{e^{(\tilde{U}_1+\tilde{U}_2)(0)}}{\int_0^L e^{\tilde{U}_1(x)} dx \int_0^L e^{\tilde{U}_2(x)} dx} \right] \\ &= L \mathbb{E} \left[ \frac{1}{\int_0^L e^{U_1(x)} dx \int_0^L e^{U_2(x)} dx} \right] \end{aligned}$$

and hence (4.2).  $\square$

*Remark 4.3.* The decay rate  $\sigma_L^2 \asymp L^{-1/2}$  results from the correlation between  $U_1$  and  $U_2$ . For two extreme cases (i)  $U_1 = U_2$  and (ii)  $U_1$  being independent of  $U_2$ , one can carry out explicit calculations. In case (i), by the above lemma, we can write

$$\sigma_L^2 = L \mathbb{E} \left[ \frac{1}{\left( \int_0^L e^{U_1(x)} dx \right)^2} \right].$$

By the calculation in [36, Proposition 4.1], we have  $\sigma_L^2 \asymp 1$ . In case (ii), similarly we have

$$\sigma_L^2 = L \left( \mathbb{E} \left[ \frac{1}{\int_0^L e^{U_1(x)} dx} \right] \right)^2.$$

By another application of the translation-invariance as in the proof of Lemma 4.2 (or see e.g. [42, Proposition 5.9]), we actually have in this case that  $\sigma_L^2 = L^{-1}$ . Therefore, the optimal decay rate in Theorem 4.1 comes from a delicate analysis which hinges on the correlation between  $U_1, U_2$ . Our analysis also applies to the case when  $U_1$  and  $U_2$  are Brownian bridges with a general positive scalar correlation  $r$  (i.e.  $\mathbb{E}[U_1(x)U_2(x)] = r\sqrt{\mathbb{E}U_1(x)^2\mathbb{E}U_2(x)^2}$ ), in which case the corresponding decay rate would be

$$(4.3) \quad L \mathbb{E} \left[ \frac{1}{\int_0^L e^{U_1(x)} dx \int_0^L e^{U_2(x)} dx} \right] \asymp L^{1 - \frac{\pi}{\pi - \arccos(r)}}.$$

**4.2. Change of variables.** We now make a further simplification by performing an affine change of coordinates so that we work with two independent standard Brownian bridges rather than two correlated Brownian bridges. From now on, we identify  $\mathbb{R}^2$  and  $\mathbb{C}$ . A straightforward covariance calculation shows that if we define, for  $k = 1, 2$ ,

$$v_k = -\frac{\sqrt{2}}{2}(\sqrt{3}, (-1)^k) = -\sqrt{2}e^{(-1)^k i\pi/6},$$

then there is a standard two-dimensional Brownian bridge  $V = (V_1, V_2)$  so that

$$v_k \cdot V = U_k, \quad k = 1, 2.$$

We define the probability measure  $\mathbb{P}_{x_1, u_1}^{x_2, u_2}$  to be the law of a standard two-dimensional Brownian bridge  $V$  on  $[x_1, x_2]$  with  $V(x_\ell) = u_\ell$ ,  $\ell = 1, 2$ . Thus  $\mathbb{P} = \mathbb{P}_{0,0}^{L,0}$ . We let  $\mathbb{E}_{x_1, u_1}^{x_2, u_2}$  be the corresponding expectation.

Given our change of variables, we can rewrite (4.2) as

$$(4.4) \quad \sigma_L^2 = L \mathbb{E} \left[ \prod_{k=1}^2 \left( \int_0^L e^{v_k \cdot V(x)} dx \right)^{-1} \right].$$

Define, for  $v \in \mathbb{R}^2$ ,

$$(4.5) \quad \omega(v) = \max_{k \in \{1, 2\}} [v_k \cdot v].$$

Thus for any  $x \in [0, L]$ , we have  $\omega(V(x)) = \max_{k=1,2} U_k(x)$ . This means that upper bounds on  $\omega(V(x))$ , such as in the barriers we construct below, translate directly to upper bounds on the correlated Brownian bridges  $U_1, U_2$ .

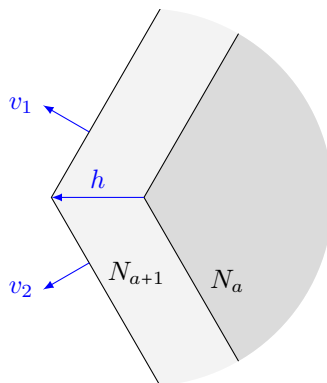


FIGURE 1. The vectors  $v_1$  and  $v_2$  are perpendicular to the edges of the wedges  $N_a$ , and the vector  $h$  is the offset between  $N_a$  and  $N_{a+1}$ .

**4.3. Brownian motion in a wedge.** Roughly speaking, the main contribution to the expectation in (4.4) comes from those paths of  $V$  such that

$$\omega(V(x)) = \max_{k \in \{1,2\}} U_k(x)$$

is bounded from above, uniformly for all  $x \in [0, L]$ . This is because, if there is even an order- $O(1)$  interval of time such that  $\omega(V(x))$  is large, this will cause the corresponding integral inside the expectation in (4.4) to be exponentially large, and thus the contribution to the expectation to be exponentially small. To understand the probability that  $\sup_{x \in [0, L]} \omega(V(x)) \leq a$  for a given barrier  $a$ , we will need some

facts about two independent Brownian motions killed upon exiting a wedge. The angle of the wedge will come from the particular correlation between  $U_1$  and  $U_2$ .

For  $a \in \mathbb{R}$ , we define the translated wedge

$$(4.6) \quad N_a := \omega^{-1}((-\infty, a]) = \{v \in \mathbb{R}^2 \mid v_k \cdot v \leq a \text{ for } k = 1, 2\}.$$

We use the convention that  $N_\infty = \mathbb{R}^2$ , and define

$$N := N_0 = \{r e^{i\theta} \mid r \geq 0 \text{ and } \theta \in [-\pi/3, \pi/3]\}.$$

Now fix the vector

$$(4.7) \quad h = (-\sqrt{6}/3, 0).$$

Since  $v_k \cdot h = 1$  for  $k = 1, 2$ , we have for all  $a \in \mathbb{R}$  and  $v \in \mathbb{R}^2$  that

$$(4.8) \quad \omega(v + ah) = \omega(v) + a.$$

Thus, we have

$$(4.9) \quad N_a = N + ah.$$

See Figure 1 for an illustration of the geometry.

With the above definitions, we see that for any  $a \in \mathbb{R}$  and any interval  $I \subseteq [0, L]$ , the event that

$$\max_{x \in I} [U_1(x) \vee U_2(x)] \leq a$$

translates to the event that  $V(x) \in N_a$  for all  $x \in I$ , i.e. that the independent Brownian bridges stay in the wedge on the interval  $I$ . Thus, for  $a > 0$  and (possibly random)  $0 \leq x_1 < x_2 \leq L$ , we define the event

$$(4.10) \quad E_{a, x_1, x_2} = \{V(x) \in N_a \text{ for all } x \in [x_1, x_2]\}.$$

By an elementary property of the Brownian bridge, we see that  $\mathbb{P}[E_{0,1,L}] = 0$ , and recalling the convention that  $N_\infty = \mathbb{R}^2$ , we see that  $\mathbb{P}[E_{\infty,x_1,x_2}] = 1$  for any  $x_1, x_2$ .

We recall the standard two-dimensional heat kernel

$$(4.11) \quad p_x(u_1, u_2) = (2\pi x)^{-1} e^{-|u_1 - u_2|^2/(2x)}, \quad x > 0, u_1, u_2 \in \mathbb{R}^2.$$

We now introduce the heat kernel of the two-dimensional standard Brownian motion killed on the boundary of the wedge  $N$ . Let  $p_x^N$  be the transition kernel of standard two-dimensional Brownian motion in  $N$  killed on  $\partial N$ , so, if  $Y$  is a Brownian motion with  $Y(0) = u \in N$ , then

$$p_x^N(u, v) = \mathbb{P}[Y(x) \in dv \text{ and } Y([0, x]) \subset N].$$

In other words, for any subset  $A \subset N$ , we have

$$\mathbb{P}[Y(x) \in A \text{ and } Y([0, x]) \subset N] = \int_A p_x^N(u, v) dv.$$

The following explicit formula for  $p_x^N$ , with the arguments in polar coordinates, is known (see [21, p. 379] or [10, Lemma 1]): for  $r_1, r_2 > 0$  and  $\theta_1, \theta_2 \in [-\pi/3, \pi/3]$ ,

$$(4.12) \quad p_x^N(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) = \frac{1}{x} \sum_{j=1}^{\infty} I_{\frac{3}{2}j} \left( \frac{r_1 r_2}{x} \right) \prod_{k=1}^2 \left[ e^{-r_k^2/(2x)} \sin \left( \frac{3}{2}j(\theta_k + \pi/3) \right) \right].$$

Here,  $I_\alpha(\nu)$  is the modified Bessel function of the first kind. We review the properties of  $I_\alpha(\nu)$  that we will need in Appendix A. At this point, it is worth emphasizing that the angle of the wedge  $N$ , which is  $2\pi/3$ , enters the formula (4.12) through the degrees of the Bessel functions and the arguments in the sine functions. For Brownian bridges with different correlations, corresponding to wedges with a different angle, the formula (4.12) would be different. We further define

$$(4.13) \quad p_x^{N_a}(u_1, u_2) = p_x^N(u_1 - ah, u_2 - ah),$$

which by (4.9) is the transition kernel for Brownian motion on  $N_a$  killed on  $\partial N_a$ .

4.3.1. *Estimates on  $p_x^N$ .* We now prove some estimates on the heat kernel  $p_x^N$ . We will use the estimates on modified Bessel functions of the first kind proved in Appendix A. The bounds here are suited to understanding  $p_x^N(u_1, u_2)$  when  $x$  is not too small relative to  $u_1$  and  $u_2$ , which is all we need in our analysis. The small- $x$  regime requires different analysis; see for example [26].

