# CHAOS EXPANSION OF 2D PARABOLIC ANDERSON MODEL

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Abstract. We prove a chaos expansion for the 2D parabolic Anderson Model in small time, with the expansion coefficients expressed in terms of the density function of the annealed polymer in a white noise environment.

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## 1. Introduction and main result

Consider the continuous parabolic Anderson model in $d = 2$ formally written as

$$\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + u \cdot (\dot{W}(x) - \infty), \quad t \geq 0\,, x \in \mathbb{R}^2\,, \tag{1.1}$$

where $\dot{W}(x)$ is a spatial white noise defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ formally satisfying

$$\mathbb{E}[\dot{W}(x)\dot{W}(y)] = \delta(x-y).$$

The equation (1.1) was analyzed in [7, 8, 9] by different approaches including the theory of regularity structures, para-controlled calculus, and the method of correctors and two-scale expansions. The main results in these references showed that a smoothed version of (1.1) converges to some limit that is independent of the mollification.

More precisely, let $\varphi : \mathbb{R}^2 \to \mathbb{R}_+$ be a smooth and compactly supported function on $\mathbb{R}^2$ satisfying $\varphi(x) = \varphi(-x)$ and $\int \varphi = 1$. Define $\varphi_\epsilon(\cdot) = \epsilon^{-2}\varphi(\cdot/\epsilon)$ and

$$\dot{W}_\epsilon(x) = \int_{\mathbb{R}^2} \varphi_\epsilon(x-y)dW(y) \tag{1.2}$$

as the mollification of $\dot{W}$. The covariance function of $\dot{W}_\epsilon$ is

$$R_\epsilon(x-y) := \mathbb{E}[\dot{W}_\epsilon(x)\dot{W}_\epsilon(y)] = \varphi_\epsilon \star \varphi_\epsilon(x-y)\,. \tag{1.3}$$

Let $u_\epsilon$ be the solution to the equation with smooth coefficients

$$\partial_t u_\epsilon(t,x) = \frac{1}{2}\Delta u_\epsilon(t,x) + u_\epsilon(\dot{W}_\epsilon(x) - C_\epsilon), \tag{1.4}$$

with the diverging constant

$$C_\epsilon = \frac{1}{\pi}\log \epsilon^{-1}. \tag{1.5}$$

Then $u_\epsilon$ converges in some weighted Hölder space to a limit $u$ that is defined to be the solution to (1.1), see [9, Theorem 4.1].

While the solution to (1.1) is well-defined, its statistical property remains a challenge. We refer to [1, 2, 5, 6, 14] for some relevant discussions. The goal of this note is to provide a Wiener chaos expansion of the solution $u$, in the short time regime. We assume $u_\epsilon(0,x) = u_0(x)$ for some bounded function $u_0$. Theorem 1.1 below shows that for small $t$, $u_\epsilon(t,x) \to u(t,x)$ in $L^2(\Omega)$ as $\epsilon \to 0$, and $u(t,x)$ is written explicitly as a Wiener chaos expansion in terms of the probability density of a polymer in a white noise environment, see (1.17). We hope that the explicit chaos expansion will provide another way of proving the convergence to (1.1), e.g. from a discrete system using the general criteria proved in [3, 14]. The tool we use is a combination of the Feynman-Kac representation and Malliavin calculus. By writing $u_\epsilon(t,x)$ in terms of a chaos expansion, it suffices to pass to the limit in each chaos.

1.1. **Elements of Malliavin calculus.** We give a brief introduction to Malliavin calculus and refer to [15] for more details. For any function $\phi \in L^2(\mathbb{R}^2)$, we define $W(\phi) = \int \phi \, dW$. Let $F$ be a smooth and cylindrical random variable of the form

$$F = f(W(\phi_1), \dots, W(\phi_n)),$$

with $\phi_i \in L^2(\mathbb{R}^2)$, $f \in C_p^\infty(\mathbb{R}^n)$ (namely $f$ and all its partial derivatives have polynomial growth), then the Malliavin derivative of $F$, denoted by $DF$, is the $L^2(\mathbb{R}^2)-$valued random variable defined by

$$DF = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(W(\phi_1), \dots, W(\phi_n))\phi_j \,. \tag{1.6}$$

For each positive integer $k$, $D^k F$ is defined to be the $k$-th iterated derivative of $F$, which is a random variable taking values in $L^2(\mathbb{R}^2)^{\otimes k}$, the $k$-th tensor product of $L^2(\mathbb{R}^2)$. The operator $D^k$ is closable from $L^2(\Omega)$ into $L^2(\Omega; L^2(\mathbb{R}^2)^{\otimes k})$ and we define the Sobolev space $\mathbb{D}^{k,2}$ as the closure of the space of smooth and cylindrical random variables under the norm

$$\|D^k F\|_{k,2} = \sqrt{\mathbb{E}\left[F^2 + \sum_{j=1}^k \|D^j F\|^2_{L^2(\mathbb{R}^2)^{\otimes j}}\right]}\,. \tag{1.7}$$

Define $\mathbb{D}^{\infty,2} = \bigcap_{k=1}^\infty \mathbb{D}^{k,2}$, and $L^2(\mathbb{R}^2)^{\odot k}$ as the $k$-th symmetric tensor product of $L^2(\mathbb{R}^2)$.

For any integer $n \geq 0$, we denote by $\mathbf{H}_n$ the $n$-th Wiener chaos of $W$. We recall that $\mathbf{H}_0$ is simply $\mathbb{R}$, and for $n \geq 1$, $\mathbf{H}_n$ is the closed linear subspace of $L^2(\Omega)$ generated by the random variables

$$\{H_n(W(h)) : h \in L^2(\mathbb{R}^2), \|h\|_{L^2(\mathbb{R}^2)} = 1\},$$

where $H_n$ is the $n$-th order Hermite polynomials. For any $n \geq 1$, the mapping

$$I_n(h^{\otimes n}) := H_n(W(h))$$

can be extended to a linear isometry between $L^2(\mathbb{R}^2)^{\odot n}$ and $\mathbf{H}_n$, with the isometric relation

$$\mathbb{E}[I_n(h^{\otimes n})^2] = n! \|h^{\otimes n}\|^2_{L^2(\mathbb{R}^2)^{\otimes n}}\,. \tag{1.8}$$

Consider now a random variable $F \in L^2(\Omega)$, it can be written as

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n)\,, \tag{1.9}$$

where the series converges in $L^2(\Omega)$, and the coefficients $f_n \in L^2(\mathbb{R}^2)^{\odot n}$ are determined by $F$. This identity is called the Wiener-chaos expansion of $F$.

When the above $F \in \mathbb{D}^{\infty,2}$, the $n-$th coefficient $f_n$ in the Wiener chaos expansion of $F$ can be explicitly written as [16, Page 3, equation (7)]

$$f_n = \frac{\mathbb{E}[D^n F]}{n!}\,. \tag{1.10}$$

1.2. **Brownian self-intersection local time and polymer in white noise.** The self-intersection local time of the planar Brownian motion is a classical subject in probability theory [4, 12, 13, 17, 18]. In the following, we discuss its connections to the parabolic Anderson model.

Using the Feynman-Kac formula, we write the solution to (1.4) as

$$u_\epsilon(t,x) = \mathbb{E}_B\left[u_0(x+B_t)\exp\left(\int_0^t \dot{W}_\epsilon(x+B_s)ds - C_\epsilon t\right)\right]\,, \tag{1.11}$$

where $B$ is a standard Brownian motion starting from the origin which is independent from $\dot{W}$, and $\mathbb{E}_B$ denotes the expectation with respect to $B$. Taking expectation with respect to $\dot{W}_\epsilon$ and using the fact that the exponent inside the expectation in (1.11) is of Gaussian distribution for each realization of the Brownian motion, we obtain

$$\mathbb{E}[u_\epsilon(t,x)] = \mathbb{E}_B\left[u_0(x+B_t)\exp\left(\int_0^t\int_0^s R_\epsilon(B_s-B_u)duds - C_\epsilon t\right)\right],$$

where we recall that $R_\epsilon$ is the covariance function of $\dot{W}_\epsilon$. It is well-known that

$$\gamma_\epsilon(t,B) := \int_0^t\int_0^s R_\epsilon(B_s-B_u)duds - \int_0^t\int_0^s \mathbb{E}_B[R_\epsilon(B_s-B_u)]duds \to \gamma(t,B) \tag{1.12}$$

almost surely, and $\gamma(t,B)$ is the so-called renormalized self-intersection local time of the planar Brownian motion formally written as

$$\gamma(t,B) = \int_0^t\int_0^s \delta(B_s-B_u)duds - \int_0^t\int_0^s \mathbb{E}_B[\delta(B_s-B_u)]duds. \tag{1.13}$$