**Lemma 4.4.** *There exists a constant  $C > 0$  so that for all  $x, r_1, r_2 > 0$  and  $\theta_1, \theta_2 \in [-\pi/3, \pi/3]$  we have*

$$(4.14) \quad p_x^N(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) \leq \frac{C(r_1 r_2)^{3/2}}{x^{5/2}} e^{(Cr_1 r_2 - r_1^2 - r_2^2)/(2x)} \prod_{k=1}^2 (\pi/3 - |\theta_k|).$$

*Proof.* Using the identity  $|\sin \theta| \leq |\theta|$ , we have for all  $\theta \in [-\pi/3, \pi/3]$  that

$$\left| \sin \left( \frac{3}{2}j(\theta + \pi/3) \right) \right| \leq \frac{3}{2}j (|\pi/3 + \theta| \wedge |\pi/3 - \theta|) = \frac{3}{2}j (\pi/3 - |\theta|).$$

Using this and Lemma A.2 in (4.12), we obtain

$$\begin{aligned} p_x^N(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) &\leq \frac{9}{4x} \left( \prod_{k=1}^2 \left[ e^{-r_k^2/(2x)} (\pi/3 - |\theta_k|) \right] \right) \sum_{j=1}^{\infty} j^2 I_{\frac{3}{2}j} \left( \frac{r_1 r_2}{x} \right) \\ &\leq \frac{C(r_1 r_2)^{3/2}}{x^{5/2}} e^{-(r_1^2 + r_2^2)/(2x) + Cr_1 r_2/x} \prod_{k=1}^2 (\pi/3 - |\theta_k|), \end{aligned}$$

which is (4.14).  $\square$

**Lemma 4.5.** *There exists a constant  $C > 0$  so that, for all  $x, a > 0$ , we have*

$$(4.15) \quad p_x^{N_a}(0, 0) = p_x^N(-ah, -ah) \geq C^{-1} a^3 x^{-5/2} e^{-Ca^2/x}.$$

*Proof.* Recall that  $h = (-\sqrt{6}/3, 0)$ . We use (4.12) to write

$$p_x^N(-ah, -ah) = \frac{1}{xe^{\frac{2a^2}{3x}}} \sum_{j=1}^{\infty} I_{\frac{3}{2}j} \left( \frac{2a^2}{3x} \right) \sin^2 \left( \frac{\pi j}{2} \right).$$

Since all of the terms of the above series are positive, it suffices to consider the first one and apply Lemma A.1 to complete the proof.  $\square$

A consequence of the above two lemmas is

**Corollary 4.6.** *There exists a constant  $C > 0$  so that, for all  $L \geq 1$ , we have*

$$(4.16) \quad \frac{a^3}{CL^{3/2}} e^{-Ca^2/L} \leq \mathbb{P}[E_{a,0,L}] \leq \frac{Ca^3}{L^{3/2}}.$$

*Proof.* First, we have

$$\mathbb{P}[E_{a,0,L}] = \frac{p_L^{N_a}(0, 0)}{p_L(0, 0)} = 2\pi L p_L^{N_a}(0, 0) = 2\pi L p_L^N(-ah, -ah).$$

where the last “=” is by (4.13). Further applying (4.14) and (4.15) yields

$$\frac{a^3}{CL^{3/2}} e^{-Ca^2/L} \leq \mathbb{P}[E_{a,0,L}] \leq C \frac{a^3}{L^{3/2}} e^{Ca^2/L}$$

with some constant  $C > 0$ . The seemingly improved upper bound in (4.16) follows when we note that a probability is always at most 1.  $\square$

**4.4. Upper bound.** In this subsection we prove the upper bound claimed in the statement of Theorem 4.1, namely the following proposition.

**Proposition 4.7.** *There exists a constant  $C > 0$  so that, for all sufficiently large  $L$ , we have  $\sigma_L^2 \leq CL^{-1/2}$ .*

In order to prove Proposition 4.7, we will partition the probability space according to the maximum value of  $\omega(V(x)) = \max_{k \in \{1,2\}} [U_k(x)]$ , as  $x$  varies in  $[0, L]$ . By Corollary 4.6, the probability that  $\omega(V(x)) \leq a$  for all  $x \in [0, L]$  is of order  $L^{-3/2} a^3$  if  $L$  is of the order  $O(a^2)$ . On the other hand, as mentioned in the previous subsection, if  $\omega(V(x))$  is of the order  $a$  at some point  $x$ , then the contribution to the expectation in (4.4) should be exponentially small in  $a$ . Thus we should be able to integrate in  $a$  to achieve a bound of the order  $L^{-3/2}$ , which will yield  $\sigma_L^2 \leq L^{-1/2}$  when multiplied by  $L$  as in (4.4).

**4.4.1. Scales and stopping times.** Our first step in carrying out our program for the upper bound is to define a sequence of scales for the maximum of  $\omega(V(x))$ , as  $x$  varies in  $[0, L]$ . Let

$$(4.17) \quad \xi_0 = 100, \quad \delta = 1/100 \quad \text{and} \quad M = \lfloor \delta \log_{\xi_0} L \rfloor + 1.$$

For  $m \in \{0, 1, \dots, M\}$ , define

$$(4.18) \quad q_m = \begin{cases} \xi_0^m - 1, & 0 \leq m < M; \\ \infty, & m = M. \end{cases}$$

In particular, this means that (whenever  $L \geq 10$ )

$$(4.19) \quad 1 \leq q_m \leq L^\delta \leq L^{1/2} \leq L/2 \quad \text{for all } m \in \{1, \dots, M-1\}.$$



For  $m \in \{1, 2, \dots, M\}$ , define (recalling the definition (4.10))

$$(4.20) \quad A_m = E_{q_m, 0, L} \setminus E_{q_{m-1}, 0, L},$$

which is the event that the maximum of  $\omega(V(x))$ ,  $x \in [0, L]$ , is between  $q_{m-1}$  and  $q_m$ . Note that the events  $A_m$  are pairwise disjoint and

$$\mathbb{P} \left[ \bigcup_{m=1}^M A_m \right] = 1 - \mathbb{P}[E_{q_0, 0, L}] = 1 - \mathbb{P}[E_{0, 0, L}] = 1.$$

The last “=” comes from the fact that the Brownian bridge  $V$  starts at 0 and thus will exit the cone  $N$  immediately with probability 1. Therefore, we have by (4.4) that

$$(4.21) \quad \sigma_L^2 = L \sum_{m=1}^M G_m,$$

where

$$(4.22) \quad G_m := \mathbb{E} \left[ \prod_{k=1}^2 \left( \int_0^L e^{v_k \cdot V(x)} dx \right)^{-1}; A_m \right].$$

Now we define a sequence of stopping times corresponding to our scales  $q_m$ . We will need to consider stopping times with respect to both the forward and the backward canonical filtrations of the Brownian bridge  $V$ . Let  $\{\mathcal{F}_x\}_{x \in [0, L]}$  be the canonical forward filtration, so  $\mathcal{F}_x$  is the  $\sigma$ -algebra generated by  $V|_{[0, x]}$ , and let  $\{\tilde{\mathcal{F}}_x\}_{x \in [0, L]}$  be the canonical backward filtration, so  $\tilde{\mathcal{F}}_x$  is the  $\sigma$ -algebra generated by  $V|_{[x, L]}$ . We note that  $V(x)/(1-x/L)$ ,  $0 \leq x < L$ , is a continuous  $\{\mathcal{F}_x\}$ -martingale with the covariation matrix of the form  $x(L-x)^{-1}I_2$ . Here  $I_2$  is the  $2 \times 2$  identity matrix. Hence it is a time-change of a two-dimensional standard Brownian motion. Likewise,  $V(x)/(x/L)$ ,  $0 < x \leq L$ , is a continuous backward  $\{\tilde{\mathcal{F}}_x\}$ -martingale, with the covariation matrix  $(L-x)x^{-1}I_2$ . We define our stopping times for these martingales rather than  $V$  because it will aid the computation of certain harmonic measures; see Lemma 4.12 below.

**Definition 4.8.** For  $1 \leq m \leq M$ , let  $\tau_m$  (resp.  $\tilde{\tau}_m$ ) be the first (resp. last) time  $x \in [0, L]$  such that  $\omega(V(x))/(1-x/L) = q_{m-1}$  (resp.  $\omega(V(x))/(x/L) = q_{m-1}$ ). It is clear that, with probability 1, such stopping times exist. Let  $k_m$  (resp.  $\tilde{k}_m$ ) be such that  $\omega(V(\tau_m)) = v_{k_m} \cdot V(\tau_m)$  (resp.  $\omega(V(\tilde{\tau}_m)) = v_{\tilde{k}_m} \cdot V(\tilde{\tau}_m)$ ).

From the definition, we see that  $\tau_m$  is a  $\{\mathcal{F}_x\}$ -stopping time and  $\tilde{\tau}_m$  is a  $\{\tilde{\mathcal{F}}_x\}$ -stopping time. With the scales and stopping times defined, we are ready to set up our framework for bounding  $G_m$ .

**Lemma 4.9.** For all  $L \geq 6$  and all  $1 \leq m \leq M$ , we have

$$(4.23) \quad G_m^2 \leq 4e^{-q_{m-1}} Q_{q_m}(L, 0) \mathbb{E}[Q_{q_m}(L - \tau_m, V(\tau_m)); \tau_m \leq L/2],$$

where

$$(4.24) \quad Q_q(J, u) := \mathbb{E}_{0, u}^{J, 0} \left[ \int_0^1 e^{2\sqrt{2}|V(x)-u|} dx; E_{q, 0, J} \right].$$

Before we prove Lemma 4.9, let us briefly explain its interpretation. By definition,  $G_m$  represents the contribution to  $\sigma_L^2$  from the paths  $V$  such that the maximum of  $\omega(V(x))$  is between  $q_{m-1}$  and  $q_m$ . The term  $e^{-q_{m-1}}$  thus bounds the size of the contribution we expect on this event. The terms  $Q_{q_m}$  encapsulate two different effects: first, the error in upper-bounding a unit-length integral by the value of the integrand at the starting point (which should be of order 1), and second, the probability of  $\omega(V(x))$  staying below  $q_m$  (which is roughly of order  $L^{-3/2}$  times a

polynomial in  $q_m$  – see Corollary 4.6). The restriction  $\tau_m \leq L/2$  in (4.23) is possible by a forward/backward symmetrization argument and will be important in the sequel.