In addition, there exists some critical $t_c > 0$ such that

$$\mathbb{E}_B[\exp(\gamma(t,B))]\begin{cases} <\infty & t<t_c, \\ =\infty & t>t_c. \end{cases} \tag{1.14}$$

The renormalization constant in (1.5) matches the expectation in (1.12) up to an $O(1)$ correction, and a calculation as in [6, Lemma 1.1] shows that there exists constants $\mu_1, \mu_2$ such that

$$\int_0^t\int_0^s \mathbb{E}_B[R_\epsilon(B_s-B_u)]duds - C_\epsilon t \to t(\mu_1+\mu_2\log t) \tag{1.15}$$

as $\epsilon \to 0$. For small $t$, it was shown in [6] that

$$\mathbb{E}[u(t,x)] = \lim_{\epsilon\to 0}\mathbb{E}[u_\epsilon(t,x)] = e^{t(\mu_1+\mu_2\log t)}\mathbb{E}_B[u_0(x+B_t)e^{\gamma(t,B)}]. \tag{1.16}$$

This motivates us to define

$$F(t) := \log \mathbb{E}_B[e^{\gamma(t,B)}],$$

so we can write

$$\mathbb{E}[u(t,x)] = e^{t(\mu_1+\mu_2\log t)+F(t)}\hat{\mathbb{E}}_{t,B}[u_0(x+B_t)],$$

where $\hat{\mathbb{E}}_{t,B}$ denotes the expectation with respect to the Wiener measure tilted by the factor $e^{\gamma(t,B)}$, i.e.,

$$\hat{\mathbb{E}}_{t,B}[X] = \frac{\mathbb{E}_B[Xe^{\gamma(t,B)}]}{\mathbb{E}_B[e^{\gamma(t,B)}]} = \mathbb{E}_B[Xe^{\gamma(t,B)}]e^{-F(t)}$$

for any bounded $X$. By the formal expression in (1.13), we can view $\hat{\mathbb{E}}_{t,B}$ as the expectation with respect to the annealed measure of a polymer in a white noise environment. By (1.14), it is clear that the measure is absolutely continuous with respect to the Wiener measure for small $t$. Applying the Radon-Nikodym theorem, for any $n \in \mathbb{Z}_+$ and $0 < s_1 < \ldots < s_n \le t < t_c$, there exists a non-negative measurable function, denoted by

$$\mathscr{F}_{s_1,\ldots,s_n} : \mathbb{R}^{2n} \to \mathbb{R},$$

such that

$$\hat{\mathbb{E}}_{t,B}[1_A(B_{s_1},\ldots,B_{s_n})] = \int_A \mathscr{F}_{s_1,\ldots,s_n}(x_1,\ldots,x_n)dx$$

for all $A \subset \mathbb{R}^{2n}$. In other words, $\mathscr{F}_{s_1,\ldots,s_n}$ is the joint spatial density function of the polymer path at $s_1 < \ldots < s_n$. We note that $\mathscr{F}$ actually depends on $t$ since the tilted measure depends on $t$. For our purpose, we use the simplified notation since $t$ is fixed. It is an elementary exercise to show that $\mathscr{F}_{s_1,\ldots,s_n}(x_1,\ldots,x_n)$ is jointly measurable in $(s_1,\ldots,s_n,x_1,\ldots,x_n)$. For the convenience of the reader, we present a proof in the appendix.

Denote $[0,t]^n_< := \{0 \le s_1 < \ldots < s_n \le t\}$, the following is our main result.

**Theorem 1.1.** *There exists $t_0 > 0$ such that for each $t \in (0,t_0), x \in \mathbb{R}^2$, the random variable $u_\epsilon(t,x)$ converges in $L^2(\Omega)$ to*

$$u(t,x) = \sum_{n=0}^{\infty} I_n(f_n(\cdot;t,x))\,. \tag{1.17}$$

*The coefficient $f_n(\cdot;t,x)$ is given by*

(1.18)

$$f_n(y_1,\ldots,y_n;t,x) = e^{t(\mu_1+\mu_2\log t)+F(t)}\int_{\mathbb{R}^2}\int_{[0,t]^n_<} u_0(x+z)\mathscr{F}_{s_1,\ldots,s_n,t}(y_1-x,\ldots,y_n-x,z)dsdz.$$

**Remark 1.2.** *The small time constraint in Theorem 1.1 seems necessary. It was shown in [6] that $\mathbb{E}[u(t,x)^2]$ is finite and admits a Feynman-Kac representation for small $t$, and we expect that $\mathbb{E}[u(t,x)^2] = \infty$ when $t$ is large, in light of (1.14).*

**Remark 1.3.** *Since the formal product $u \cdot \dot{W}$ in (1.1) comes from the classical physical product $u_\epsilon \dot{W}_\epsilon$ in (1.4), we may interpret it in the Stratonovich's sense. If it is replaced by the Wick product:*

$$\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + u(t,x) \diamond \dot{W}(x), \tag{1.19}$$

*a different chaos expansion was proved in* [10]*. Compared with* (1.17)*, the only difference is the lack of the weight $e^{\gamma(t,B)}$ in the definition of $\mathscr{F}$. This reduces the polymer measure to the original Wiener measure, in which case we have*

$$\mathscr{F}_{s_1,\ldots,s_n}(x_1,\ldots,x_n) = q_{s_1}(x_1)q_{s_2-s_1}(x_2-x_1)\ldots q_{s_n-s_{n-1}}(x_n-x_{n-1}), \tag{1.20}$$

*where $q_t(x) := (2\pi t)^{-1}e^{-|x|^2/2t}$ is the standard heat kernel. With $\mu_1 = \mu_2 = F = 0$, the expansion coefficient is given by*

$$\begin{aligned} &f_n(y_1,\ldots,y_n;t,x) \\ &= \int_{\mathbb{R}^2}\int_{[0,t]^n_<} u_0(x+z)\mathscr{F}_{s_1,\ldots,s_n,t}(y_1-x,\ldots,y_n-x,z)dsdz \\ &= \int_{[0,t]^n}\int_{\mathbb{R}^2} 1_{\{0<s_n<\ldots<s_1<t\}}\prod_{j=0}^n q_{s_j-s_{j+1}}(y_j-y_{j+1})u_0(z)dzds, \end{aligned} \tag{1.21}$$

*with the convention that $y_0 = x, y_{n+1} = z, s_{n+1} = 0$. Thus, the resulting chaos expansion is obtained by iterating the mild formulation of* (1.19)*. The missing exponential weight $e^{\gamma(t,B)}$ favors self-attracting of the polymer paths, which prevails in the intermittency behaviors of parabolic Anderson model. We refer to the recent monograph* [11] *for more details.*

**Remark 1.4.** *The same proof works in the one dimensional case, where the small time constraint can be removed, and there is no need to renormalize. A similar expansion coefficient as* (1.18) *holds.*

## 2. Proof of the main result

For fixed $t > 0, x \in \mathbb{R}^2, \epsilon > 0$ and each realization of the Brownian motion, we write the exponent in (1.11) as

$$\begin{aligned} \int_0^t \dot{W}_\epsilon(x+B_s)ds &= \int_0^t\int_{\mathbb{R}^2}\varphi_\epsilon(x+B_s-y)dW(y)ds \\ &= \int_{\mathbb{R}^2}\left(\int_0^t \varphi_\epsilon(x+B_s-y)ds\right)dW(y) \\ &= \int_{\mathbb{R}^2}\Phi^\epsilon_{t,x,B}(y)dW(y), \end{aligned}$$