*Proof of Lemma 4.9.* For any  $x_1, x_2 \in \mathbb{R}$ , define

$$Z_k(x_1, x_2) = \left( \int_{x_1 \vee 0}^{x_2 \wedge L} e^{v_k \cdot V(x)} dx \right)^{-1}.$$

We first note that

$$Z_1(0, L) \vee Z_2(0, L) \leq Z_1(0, 1) \vee Z_2(0, 1)$$

and

$$Z_1(0, L) \wedge Z_2(0, L) \leq Z_{k_m}(\tau_m, \tau_m + 1) \wedge Z_{\tilde{k}_m}(\tilde{\tau}_m - 1, \tilde{\tau}_m)$$

for all  $1 \leq m \leq M$ . Thus we can estimate the random variable inside the expectation in (4.22) by

$$\begin{aligned} \prod_{k=1}^2 \left( \int_0^L e^{v_k \cdot V(x)} dx \right)^{-1} &= (Z_1(0, L) \vee Z_2(0, L)) (Z_1(0, L) \wedge Z_2(0, L)) \\ &\leq (Z_1(0, 1) \vee Z_2(0, 1)) (Z_{k_m}(\tau_m, \tau_m + 1) \wedge Z_{\tilde{k}_m}(\tilde{\tau}_m - 1, \tilde{\tau}_m)). \end{aligned}$$

Using this in (4.22) and then applying the Cauchy–Schwarz inequality, we obtain

$$(4.25) \quad \begin{aligned} G_m^2 &\leq \mathbb{E}[Z_1(0, 1)^2 + Z_2(0, 1)^2; A_m] \\ &\quad \times \mathbb{E}[(Z_{k_m}(\tau_m, \tau_m + 1) \wedge Z_{\tilde{k}_m}(\tilde{\tau}_m - 1, \tilde{\tau}_m))^2; A_m]. \end{aligned}$$

Applying Jensen’s inequality (recalling that  $|v_k| = \sqrt{2}$ ), we bound the first factor on the r.h.s. of (4.25) by

$$(4.26) \quad \mathbb{E}[Z_1(0, 1)^2 + Z_2(0, 1)^2; A_m] \leq 2Q_{q_m}(L, 0).$$

Now we consider the second factor on the r.h.s. of (4.25). By definition, on the event  $E_{q_{m-1}, 0, L}^c$ , there exists a time  $x \in [0, L]$  such that  $\omega(V(x)) > q_{m-1}$ . This means in particular that  $\omega(V(x))/(1 - x/L) > q_{m-1}$  and  $\omega(V(x))/(x/L) > q_{m-1}$ , so if  $x \in [0, L/2]$ , then  $\tau_m \leq L/2$ , while if  $x \in [L/2, L]$ , then  $\tilde{\tau}_m \geq L/2$ . So in fact

$$(4.27) \quad \mathbb{1}_{E_{q_{m-1}, 0, L}^c} \leq \mathbb{1}_{\tau_m \leq L/2} + \mathbb{1}_{\tilde{\tau}_m \geq L/2}.$$

Using (4.27) and recalling that  $A_m = E_{q_m, 0, L} \setminus E_{q_{m-1}, 0, L}$ , we obtain

$$(4.28) \quad \begin{aligned} &\mathbb{E}[(Z_{k_m}(\tau_m, \tau_m + 1) \wedge Z_{\tilde{k}_m}(\tilde{\tau}_m - 1, \tilde{\tau}_m))^2; A_m] \\ &\leq \mathbb{E}[Z_{k_m}(\tau_m, \tau_m + 1)^2 \mathbb{1}_{\tau_m \leq L/2}; E_{q_m, 0, L}] + \mathbb{E}[Z_{\tilde{k}_m}(\tilde{\tau}_m - 1, \tilde{\tau}_m)^2 \mathbb{1}_{\tilde{\tau}_m \geq L/2}; E_{q_m, 0, L}] \\ &= 2\mathbb{E}[Z_{k_m}(\tau_m, \tau_m + 1)^2 \mathbb{1}_{\tau_m \leq L/2}; E_{q_m, 0, L}], \end{aligned}$$

where the last “=” is by the time-reversal symmetry of the Brownian bridge. Note that when  $\tau_m \leq L/2$ , we have  $[\tau_m, \tau_m + 1] \subset [0, L]$ , in which case

$$Z_{k_m}(\tau_m, \tau_m + 1) = \left( \int_{\tau_m}^{\tau_m + 1} e^{v_k \cdot V(x)} dx \right)^{-1}.$$

Therefore, we apply Jensen’s inequality again, which leads to

$$(4.29) \quad \begin{aligned} &\mathbb{E}[(Z_{k_m}(\tau_m, \tau_m + 1) \wedge Z_{\tilde{k}_m}(\tilde{\tau}_m - 1, \tilde{\tau}_m))^2; A_m] \\ &\leq 2\mathbb{E}[Z'_{k_m}(\tau_m, \tau_m + 1); E_{q_m, 0, L} \cap \{\tau_m \leq L/2\}] \\ &= 2\mathbb{E}[Z'_{k_m}(\tau_m, \tau_m + 1); E_{q_m, \tau_m, L} \cap \{\tau_m \leq L/2\}], \end{aligned}$$

with

$$Z'_k(x_1, x_2) = \int_{x_1}^{x_2} e^{-2v_k \cdot V(x)} dx.$$

For the last “=”, it is because, if  $x \leq \tau_m \leq L/2$ , then (i) if  $\omega(V(x)) \geq 0$ , we have

$$(4.30) \quad \omega(V(x)) \leq \frac{\omega(V(x))}{1-x/L} \leq q_{m-1} \leq q_m,$$

and (ii) if  $\omega(V(x)) < 0$ , we also have  $\omega(V(x)) < q_m$ . This implies

$$E_{q_m,0,L} \cap \{\tau_m \leq L/2\} = E_{q_m,\tau_m,L} \cap \{\tau_m \leq L/2\}.$$

Now since (still on the event  $\tau_m \leq L/2$ ) we have

$$v_{k_m} \cdot V(\tau_m) = (1 - \tau_m/L)q_{m-1} \geq \frac{1}{2}q_{m-1},$$

we can further bound from above the r.h.s. of (4.29) by (recalling that  $|v_k| = \sqrt{2}$  and using the strong Markov property of  $V$ )

$$\begin{aligned} & 2\mathbb{E} \left[ e^{-2v_{k_m} \cdot V(\tau_m)} \int_{\tau_m}^{\tau_m+1} e^{2\sqrt{2}|V(x)-V(\tau_m)|} dx; E_{q_m,\tau_m,L} \cap \{\tau_m \leq L/2\} \right] \\ & \leq 2e^{-q_{m-1}} \mathbb{E}[Q_{q_m}(L - \tau_m, V(\tau_m)); \tau_m \leq L/2]. \end{aligned}$$

Using this in (4.29) and combining with (4.26), we complete the proof.  $\square$

4.4.2. *The quantity  $Q_q$ .* In order to use (4.23) to prove Proposition 4.7, we need to estimate  $Q_q(J, u)$ . This is done in the next lemma.

**Lemma 4.10.** *There is a constant  $C$  so that for any  $J \geq 2$ , any  $q \in [1, J]$ , and any  $u \in \mathbb{R}^2$  we have*

$$(4.31) \quad Q_q(J, u) \leq C J^{-3/2} q^{3/2} e^{CqJ^{-1}(q+|u|)} (q^{1/2} + |u|^{1/2}) (q + |\omega(u)|).$$

Moreover, for any  $q \in (0, \infty]$ ,  $J \geq 2$ , and  $u \in \mathbb{R}^2$ , we have

$$(4.32) \quad Q_q(J, u) \leq C e^{C|u|/J}.$$

Thanks to our symmetrization procedure in the proof of Lemma 4.9,  $Q_q(J, u)$  is only applied in (4.23) with  $J \in [L/2, L]$ , so  $J$  is of order  $O(L)$ . The dependence on  $u$  is somewhat more problematic, as in (4.23) we must take  $u = V(\tau_m)$ . While we know that  $V(\tau_m)/(1 - \tau_m/L) \in \partial N_{q_{m-1}}$ , the boundary  $\partial N_{q_{m-1}}$  of the wedge is non-compact, and some additional control will be needed. That will be the goal of Section 4.4.3 below.

*Proof of Lemma 4.10.* Using the definition (4.24) and the Markov property of the Brownian bridge, we have

$$\begin{aligned} Q_q(J, u) & \leq \mathbb{E}_{0,u}^{J,0} \left[ \int_0^1 e^{2\sqrt{2}|V(x)-u|} dx; E_{q,1,J} \right] \\ & = \mathbb{E}_{0,u}^{J,0} \left[ \int_0^1 e^{2\sqrt{2}|V(x)-u|} dx \times \mathbb{P}_{1,V(1)}^{J,0} [E_{q,1,J}] \right] \\ (4.33) \quad & = \int_{N_q} p_{\frac{J-1}{J}}(w - (1 - 1/J)u) \mathbb{E}_{0,u}^{1,w} \left[ \int_0^1 e^{2\sqrt{2}|V(x)-u|} dx \right] \mathbb{P}_{1,w}^{J,0} [E_{q,1,J}] dw. \end{aligned}$$

The second term in the product on the last line of (4.33) is bounded as

$$(4.34) \quad \mathbb{E}_{0,u}^{1,w} \left[ \int_0^1 e^{2\sqrt{2}|V(x)-u|} dx \right] = \int_0^1 \mathbb{E}_{0,u}^{1,w} e^{2\sqrt{2}|V(x)-u|} dx \leq C e^{C|w-u|}$$

for a universal constant  $C$ . Using (4.34) and the trivial bound  $\mathbb{P}_{1,w}^{J,0} [E_{q,1,J}] \leq 1$  in (4.33) implies that

$$Q_q(J, u) \leq C \int_{\mathbb{R}^2} p_{\frac{J-1}{J}}(w - (1 - 1/J)u) e^{C|w-u|} dw \leq C e^{C|u|/J} \int_{\mathbb{R}^2} p_{\frac{J-1}{J}}(w) e^{C|w|} dw$$

and hence (4.32).

To derive (4.31), we proceed by noting that the third term in the product on the farthest right-hand side of (4.33) is bounded by Lemma 4.11 below as (when  $J \geq 2$  and  $w \in N_q$ )

$$(4.35) \quad \mathbb{P}_{1,w}^{J,0}[E_{q,1,J}] \leq C J^{-3/2} |w - qh|^{1/2} q^{3/2} (q - \omega(w)) e^{C J^{-1} q |w - qh|}.$$

Using (4.34) and (4.35) in (4.33), we obtain

$$\begin{aligned} Q_q(J, u) &\leq \frac{C q^{3/2}}{J^{3/2}} \int_{N_q} p_{\frac{J-1}{J}}(w - (1 - 1/J)u) |w - qh|^{1/2} (q - \omega(w)) \\ &\quad \times e^{C[|u-w| + J^{-1}q|w-qh|]} dw \\ &\leq \frac{C q^{3/2}}{J^{3/2}} \int_{\mathbb{R}^2} p_{\frac{J-1}{J}}(w) |w + (1 - 1/J)u - qh|^{1/2} (q - \omega(w + (1 - 1/J)u))_+ \\ &\quad \times e^{C[|u/J-w| + J^{-1}q|w+(1-1/J)u-qh|]} dw. \end{aligned}$$

Then we collect the following estimates (which use the assumption  $q \in [1, J]$ )

$$\begin{aligned} |w + (1 - 1/J)u - qh|^{1/2} &\leq C[|w|^{1/2} + q^{1/2} + |u|^{1/2}], \\ (q - \omega(w + (1 - 1/J)u))_+ &\leq C[|w| + q + |\omega(u)|], \text{ and} \\ |u/J - w| + J^{-1}q|w + (1 - 1/J)u - qh| &\leq C[|w| + qJ^{-1}(q + |u|)]. \end{aligned}$$

The first and the third inequalities are rather straightforward to obtain. For the second one, it is enough to write

$$\begin{aligned} q - \omega(w + (1 - 1/J)u) &= q - \max_{k=1,2} v_k \cdot [w + (1 - 1/J)u] \\ &= q + \min_{k=1,2} [-v_k \cdot w - (1 - 1/J)v_k \cdot u] \\ &\leq q - v_{k_u} \cdot w - (1 - 1/J)\omega(u) \\ &\leq q + C|w| + |\omega(u)|, \end{aligned}$$

where  $k_u \in \{1, 2\}$  is chosen so that  $v_{k_u} \cdot u = \omega(u)$ . Using the above estimates, we conclude that

$$\begin{aligned} Q_q(J, u) &\leq C J^{-3/2} q^{3/2} e^{CqJ^{-1}(q+|u|)} \int_{\mathbb{R}^2} p_{\frac{J-1}{J}}(w) [|w|^{1/2} + q^{1/2} + |u|^{1/2}] \\ &\quad \times [|w| + q + |\omega(u)|] e^{C|w|} dw \\ &\leq C J^{-3/2} q^{3/2} e^{CqJ^{-1}(q+|u|)} [q^{1/2} + |u|^{1/2}] [q + |\omega(u)|], \end{aligned}$$

where in the last inequality we again used that  $q \geq 1$ . This completes the proof of (4.31).  $\square$

Now we must prove the lemma used above, which is a generalization of Corollary 4.6.