with

$$\Phi^\epsilon_{t,x,B}(y) := \int_0^t \varphi_\epsilon(x+B_s-y)ds.$$

Then it is easy to see that $u_\epsilon(t,x)\in\mathbb{D}^{\infty,2}$, and

(2.1)
$$\begin{aligned}D^n u_\epsilon(t,x) =&\mathbb{E}_B\left[u_0(x+B_t)D^n\exp\left(\int_{\mathbb{R}^2}\Phi^\epsilon_{t,x,B}(y)dW(y)-C_\epsilon t\right)\right]\\ =&\mathbb{E}_B\left[u_0(x+B_t)\exp\left(\int_{\mathbb{R}^2}\Phi^\epsilon_{t,x,B}(y)dW(y)-C_\epsilon t\right)(\Phi^\epsilon_{t,x,B}(\cdot))^{\otimes n}\right].\end{aligned}$$

By the Stroock's formula (1.10), we can write the Wiener chaos expansion of $u_\epsilon(t,x)$ as

$$u_\epsilon(t,x)=\sum_{n=0}^\infty I_n\left(f_{\epsilon,n}(\cdot;t,x)\right), \tag{2.2}$$

with

(2.3)
$$\begin{aligned}f_{\epsilon,n}(\cdot;t,x) =&\frac{1}{n!}\mathbb{E}[D^n u_\epsilon(t,x)]\\ =&\frac{1}{n!}\mathbb{E}_B\left[u_0(x+B_t)\exp\left(\int_0^t\int_0^s R_\epsilon(B_s-B_u)duds-C_\epsilon t\right)\left(\Phi^\epsilon_{t,x,B}(\cdot)\right)^{\otimes n}\right].\end{aligned}$$

By (1.15), we define

$$r_\epsilon:=\int_0^t\int_0^s\mathbb{E}_B[R_\epsilon(B_s-B_u)]duds-C_\epsilon t-t(\mu_1+\mu_2\log t), \tag{2.4}$$

which goes to zero as $\epsilon\to0$, and rewrite

$$f_{\epsilon,n}(\cdot;t,x)=\frac{e^{t(\mu_1+\mu_2\log t)+r_\epsilon}}{n!}\mathbb{E}_B\left[u_0(x+B_t)\exp(\gamma_\epsilon(t,B))\left(\Phi^\epsilon_{t,x,B}(\cdot)\right)^{\otimes n}\right]. \tag{2.5}$$

To prove Theorem 1.1, it suffices to show that as $\epsilon\to0$,

$$\sum_{n=0}^\infty n!\|f_{\epsilon,n}(\cdot;t,x)-f_n(\cdot;t,x)\|^2_{L^2(\mathbb{R}^2)^{\otimes n}}\to0. \tag{2.6}$$

Define

$$\tilde f_{\epsilon,n}(\cdot;t,x):=\frac{e^{t(\mu_1+\mu_2\log t)+r_\epsilon}}{n!}\mathbb{E}_B\left[u_0(x+B_t)\exp(\gamma(t,B))\left(\Phi^\epsilon_{t,x,B}(\cdot)\right)^{\otimes n}\right]. \tag{2.7}$$

Since

$$\begin{aligned}&\|f_{\epsilon,n}(\cdot;t,x)-f_n(\cdot;t,x)\|^2_{L^2(\mathbb{R}^2)^{\otimes n}}\\ &\le2\|f_{\epsilon,n}(\cdot;t,x)-\tilde f_{\epsilon,n}(\cdot;t,x)\|^2_{L^2(\mathbb{R}^2)^{\otimes n}}+2\|\tilde f_{\epsilon,n}(\cdot;t,x)-f_n(\cdot;t,x)\|^2_{L^2(\mathbb{R}^2)^{\otimes n}},\end{aligned}$$

the proof of (2.6) reduces to the following three lemmas.