**Lemma 4.11.** *There exists a constant  $C > 0$  so that for every  $J > 0$ ,  $q > 0$ , and  $w \in N_q$ , we have that*

$$(4.36) \quad \mathbb{P}_{0,w}^{J,0}[E_{q,0,J}] \leq C J^{-3/2} |w - qh|^{1/2} q^{3/2} (q - \omega(w)) e^{C J^{-1} q |w - qh|}.$$

*Proof.* We note that

$$(4.37) \quad \mathbb{P}_{0,w}^{J,0}[E_{q,0,J}] = \frac{p_J^{N_q}(w, 0)}{p_J(w, 0)}.$$

We bound the numerator using (4.13) and (4.14) (recalling that  $h = (-\sqrt{6}/3, 0)$ )

$$(4.38) \quad \begin{aligned} p_J^{N_q}(w, 0) &= p_J^N(w - qh, -qh) \\ &\leq \frac{C|w - qh|^{3/2} q^{3/2}}{J^{5/2}} |\sin(\pi/3 - |\arg(w - qh)|)| e^{(Cq|w - qh| - |w - qh|^2 - q^2|h|^2)/(2J)}, \end{aligned}$$

where we used the inequality  $|\pi/3 - \theta| \leq C \sin(\pi/3 - \theta)$  for  $\theta \in [0, \pi/3]$ . Here the convention is that  $\arg$  takes values in  $[-\pi/2, \pi/2]$ .

Assume now without loss of generality that the second coordinate of  $w$  is non-negative, i.e.,  $w_2 \geq 0$ . This implies  $\arg(w - qh) \in [0, \pi/3]$  given the assumption of  $w \in N_q$ . Since  $\arg(v_1) = 5\pi/6$  and  $|v_1| = \sqrt{2}$ , we have

$$\begin{aligned} v_1 \cdot (w - qh) &= \sqrt{2}|w - qh| \cos(5\pi/6 - \arg(w - qh)) \\ &= -\sqrt{2}|w - qh| \sin(\pi/3 - \arg(w - qh)). \end{aligned}$$

This means that

$$0 \leq \sin(\pi/3 - \arg(w - qh)) = \frac{v_1 \cdot (qh - w)}{\sqrt{2}|w - qh|} = \frac{q - v_1 \cdot w}{\sqrt{2}|w - qh|} = \frac{q - \omega(w)}{\sqrt{2}|w - qh|}.$$

In the last “=”, we used the fact that  $v_1 \cdot w \geq v_2 \cdot w$  if  $w_2 \geq 0$ . Using this in (4.38), we obtain

$$(4.39) \quad p_J^{N_q}(w, 0) \leq \frac{C|w - qh|^{1/2} q^{3/2}}{J^{5/2}} (q - \omega(w)) e^{(Cq|w - qh| - |w - qh|^2 - q^2|h|^2)/(2J)}.$$

Using this, along with the expression of the standard heat kernel, in (4.37), we obtain

$$\mathbb{P}_{0,w}^{J,0}[E_{q,0,J}] \leq \frac{C|w - qh|^{1/2} q^{3/2}}{J^{3/2}} (q - \omega(w)) e^{(Cq|w - qh| + |w|^2 - |w - qh|^2 - q^2|h|^2)/(2J)}.$$

For the term in the exponent, we note that

$$\begin{aligned} Cq|w - qh| + |w|^2 - |w - qh|^2 - q^2|h|^2 &= Cq|w - qh| + 2qh \cdot w - 2q^2|h|^2 \\ &= Cq|w - qh| + 2qh \cdot [w - qh] \leq Cq|w - qh| \end{aligned}$$

for some larger constant  $C$ , so we obtain (4.36).  $\square$

**4.4.3. The location of the Brownian bridge at the stopping time.** In order to apply Lemma 4.10 for the expectation in (4.23), we need some control on  $|V(\tau_m)|$ . We recall the Gaussian tail bound

$$(4.40) \quad \mathbb{P}[|V(\tau_m)| \geq \xi] \leq \mathbb{P}\left[\sup_{x \in [0, L]} |V(x)| \geq \xi\right] \leq C e^{-C^{-1} L^{-1} \xi^2}.$$

We will also need an  $L$ -independent bound, which we obtain in the following lemma through a conformal transformation argument.

**Lemma 4.12.** *Let  $q > 0$  and let  $\tau$  be the first time  $x$  such that  $V(x)/(1-x/L) \in \partial N_q$ . Recall that  $h$  has been defined in (4.7). Then,*

$$(4.41) \quad \mathbb{P}[|V(\tau)| \geq q|h| + \xi] \leq \frac{2}{\pi} \arctan\left(\frac{q^{3/2}|h|^{3/2}}{\xi^{3/2}}\right) \quad \text{for all } \xi > 0.$$

*Proof.* Define

$$Y(y) = \frac{L+y}{L} V\left(\frac{Ly}{L+y}\right), \quad y \geq 0.$$

It is a standard fact, and can be checked by a covariance calculation, that  $Y$  is a standard planar Brownian motion with  $Y(0) = 0$ . Let  $\kappa$  be the first  $y$  such that  $Y(y) \in \partial N_q$ . Now define

$$Z(y) = (Y(y) - qh)^{3/2}, \quad y \geq 0,$$

where we raise an element of  $\mathbb{R}^2$  to the  $3/2$  power by thinking of it as an element of  $\mathbb{C}$ . The map  $z \mapsto z^{3/2}$  is a conformal isomorphism between the interior of  $N$  and the interior of the right half plane  $-i\mathbb{H}$ . Here we use the standard notation that  $\mathbb{H} = \{z : \text{Im } z \geq 0\}$  is the upper half plane and so  $-i\mathbb{H} = \{z : \text{Re } z \geq 0\}$  is the right half plane. Now  $\kappa$  is the first  $y$  such that  $Z(y) \in \partial(-i\mathbb{H}) = i\mathbb{R}$ . By the conformal invariance of the planar Brownian motion, we know that  $Z(\cdot)$  is a time changed planar Brownian motion starting at  $Z(0) = (-qh)^{3/2} = (q|h|)^{3/2}$ , so the probability density function of  $Z(\kappa)$  is given by the Poisson kernel: for any  $z \in \mathbb{R}$ ,

$$\mathbb{P}[Z(\kappa) \in idz] = \frac{q^{3/2}|h|^{3/2}}{\pi(q^3|h|^3 + z^2)}.$$

Undoing the change of variables, we find that for all  $\xi \geq 0$ ,

$$\mathbb{P}[|Y(\kappa) - qh| \geq \xi] = \mathbb{P}[|Z(\kappa)| \geq \xi^{3/2}] = \frac{2}{\pi} \arctan\left(\frac{q^{3/2}|h|^{3/2}}{\xi^{3/2}}\right).$$

Using the time-change  $x = \frac{Ly}{L+y}$ , which leads to  $\frac{L+y}{L} = \frac{1}{1-x/L}$ , we see that  $\tau = \frac{L\kappa}{L+\kappa}$  is the first  $x$  such that  $V(x)/(1-x/L)$  hits the boundary of  $N_q$ , and moreover that  $\frac{V(\tau)}{1-\tau/L} = Y(\kappa)$ . This means that

$$\begin{aligned} \mathbb{P}[|V(\tau)| \geq q|h| + \xi] &\leq \mathbb{P}[|V(\tau)|/(1-\tau/L) \geq q|h| + \xi] \\ &\leq \mathbb{P}[|V(\tau)/(1-\tau/L) - qh| \geq \xi] \\ &= \mathbb{P}[|Y(\kappa) - qh| \geq \xi] \\ &= \frac{2}{\pi} \arctan\left(\frac{q^{3/2}|h|^{3/2}}{\xi^{3/2}}\right), \end{aligned}$$

as claimed.  $\square$

4.4.4. *Proof of the upper bound  $\sigma_L^2 \leq CL^{-1/2}$ .* We now have all of the necessary ingredients to prove Proposition 4.7. Recall that  $\sigma_L^2 = L \sum_{m=1}^M G_m$ . (See (4.21).) By Lemma 4.9, we need to control the factors on the r.h.s. of the following:

$$(4.42) \quad G_m^2 \leq 4e^{-q_m-1} Q_{q_m}(L, 0) \mathbb{E}[Q_{q_m}(L - \tau_m, V(\tau_m)); \tau_m \leq L/2].$$

First we consider the case  $1 \leq m \leq M-1$ . By our assumption of  $1 \leq q_m \leq L^{1/2}$  in (4.19), the inequality (4.31) tells us that

$$(4.43) \quad Q_{q_m}(L, 0) \leq CL^{-3/2} q_m^3 e^{Cq_m/L} \leq CL^{-3/2} q_m^3.$$

Similarly, we have

$$(4.44) \quad \begin{aligned} Q_{q_m}(L - \tau_m, V(\tau_m)) &\leq C(L - \tau_m)^{-3/2} q_m^{3/2} e^{Cq_m(L - \tau_m)^{-1}(q_m + |V(\tau_m)|)} \\ &\quad \times [q_m^{1/2} + |V(\tau_m)|^{1/2}][q_m + |\omega(V(\tau_m))|]. \end{aligned}$$

On the event that  $\tau_m \leq L/2$ , we have  $L - \tau_m \geq L/2$ , and by the fact that

$$\omega(V(\tau_m)) = q_{m-1}(1 - \tau_m/L)$$

we have

$$\omega(V(\tau_m))/q_m = q_{m-1}(1 - \tau_m/L)/q_m \in [C^{-1}, C], \quad \text{if } m \geq 2,$$

where  $C > 0$  is a universal constant. Also, we have  $|V(\tau_m)| \geq C^{-1}\omega(V(\tau_m))$ . Using these observations, or a simpler argument if  $m = 1$ , in (4.44), we obtain

$$(4.45) \quad Q_{q_m}(L - \tau_m, V(\tau_m)) \leq CL^{-3/2}q_m^{5/2}[q_m^{1/2} + |V(\tau_m)|^{1/2}]e^{Cq_mL^{-1}|V(\tau_m)|}.$$

On the other hand, we also have by (4.32) that (still on the event  $\tau_m \leq L/2$ )

$$(4.46) \quad Q_{q_m}(L - \tau_m, V(\tau_m)) \leq Ce^{C|V(\tau_m)|/L}.$$

Then we can split up the expectation on the r.h.s. of (4.23) as (recalling that  $q_m \leq L^\delta$  by (4.19))

$$(4.47) \quad \begin{aligned} & \mathbb{E}[Q_{q_m}(L - \tau_m, V(\tau_m)); \tau_m \leq L/2] \\ &= \mathbb{E}[Q_{q_m}(L - \tau_m, V(\tau_m)); \tau_m \leq L/2 \text{ and } |V(\tau_m)| \leq L^{1-\delta}] \\ & \quad + \mathbb{E}[Q_{q_m}(L - \tau_m, V(\tau_m)); \tau_m \leq L/2 \text{ and } |V(\tau_m)| > L^{1-\delta}] \\ & \leq CL^{-3/2}q_m^{5/2}(q_m^{1/2} + \mathbb{E}|V(\tau_m)|^{1/2}) + C\mathbb{E}[e^{C|V(\tau_m)|/L}; |V(\tau_m)| > L^{1-\delta}]. \end{aligned}$$

The first term in the last expression comes from (4.45), and the second term comes from (4.46).

To deal with the first term in (4.47), we note that, by Lemma 4.12, we have

$$(4.48) \quad \begin{aligned} \mathbb{E}|V(\tau_m)|^{1/2} &= \int_0^\infty \mathbb{P}[|V(\tau_m)| \geq \xi^2]d\xi \leq q_m^{1/2}|h|^{1/2} + \int_0^\infty \mathbb{P}[|V(\tau_m)| \geq q_m|h| + \xi^2]d\xi \\ &\leq q_m^{1/2}|h|^{1/2} + \frac{2}{\pi} \int_0^\infty \arctan\left(\frac{q_m^{3/2}|h|^{3/2}}{\xi^3}\right)d\xi \leq Cq_m^{1/2}. \end{aligned}$$

For the second term in (4.47), note that for any  $b \in [1, \infty)$  we have

$$\begin{aligned} & \mathbb{E}[e^{b|V(\tau_m)|/L}; |V(\tau_m)| > L^{1-\delta}] \\ &= \int_0^\infty \mathbb{P}[e^{b|V(\tau_m)|/L} \geq \xi \text{ and } |V(\tau_m)| > L^{1-\delta}]d\xi \\ &\leq \int_0^\infty \mathbb{P}[|V(\tau_m)| \geq \max\{b^{-1}L \log \xi, L^{1-\delta}\}]d\xi \\ &= e^{bL^{-\delta}} \mathbb{P}[|V(\tau_m)| \geq L^{1-\delta}] + \int_{e^{bL^{-\delta}}}^\infty \mathbb{P}[|V(\tau_m)| \geq b^{-1}L \log \xi]d\xi \\ &\leq Ce^{bL^{-\delta} - L^{1-2\delta}/C} + C \int_{e^{bL^{-\delta}}}^\infty e^{-C^{-1}b^{-2}L(\log \xi)^2}d\xi, \end{aligned}$$

with the last inequality by two applications of (4.40). The last integral we can estimate as

$$\begin{aligned} \int_{e^{bL^{-\delta}}}^\infty e^{-C^{-1}L(\log \xi)^2}d\xi &= \int_{bL^{-\delta}}^\infty e^{\eta - C^{-1}L\eta^2}d\eta \leq \int_{bL^{-\delta}}^\infty e^{(1-C^{-1}bL^{1-\delta})\eta}d\eta \\ &= \frac{e^{(1-C^{-1}bL^{1-\delta})bL^{-\delta}}}{C^{-1}bL^{1-\delta} - 1}, \end{aligned}$$

provided that  $L$  is large enough that  $1 < C^{-1}bL^{1-\delta}$ . (Recalling our assumption that  $b \geq 1$ , it is sufficient to assume that  $L^{1-\delta} \geq C$ .) Since  $\delta < 1/2$ , this means that (taking  $b$  to be a fixed constant  $C$ )

$$(4.49) \quad \mathbb{E}[e^{C|V(\tau_m)|/L}; |V(\tau_m)| > L^{1-\delta}] \leq Ce^{-C^{-1}L^{1-2\delta}}$$

as long as  $L$  is greater than some absolute constant.

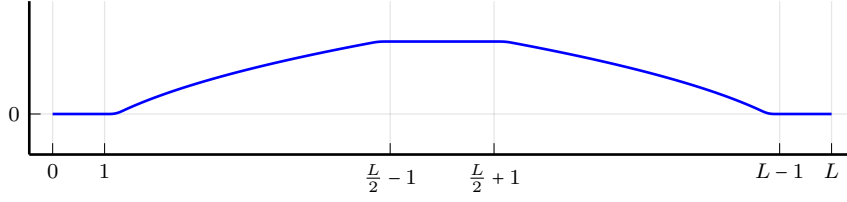
Now using (4.48) and (4.49) in (4.47), we obtain

$$(4.50) \quad \mathbb{E}[Q_{q_m}(L - \tau_m, V(\tau_m)); \tau_m \leq L/2] \leq CL^{-3/2}q_m^3 + Ce^{-C^{-1}L^{1-2\delta}} \leq CL^{-3/2}q_m^3.$$

Further using (4.43) and (4.50) in (4.42), we obtain, for  $1 \leq m \leq M - 1$ , that

$$(4.51) \quad G_m^2 \leq Ce^{-q_{m-1}}L^{-3}q_m^6.$$




 FIGURE 2. The function  $q_{\kappa, L}$  ( $\kappa = 0.4$ ).

We still need to bound  $G_M$ . For this we use (4.42) again, but only with the bound (4.32) (eschewing (4.31) which is not useful for  $q = \infty$ ). The resulting bound is

$$(4.52) \quad \begin{aligned} G_M^2 &\leq 4e^{-qM-1} Q_\infty(L, 0) \mathbb{E}[Q_\infty(L - \tau_M, V(\tau_M)); \tau_M \leq L/2] \\ &\leq Ce^{-qM-1} \mathbb{E}[e^{CL^{-1}|V(\tau_M)|}] \leq Ce^{-qM-1}, \end{aligned}$$

where the last step comes from another application of (4.40).

Now we use (4.51) and (4.52) in (4.21), and recall (4.18) and (4.19), to obtain

$$\sigma_L^2 \leq CL^{-1/2} \sum_{m=1}^{M-1} e^{-\frac{1}{2}q_{m-1}} q_m^3 + CL e^{-\frac{1}{2}q_{M-1}} \leq CL^{-1/2} \left[ \sum_{m=1}^{\infty} e^{-\frac{1}{2}(\xi_0^{m-1}-1)} (\xi_0^m - 1)^3 + 1 \right].$$

The infinite series is summable, so this implies that  $\sigma_L^2 \leq CL^{-1/2}$ , as claimed.  $\square$

**4.5. Lower bound.** We now complete the proof of Theorem 4.1 by proving the lower bound. We will use the framework, notations, and estimates established in Sections 4.1–4.3. On the other hand, this section is independent of the proof of the upper bound in Section 4.4, and we will use some different notations from what is used there.

**Proposition 4.13.** *There exists a constant  $C > 0$  so that, for all  $L \geq 1$ , we have  $\sigma_L^2 \geq C^{-1}L^{-1/2}$ .*

Fix a smooth, symmetric function  $\chi$  so that  $\chi|_{[-1,1]} \equiv 0$  and  $\chi|_{[-2,2]^c} \equiv 1$ . Define  $\chi_L(x) = \chi(x)\chi(x - L/2)\chi(x - L)$ . For any  $\kappa > 0$ , define

$$(4.53) \quad q_{\kappa, L}(y) = \kappa \int_0^y \chi_L(x) \operatorname{sgn}(L/2 - x) [x \wedge (L - x)]^{\kappa-1} dx, \quad y \in [0, L].$$

(See Figure 2.) It is clear that  $q_{\kappa, L}(\cdot)$  is smooth,  $q_{\kappa, L}(L/2 + y) = q_{\kappa, L}(L/2 - y)$  for all  $y \in [0, L/2]$ , and that

$$(4.54) \quad q_{\kappa, L}(y) \leq \kappa \int_0^y x^{\kappa-1} dx = y^\kappa \quad \text{for } y \in [0, L/2].$$

Now fix  $\gamma \in (0, 1/2)$  and define the event

$$F = \{\omega(V(x)) \leq 1 - q_{\gamma, L}(x) \text{ for all } x \in [0, L]\}.$$

Note that  $F \subset E_{1,0,L}$ . The key to our proof of the lower bound will be the following “entropic repulsion” lemma, which states that conditional on  $\omega(V(x))$  staying below 1, it in fact has a positive probability of staying below  $1 - q_{\gamma, L}(x)$ .

**Lemma 4.14.** *There exists a constant  $C > 0$  such that whenever  $L \geq 1$  we have  $\mathbb{P}[F | E_{1,0,L}] \geq C^{-1}$ .*

Before proving Lemma 4.14, we show how to use it to prove Proposition 4.13.

*Proof of Proposition 4.13.* On the event  $F$ , we have

$$\prod_{k=1}^2 \left( \int_0^L e^{v_k \cdot V(x)} dx \right)^{-1} \geq \left( \int_0^L e^{1-q_{\gamma,L}(x)} dx \right)^{-2} \geq C^{-1},$$

where  $C$  is a positive constant independent of  $L$  and the last step comes from (4.54). Therefore, we have

$$(4.55) \quad \sigma_L^2 = \mathbb{E} \left[ \prod_{k=1}^2 \left( \int_0^L e^{v_k \cdot V(x)} dx \right)^{-1} \right] \geq C^{-1} \mathbb{P}_{0,0}^{L,0}[F] = C^{-1} \mathbb{P}_{0,0}^{L,0}[F | E_{1,0,L}] \mathbb{P}_{0,0}^{L,0}[E_{1,0,L}].$$

By (4.13) and (4.15), we have

$$(4.56) \quad \mathbb{P}_{0,0}^{L,0}[E_{1,0,L}] = \frac{p_L^N(-h, -h)}{p_L(0,0)} = 2\pi L p_L^N(-h, -h) \geq C^{-1} L^{-3/2}.$$

The claimed lower bound follows from applying Lemma 4.14 and (4.56) in (4.55).  $\square$

Now we prove Lemma 4.14. We follow the procedure of [17] as explicated in the proof of [27, Proposition 2.1]. The works [17, 27] considered a one-dimensional Brownian bridge conditioned to stay above a line. The main ingredient is the Cameron–Martin theorem. Due to the two-dimensional nature of our problem, our proof requires an estimate for the expectation of the modulus of a two-dimensional Brownian bridge conditioned to stay in the cone; see Lemma 4.15 below.

*Proof of Lemma 4.14.* In this proof we assume that our probability space is the space of  $\mathbb{R}^2$ -valued continuous functions on  $[0, L]$ , and the measure  $\mathbb{P}$  is the standard two-dimensional Brownian bridge measure on this space, defined on the Borel  $\sigma$ -algebra, with  $\mathbb{E}$  the associated expectation. For a continuous function  $Y : [0, L] \rightarrow \mathbb{R}$ , we define

$$g(Y) = - \int_0^L q_{\gamma,L}''(x) Y(x) dx.$$

Now we have, using (4.8), that

$$(4.57) \quad \begin{aligned} \mathbb{P}[F] &= \mathbb{P}[(V + q_{\gamma,L}h) \in E_{1,0,L}] \\ &\geq \mathbb{P}[(V + q_{\gamma,L}h) \in E_{1,0,L} \text{ and } g(V_1 + q_{\gamma,L}h_1) \leq M], \end{aligned}$$

for any  $M < \infty$ . Recall that  $h = (-\sqrt{6}/3, 0)$ . Here  $V_1, h_1$  are the first components of  $V$  and  $h$ , respectively. We have by the Cameron–Martin theorem (see e.g. [43, Theorem 2.6]) that

$$(4.58) \quad \frac{d\text{Law}(V + q_{\gamma,L}h)}{d\mathbb{P}} = \exp \left\{ h_1 \int_0^L q'_{\gamma,L}(x) dV_1(x) - \frac{h_1^2}{2} \int_0^L |q'_{\gamma,L}(x)|^2 dx \right\}.$$

From (4.57) and (4.58) we have

$$(4.59) \quad \begin{aligned} \mathbb{P}[F] &\geq \exp \left\{ -\frac{h_1^2}{2} \int_0^L |q'_{\gamma,L}(x)|^2 dx \right\} \\ &\quad \times \mathbb{E} \left[ \exp \left\{ h_1 \int_0^L q'_{\gamma,L}(x) dV_1(x) \right\}; E_{1,0,L} \cap \{g(V_1) \leq M\} \right]. \end{aligned}$$

We see from the definition (4.53) that

$$(4.60) \quad \int_0^L |q'_{\gamma,L}(x)|^2 dx \leq 2\gamma^2 \int_1^\infty x^{2(\gamma-1)} dx \leq C,$$

by our assumption of  $\gamma < 1/2$ . Moreover, by using integration by parts (and the fact that  $q_{\gamma,L}$  is smooth and supported on a compact subset of the interior of  $[0, L]$  by definition), we have

$$(4.61) \quad h_1 \int_0^L q'_{\gamma,L}(x) dV_1(x) = -h_1 \int_0^L q''_{\gamma,L}(x) V_1(x) dx = -\frac{\sqrt{6}}{3} g(V_1).$$

Using (4.60) and (4.61) in (4.59), we obtain

$$(4.62) \quad \mathbb{P}[F] \geq c(M) \mathbb{P}[E_{1,0,L} \cap \{g(V_1) \leq M\}],$$

for a constant  $c(M) > 0$  depending on  $M$  but not on  $L$ .

Now since  $F \subset E_{1,0,L}$ , we can divide through by  $\mathbb{P}[E_{1,0,L}]$  in (4.62) to obtain

$$(4.63) \quad \mathbb{P}[F | E_{1,0,L}] \geq c(M) \mathbb{P}[g(V_1) \leq M | E_{1,0,L}].$$

So it remains to estimate the last conditional probability. By Lemma 4.15 below, we have whenever  $x \in [1, L-1]$  that

$$\mathbb{E}[|V_1(x)| | E_{1,0,L}] \leq C[x \wedge (L-x)]^{1/2},$$

and so by linearity of conditional expectation and the fact that  $q''_L$  is supported on  $[1, L-1]$ , we have

$$(4.64) \quad \mathbb{E}\left[\int_0^L |q''_{\gamma,L}(x) V_1(x)| dx \mid E_{1,0,L}\right] \leq C \int_1^{L-1} |q''_{\gamma,L}(x)| [x \wedge (L-x)]^{1/2} dx \leq C,$$

for a constant  $C < \infty$  independent of  $L$ . The last inequality in (4.64) is derived as follows. By symmetry and the assumption on  $q_{\gamma,L}$ , we only need to consider the integration domain of  $x \in [1, \frac{L}{2} - 1]$ . Recalling (4.53), we have  $q''_{\gamma,L}(x) = \gamma(\gamma-1)x^{\gamma-2}$  in  $x \in [2, \frac{L}{2} - 2]$ , and since  $\gamma - 2 + \frac{1}{2} < -1$ , the corresponding integral is bounded independent of  $L$ . For  $x \in [1, 2] \cup [\frac{L}{2} - 2, \frac{L}{2} - 1]$ , we also have  $|q''_{\gamma,L}(x)| x^{1/2} \leq C$ , and the last inequality in (4.64) follows.

By the definition of  $g$ , the inequality (4.64), and Markov's inequality, we obtain

$$(4.65) \quad \mathbb{P}[g(V_1) \geq M | E_{1,0,L}] \leq \mathbb{P}\left[\int_0^L |q''_{\gamma,L}(x) V_1(x)| dx \geq M \mid E_{1,0,L}\right] \leq C/M.$$

Substituting (4.65) into (4.63) and taking  $M$  sufficiently large (independent of  $L$ ), we obtain

$$\mathbb{P}[F | E_{1,0,L}] \geq \frac{1}{2} c(M).$$

This completes the proof.  $\square$

It remains to prove the lemma we used in the last proof.

**Lemma 4.15.** *There exists a constant  $C > 0$ , independent of  $L$ , so that whenever  $L \geq 2$  and  $x \in [1, L-1]$  we have*

$$(4.66) \quad \mathbb{E}[|V(x)| | E_{1,0,L}] \leq C[x \wedge (L-x)]^{1/2}.$$

*Proof.* We note that (using (4.13))

$$(4.67) \quad \mathbb{E}[|V(x)| | E_{1,0,L}] = \frac{\int_N |u+h| p_x^N(-h, u) p_{L-x}^N(u, -h) du}{p_L^N(-h, -h)}.$$

For the denominator we use (4.15) to write

$$(4.68) \quad p_L^N(-h, -h) \geq C^{-1} L^{-5/2}.$$

To investigate the numerator of (4.67), we note that

$$(4.69) \quad \int_N |u+h| p_x^N(-h, u) p_{L-x}^N(u, -h) du \leq |h| p_L^N(-h, -h) + S,$$

with

$$\begin{aligned}
S &:= \int_N |u| p_x^N(-h, u) p_{L-x}^N(u, -h) du \\
&= \int_0^\infty \int_{-\pi/3}^{\pi/3} r^2 p_x^N(-h, r e^{i\theta}) p_{L-x}^N(r e^{i\theta}, -h) d\theta dr \\
&\leq \frac{C e^{-\frac{1}{3}[x^{-1}+(L-x)^{-1}]}}{[x(L-x)]^{5/2}} \int_0^\infty r^5 e^{-\frac{1}{2}r^2[x^{-1}+(L-x)^{-1]}+Cr[x^{-1}+(L-x)^{-1}]} dr
\end{aligned}$$

by (4.14). Without loss of generality, we consider the case of  $x \leq L-x$ . By the change of variables  $r \rightarrow rx^{1/2}$ , we obtain

$$\begin{aligned}
(4.70) \quad S &\leq \frac{Cx^{1/2}}{(L-x)^{5/2}} \int_0^\infty r^5 e^{-\frac{1}{2}r^2x[x^{-1}+(L-x)^{-1]}+Crx^{1/2}[x^{-1}+(L-x)^{-1}]} dr \\
&\leq \frac{Cx^{1/2}}{L^{5/2}} \int_0^\infty r^5 e^{-\frac{1}{2}r^2+Crx^{-1/2}} dr \\
&\leq \frac{Cx^{1/2}e^{Cx^{-1}}}{L^{5/2}} \int_0^\infty r^5 e^{-\frac{1}{4}r^2} dr \leq Cx^{1/2}L^{-5/2},
\end{aligned}$$

where we used the assumption that  $x \geq 1$  and Young's inequality. Using (4.68), (4.69) and (4.70) in (4.67) gives us (4.66).  $\square$

## 5. FURTHER DISCUSSIONS

**5.1. Another proof of CLT.** The Clark-Ocone representation suggests another way of proving the central limit theorem, which says that for fixed  $L > 0$ ,

$$(5.1) \quad \frac{\log \mathcal{U}(t, x) + \gamma_L t}{\sqrt{t}} \Rightarrow N(0, \sigma_L^2), \quad \text{as } t \rightarrow \infty.$$

We sketch a heuristic argument here. Throughout the section,  $L > 0$  is fixed (so the estimates and proportionality constants depend on  $L$ ).

In (2.5),  $f, g$  can be chosen as arbitrary measures, and by choosing  $f = \delta_x$  and  $g = \nu$  (with  $\nu$  an arbitrary probability measure on  $\mathbb{T}_L$ ) we have

$$(5.2) \quad \log \mathcal{U}(t, x) - \mathbb{E} \log \mathcal{U}(t, x) = \int_0^t \int_0^L \mathcal{G}_{t,s}(f, \rho(s; g), y) \rho(s, y; g) \eta(s, y) dy ds.$$

Here  $\mathcal{U}$  is the solution to SHE starting from  $g$ .

Since  $L$  is fixed, by [35, Theorem 2.3], we know that the process  $\{\rho(t; g)\}_{t \geq 0}$  mixes exponentially fast. This would imply, see [35, (2.24)],

$$(5.3) \quad \mathbb{E} \log \mathcal{U}(t, x) = -\gamma_L t + O(1).$$

With  $f = \delta_x$ , we have

$$\begin{aligned}
\mathcal{G}_{t,s}(f, h, y) &= \mathbb{E} \left[ \frac{\int_0^L f(x') \mathcal{Z}(t, x'; s, y) dx'}{\int_0^L \int_0^L f(x') \mathcal{Z}(t, x'; s, y') h(y') dx' dy'} \right] \\
&= \mathbb{E} \left[ \frac{\mathcal{Z}(t, x; s, y)}{\int_0^L \mathcal{Z}(t, x; s, y') h(y') dy'} \right] = \mathbb{E} \left[ \frac{\tilde{\rho}_{t,x}(s, y)}{\int_0^L \tilde{\rho}_{t,x}(s, y') h(y') dy'} \right],
\end{aligned}$$

where

$$\tilde{\rho}_{t,x}(s, y) := \frac{\mathcal{Z}(t, x; s, y)}{\int_0^L \mathcal{Z}(t, x; s, y') dy'}.$$

It is straightforward to check that for each  $t > 0$  and  $x \in \mathbb{T}_L$  fixed,

$$(\tilde{\rho}_{t,x}(s, \cdot))_{s \in [0, t]} \stackrel{\text{law}}{=} (\rho(t-s, \cdot; \delta_x))_{s \in [0, t]}$$

where the r.h.s. is the endpoint distribution of the directed polymer of length  $t - s$ , starting from  $x$ . Thus, by the exponential mixing of  $\rho(t, \cdot; \delta_x)$ , we expect the following approximation

$$\mathcal{G}_{t,s}(f, h, y) \approx \mathbb{E}_B \left[ \frac{e^{B(y)}}{\int_0^L e^{B(y')} h(y') dy'} \right] = \mathcal{A}(h, y), \quad \text{if } t - s \gg 1;$$

see (2.7). Recall that when  $f$  is sampled from the invariant distribution  $\mu$ , the above approximation is exact, as in (2.7).

With the above approximation, we expect that, for any  $\beta \in (0, 1)$ ,

$$\begin{aligned} \log \mathcal{U}(t, x) - \mathbb{E} \log \mathcal{U}(t, x) &= \int_0^t \int_0^L \mathcal{G}_{t,s}(f, \rho(s; g), y) \rho(s, y; g) \eta(s, y) dy ds \\ &\approx \int_0^{t-t^\beta} \int_0^L \mathcal{G}_{t,s}(f, \rho(s; g), y) \rho(s, y; g) \eta(s, y) dy ds \\ (5.4) \quad &\approx \int_0^{t-t^\beta} \int_0^L \mathcal{A}(\rho(s; g), y) \rho(s, y; g) \eta(s, y) dy ds \\ &\approx \int_0^t \int_0^L \mathcal{A}(\rho(s; g), y) \rho(s, y; g) \eta(s, y) dy ds =: M_t. \end{aligned}$$

The above “ $\approx$ ”s mean that the  $L^2$  norm of the error is of order  $o(\sqrt{t})$ . The  $M_t$  defined above is a martingale. An application of the martingale central limit theorem, as in [35, Lemma 5.29], implies that

$$\frac{\log \mathcal{U}(t, x) - \mathbb{E} \log \mathcal{U}(t, x)}{\sqrt{t}} \Rightarrow N(0, \sigma_L^2), \quad \text{as } t \rightarrow \infty.$$

Combining with (5.3), this completes the proof of (5.1).

**5.2. Connection to Liouville quantum mechanics.** Recall that from (2.10) and Lemma 4.2, we have

$$\sigma_L^2 = L \mathbb{E} \frac{1}{\int_0^L e^{B_1(x)+B_3(x)} dx \int_0^L e^{B_2(x)+B_3(x)} dx},$$

where  $B_j$  are independent Brownian bridges with  $B_j(0) = B_j(L) = 0$ ,  $j = 1, 2, 3$ . Random variables of the form  $\int_0^L e^{B(x)} dx$  have been studied extensively due to the applications in physics and mathematical finance, where  $B$  could be a Brownian motion or a Brownian bridge. The densities of these random variables are known; see the survey [42]. One way to derive the density of such a random variable is through the connection to the Schrödinger operator with the so-called Liouville potential  $-\frac{1}{2}\partial_x^2 + e^x$ , whose Green’s function can be written down explicitly, see [42, Theorem 4.1] and its proof. It suggests that, for our problem, to study the asymptotics of  $\sigma_L^2$ , another possible approach is to try to derive the density of the joint distribution of  $\int_0^L e^{W_1(x)} dx$  and  $\int_0^L e^{W_2(x)} dx$ , with  $W_1, W_2$  two correlated Brownian bridges. The corresponding Schrödinger operator would take the form

$$-\partial_{x_1}^2 - \partial_{x_1 x_2} - \partial_{x_2}^2 + e^{x_1} + e^{x_2},$$

which, after a change of variables, becomes a generalized Toda lattice. It would be interesting to see whether there are integrable tools that could lead to a good understanding of the spectral properties of the above operator, using which we may compute the value of  $\sigma_L^2$  explicitly. Similar approaches appeared in e.g. [13, 44] to study the distributions of Brownian functionals. It is also worth mentioning that, by the replica method and the study of the ground state energy of the delta-Bose gas on a torus, there is actually a prediction of the value of  $\sigma_L^2$ , as an explicit integral involving the hyperbolic tangent function; see [19, Equation (33)].

**5.3. Noise with general covariance function in arbitrary dimensions.** The approach developed in this paper suggests a possible way of understanding the universality. The noise we considered is the  $1 + 1$  spacetime white noise in which case the invariant measure for the KPZ equation is explicit, given by the Brownian bridge. Suppose we consider a noise that is white in time and colored in space, with a spatial covariance function  $R(\cdot)$  that is independent of  $L$ :

$$\mathbb{E}\eta(t, x)\eta(s, y) = \delta(t - s)R(x - y).$$

As long as  $R(\cdot)$  decays fast enough, one expects a similar result as (1.3) in  $d = 1$ . Denote the solution to the KPZ equation by  $h_L$  and assume that  $h_L(0, x) - h_L(0, 0)$  is sampled from the unique invariant measure  $\tilde{\pi}$ , for which we do not have the explicit formula. By following the same proof verbatim as presented in Sections 2 and 3, one arrives at

$$(5.5) \quad h_L(t, 0) - \mathbb{E}h_L(t, 0) = I_L(t) + (Y_L(0) - Y_L(t)),$$

with  $\text{Var } I_L(t) = t\tilde{\sigma}_L^2$  and  $\{Y_L(t)\}_{t \geq 0}$  a stationary process. The variance takes the form

$$(5.6) \quad \tilde{\sigma}_L^2 = \int_0^L \int_0^L \mathbb{E} \tilde{\rho}(x_1)\tilde{\rho}(x_2)R(x_1 - x_2)dx_1dx_2,$$

where

$$\tilde{\rho}(x) = \mathbb{E} \left[ \frac{e^{H_1(x) + H_2(x)}}{\int_0^L e^{H_1(x') + H_2(x')} dx'} \middle| H_1 \right],$$

with  $H_1$  and  $H_2$  sampled independently from  $\tilde{\pi}$ . In addition, for each  $t \geq 0$ ,

$$(5.7) \quad Y_L(t) \stackrel{\text{law}}{=} \mathbb{E} \left[ \log \int_0^L e^{H_1(x) + H_2(x)} dx \middle| H_1 \right].$$

A natural conjecture is that, in  $d = 1$  and with  $R(\cdot)$  decaying sufficiently fast, we have  $\tilde{\sigma}_L^2 \asymp L^{-1/2}$  and  $\text{Var } Y_L(t) \asymp L$ . This would imply (1.3) for the noise with a general spatial covariance function. In some sense, a contribution of this paper is to reduce the “dynamic” problem of estimating the variance of the height function, in the super-relaxation and relaxation regime, to the “static” problem of studying the integrals (5.6) and (5.7), which only involve the invariant measure. In the case of  $1 + 1$  spacetime white noise,  $H_j$  are independent copies of Brownian bridges for which we did some explicit calculations. The decomposition (5.5) and the identities (5.6) and (5.7) hold in high dimensions as well. It is unclear what would be the picture there, although another natural conjecture would be that the critical length scale in high dimensions still comes from a balance of the two terms  $t\tilde{\sigma}_L^2 \sim \text{Var } Y_L(t)$ .

#### APPENDIX A. ESTIMATES FOR MODIFIED BESSEL FUNCTIONS

In this brief appendix we recall standard definitions and estimates, used in Section 4, on the modified Bessel functions of the first kind. We will use the formula

$$(A.1) \quad I_\nu(z) = \frac{(z/2)^\nu}{\pi^{1/2}\Gamma(\nu + 1/2)} \int_0^\pi e^{z \cos \eta} \sin^{2\nu} \eta d\eta$$

for these functions; see [1, (9.8.18) on p. 376].

**Lemma A.1.** *For all  $\nu, z \geq 0$  we have*

$$\frac{(z/2)^\nu e^{-z}}{\Gamma(\nu + 1/2)} \leq I_\nu(z) \leq \frac{(z/2)^\nu e^z}{\Gamma(\nu + 1/2)}.$$

*Proof.* We first note that, for all  $\nu \geq 0$ , we have

$$(A.2) \quad \int_0^\pi \sin^{2\nu} \eta d\eta = \frac{\sqrt{\pi}\Gamma(\nu + 1/2)}{\Gamma(\nu + 1)}.$$

From (A.1) we have for all  $z \geq 0$  that

$$I_\nu(z) \in \left( \frac{(z/2)^\nu}{\pi^{1/2}\Gamma(\nu + 1/2)} \int_0^\pi \sin^{2\nu} \eta d\eta \right) \cdot [e^{-z}, e^z] = \frac{(z/2)^\nu}{\Gamma(\nu + 1)} \cdot [e^{-z}, e^z],$$

which was the claim.  $\square$

**Lemma A.2.** *For any  $\alpha > 1$  there is a constant  $C < \infty$  (depending on  $\alpha$ ) so that, for all  $q \geq 0$ , we have*

$$(A.3) \quad \sum_{j=1}^{\infty} j^2 I_{\alpha j}(q) \leq C e^q [e^{(q/2)^\alpha} - 1] \leq C q^\alpha e^{Cq}.$$

*Proof.* By Lemma A.1, we have for any  $q \geq 0$  that

$$\sum_{j=1}^{\infty} j^2 I_{\alpha j}(q) \leq e^q \sum_{j=1}^{\infty} \frac{j^2 (q/2)^{\alpha j}}{\Gamma(\alpha j + 1/2)}.$$

This sum evidently converges absolutely and uniformly on compact subsets of  $[0, \infty)$ , and so in particular we have a constant  $C$  so that

$$(A.4) \quad \sum_{j=1}^{\infty} j^2 I_{\alpha j}(q) \leq C q^\alpha \text{ for all } q \in [0, 1].$$

On the other hand, for  $q \geq 1$ , we have

$$(A.5) \quad \sum_{j=1}^{\infty} \frac{j^2 (q/2)^{\alpha j}}{\Gamma(\alpha j + 1/2)} \leq C \sum_{j=1}^{\infty} \frac{j^2 q^{\alpha j}}{[\alpha j - 1/2]!} \leq C \sum_{k=0}^{\infty} \frac{(k+1)^2 q^{k+3/2}}{k!} \leq C e^{Cq}.$$

Combining (A.4) and (A.5), we get (A.3).  $\square$

## REFERENCES

- [1] M. ABRAMOWITZ AND I. A. STEGUN, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series, No. 55, U. S. Government Printing Office, Washington, D.C., 1964.
- [2] G. AMIR, I. CORWIN, AND J. QUASTEL, *Probability distribution of the free energy of the continuum directed random polymer in 1+1 dimensions*, Comm. Pure Appl. Math., 64 (2011), 466–537.
- [3] J. BAIK AND Z. LIU, *TASEP on a ring in sub-relaxation time scale*, J. Stat. Phys., 165 (2016), 1051–1085.
- [4] J. BAIK AND Z. LIU, *Fluctuations of TASEP on a ring in relaxation time scale*, Comm. Pure Appl. Math., 71 (2018), 747–813.
- [5] J. BAIK AND Z. LIU, *Multipoint distribution of periodic TASEP*, J. Amer. Math. Soc., 32 (2019), 609–674.
- [6] J. BAIK AND Z. LIU, *Periodic TASEP with general initial conditions*, Probab. Theory Related Fields, 179 (2021), 1047–1144.
- [7] J. BAIK, Z. LIU, AND G. L. F. SILVA, *Limiting one-point distribution of periodic TASEP*, Aug. 2020, arXiv preprint [2008.07024v1](https://arxiv.org/abs/2008.07024v1).
- [8] Y. BAKHTIN AND D. SEO, *Localization of directed polymers in continuous space*, Electron. J. Probab., 25 (2020), Paper No. 142, 56.
- [9] M. BALÁZS, J. QUASTEL, AND T. SEPPÄLÄINEN, *Fluctuation exponent of the KPZ/stochastic Burgers equation*, J. Amer. Math. Soc., 24 (2011), 683–708.
- [10] R. BAÑUELOS AND R. G. SMITS, *Brownian motion in cones*, Probab. Theory Related Fields, 108 (1997), 299–319.
- [11] G. BARRAQUAND AND P. L. DOUSSAL, *Steady state of the KPZ equation on an interval and Liouville quantum mechanics*, Sept. 2021, arXiv preprint [2105.15178v4](https://arxiv.org/abs/2105.15178v4).
- [12] E. BATES AND S. CHATTERJEE, *The endpoint distribution of directed polymers*, Ann. Probab., 48 (2020), 817–871.



- [13] F. BAUDOIN AND N. O'CONNELL, *Exponential functionals of Brownian motion and class-one Whittaker functions*, Ann. Inst. Henri Poincaré Probab. Stat., 47 (2011), 1096–1120.
- [14] L. BERTINI AND G. GIACOMIN, *Stochastic Burgers and KPZ equations from particle systems*, Comm. Math. Phys., 183 (1997), 571–607.
- [15] A. BORODIN, I. CORWIN, AND P. FERRARI, *Free energy fluctuations for directed polymers in random media in 1 + 1 dimension*, Comm. Pure Appl. Math., 67 (2014), 1129–1214.
- [16] A. BORODIN, I. CORWIN, P. FERRARI, AND B. VETÓ, *Height fluctuations for the stationary KPZ equation*, Math. Phys. Anal. Geom., 18 (2015), Art. 20, 95.
- [17] M. BRAMSON, *Convergence of solutions of the Kolmogorov equation to travelling waves*, Mem. Amer. Math. Soc., 44 (1983), iv+190.
- [18] Y. BRÖKER AND C. MUKHERJEE, *Localization of the Gaussian multiplicative chaos in the Wiener space and the stochastic heat equation in strong disorder*, Ann. Appl. Probab., 29 (2019), 3745–3785.
- [19] É. BRUNET AND B. DERRIDA, *Ground state energy of a non-integer number of particles with  $\delta$  attractive interactions*, Phys. A, 279 (2000), 398–407.
- [20] W. BRYC, A. KUZNETSOV, Y. WANG, AND J. WESOLOWSKI, *Markov processes related to the stationary measure for the open KPZ equation*, July 2021, arXiv preprint [2105.03946v2](https://arxiv.org/abs/2105.03946).
- [21] H. S. CARSLAW AND J. C. JAEGER, *Conduction of heat in solids*, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, second ed., 1988.
- [22] S. CHATTERJEE, *Superconcentration and related topics*, Springer Monogr. Math., Springer, Cham, 2014.
- [23] S. CHATTERJEE AND P. S. DEY, *Central limit theorem for first-passage percolation time across thin cylinders*, Probab. Theory Related Fields, 156 (2013), 613–663.
- [24] L. CHEN, D. KHOSHNEVISAN, D. NUALART, AND F. PU, *Spatial ergodicity for SPDEs via Poincaré-type inequalities* July 2019, arXiv preprint [1907.11553v1](https://arxiv.org/abs/1907.11553).
- [25] L. CHEN, D. KHOSHNEVISAN, D. NUALART, AND F. PU, *Central limit theorems for spatial averages of the stochastic heat equation via Malliavin-Stein's method*, Aug. 2020, arXiv preprint [2008.02408v1](https://arxiv.org/abs/2008.02408).
- [26] M. CHUPEAU, O. BÉNICHOU, AND S. N. MAJUMDAR, *Survival probability of a Brownian motion in a planar wedge of arbitrary angle*, Phys. Rev. E, 91 (2015), 032106, 8.
- [27] A. CORTINES, L. HARTUNG, AND O. LOUIDOR, *Decorated random walk restricted to stay below a curve*, Ann. Probab., 47 (2019). Supplement to [28].
- [28] A. CORTINES, L. HARTUNG, AND O. LOUIDOR, *The structure of extreme level sets in branching Brownian motion*, Ann. Probab., 47 (2019), 2257–2302.
- [29] I. CORWIN, *The Kardar-Parisi-Zhang equation and universality class*, Random Matrices Theory Appl., 1 (2012), 1130001, 76.
- [30] I. CORWIN AND A. KNIZEL, *Stationary measure for the open KPZ equation*, Apr. 2021, arXiv preprint [2103.12253v2](https://arxiv.org/abs/2103.12253).
- [31] D. DAUVERGNE, J. ORTMANN, AND B. VIRAG, *The directed landscape*, July 2021, arXiv preprint [1812.00309v6](https://arxiv.org/abs/2107.1812).
- [32] B. DERRIDA, M. R. EVANS, AND D. MUKAMEL, *Exact diffusion constant for one-dimensional asymmetric exclusion models*, J. Phys. A, 26 (1993), 4911–4918.
- [33] P. DEY, M. JOSEPH, AND R. PELED, *Longest increasing path within the critical strip*, Aug. 2018, arXiv preprint [1808.08407v1](https://arxiv.org/abs/1808.08407).
- [34] T. FUNAKI AND J. QUASTEL, *KPZ equation, its renormalization and invariant measures*, Stoch. Partial Differ. Equ. Anal. Comp., 3 (2015), 159–220.
- [35] Y. GU AND T. KOMOROWSKI, *KPZ on torus: Gaussian fluctuations*, June 2021, arXiv preprint [2104.13540v2](https://arxiv.org/abs/2104.13540).
- [36] Y. GU AND T. KOMOROWSKI, *High temperature behaviors of the directed polymer on a cylinder*, Oct. 2021, arXiv preprint [2110.07368](https://arxiv.org/abs/2110.07368).
- [37] M. GUBINELLI AND N. PERKOWSKI, *KPZ reloaded*, Comm. Math. Phys., 349 (2017), 165–269.
- [38] M. HAIRER AND J. MATTINGLY, *The strong Feller property for singular stochastic PDEs*, Ann. Inst. Henri Poincaré Probab. Stat., 54 (2018), 1314–1340.
- [39] Z. LIU, *Height fluctuations of stationary TASEP on a ring in relaxation time scale*, Ann. Inst. H. Poincaré B, 54 (2018), 1031–1057.
- [40] Z. LIU, A. SAENZ, AND D. WANG, *Integral formulas of ASEP and  $q$ -TAZRP on a ring*, Commun. Math. Phys., 379 (2020), 261–325.
- [41] K. MATETSKI, J. QUASTEL, AND D. REMENIK, *The KPZ fixed point*, Oct. 2020, arXiv preprint [1701.00018v3](https://arxiv.org/abs/2010.1701).
- [42] H. MATSUMOTO AND M. YOR, *Exponential functionals of Brownian motion. I. Probability laws at fixed time*, Probab. Surv., 2 (2005), 312–347.
- [43] A. R. NAHMOD, L. REY-BELLET, S. SHEFFIELD, AND G. STAFFILANI, *Absolute continuity of Brownian bridges under certain gauge transformations*, Math. Res. Lett., 18 (2011), 875–887.

- [44] N. O'CONNELL, *Directed polymers and the quantum Toda lattice*, Ann. Probab., 40 (2012), 437–458.
- [45] J. QUASTEL, *Introduction to KPZ*, in Current Developments in Mathematics, 2011, Int. Press, Somerville, MA, 2012, 125–194.
- [46] J. QUASTEL AND S. SARKAR, *Convergence of exclusion processes and KPZ equation to the KPZ fixed point*, Oct. 2020, arXiv preprint [2008.06584v4](https://arxiv.org/abs/2008.06584v4).
- [47] J. QUASTEL AND H. SPOHN, *The one-dimensional KPZ equation and its universality class*, J. Stat. Phys., 160 (2015), 965–984.
- [48] T. C. ROSATI, *Synchronization for KPZ*, Sept. 2021, arXiv preprint [1907.06278v3](https://arxiv.org/abs/1907.06278v3).
- [49] T. SASAMOTO AND H. SPOHN, *Exact height distributions for the KPZ equation with narrow wedge initial condition*, Nuclear Phys. B, 834 (2010), 523–542.
- [50] T. SASAMOTO AND H. SPOHN, *One-dimensional Kardar-Parisi-Zhang equation: an exact solution and its universality*, Phys. Rev. Lett., 104 (2010), 230602.
- [51] Y. G. SINAI, *Two results concerning asymptotic behavior of solutions of the Burgers equation with force*, J. Stat. Phys., 64 (1991), 1–12.
- [52] B. VIRAG, *The heat and the landscape I*, Aug. 2020, arXiv preprint [2008.07241v1](https://arxiv.org/abs/2008.07241v1).
- [53] M. YOR, *On some exponential functionals of Brownian motion*, Adv. Appl. Probab., 24 (1992), 509–531.

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