**Lemma 2.1.** *There exists $t_0,C>0$ independent of $\epsilon,n$ such that if $t<t_0$,*

$$\|f_{\epsilon,n}(\cdot;t,x)\|^2_{L^2(\mathbb{R}^2)^{\otimes n}}+\|\tilde f_{\epsilon,n}(\cdot;t,x)\|^2_{L^2(\mathbb{R}^2)^{\otimes n}}+\|f_n(\cdot;t,x)\|^2_{L^2(\mathbb{R}^2)^{\otimes n}}\le\frac{(Ct)^n}{n!}.$$

**Lemma 2.2.** *There exists $t_0>0$ such that if $t<t_0$,*

$$\|f_{\epsilon,n}(\cdot;t,x)-\tilde f_{\epsilon,n}(\cdot;t,x)\|^2_{L^2(\mathbb{R}^2)^{\otimes n}}\to0,\quad \text{as } \epsilon\to0.$$

**Lemma 2.3.** *There exists $t_0 > 0$ such that if $t < t_0$,*

$$\|\tilde{f}_{\epsilon,n}(\cdot;t,x) - f_n(\cdot;t,x)\|^2_{L^2(\mathbb{R}^2)^{\otimes n}} \to 0, \quad \text{as } \epsilon \to 0.$$

In the following, we use the notation $a \lesssim b$ when $a \leq Cb$ for some constant $C > 0$ independent of $\epsilon, n$.

*Proof of Lemma 2.1.* The proof of $f_{\epsilon,n}$ and $\tilde{f}_{\epsilon,n}$ is the same. Take $f_{\epsilon,n}$ for example:

$$\begin{aligned}&\|f_{\epsilon,n}(\cdot;t,x)\|^2_{L^2(\mathbb{R}^2)^{\otimes n}}\\ \lesssim&\frac{1}{(n!)^2}\int_{\mathbb{R}^{2n}}\mathbb{E}_{B^1,B^2}\left[\prod_{j=1}^{2}\left(e^{\gamma_\epsilon(t,B^j)}\prod_{k=1}^{n}\Phi^\epsilon_{t,x,B^j}(y_k)\right)\right]dy,\end{aligned}$$

where $B^1, B^2$ stand for independent Brownian motions. Performing the integral in the $y$ variable, the r.h.s. of the above display is bounded by

$$\frac{1}{(n!)^2}\mathbb{E}_{B^1,B^2}\left[e^{\gamma_\epsilon(t,B^1)+\gamma_\epsilon(t,B^2)}\left(\int_{[0,t]^2}R_\epsilon(B^1_s - B^2_u)dsdu\right)^n\right].$$

Now we use Cauchy-Schwarz inequality and (A.1)-(A.2) to derive

$$\begin{aligned}&\mathbb{E}_{B^1,B^2}\left[e^{\gamma_\epsilon(t,B^1)+\gamma_\epsilon(t,B^2)}\left(\int_{[0,t]^2}R_\epsilon(B^1_s - B^2_u)dsdu\right)^n\right]\\ \leq&\sqrt{\mathbb{E}_{B^1,B^2}\left[e^{2\gamma_\epsilon(t,B^1)+2\gamma_\epsilon(t,B^2)}\right]\mathbb{E}_{B^1,B^2}\left[\left(\int_{[0,t]^2}R_\epsilon(B^1_s - B^2_u)dsdu\right)^{2n}\right]}\\ \lesssim&(Ct)^n\sqrt{(2n)!}\,.\end{aligned}$$

An application of Stirling's approximation yields the desired result. By Lemma 2.3, the same estimate holds for $f_n$. The proof is complete.

*Proof of Lemma 2.2.* By the same discussion as in the proof of Lemma 2.1, we have

$$\begin{aligned}&\|f_{\epsilon,n}(\cdot;t,x) - \tilde{f}_{\epsilon,n}(\cdot;t,x)\|^2_{L^2(\mathbb{R}^2)^{\otimes n}}\\ \leq&\frac{1}{(n!)^2}\mathbb{E}_{B^1,B^2}\left[\left|(e^{\gamma_\epsilon(t,B^1)} - e^{\gamma(t,B^1)})(e^{\gamma_\epsilon(t,B^2)} - e^{\gamma(t,B^2)})\right|\left(\int_0^t\int_0^t R_\epsilon(B^1_s - B^2_u)dsdu\right)^n\right].\end{aligned}$$

By the fact that $\gamma_\epsilon \to \gamma$ a.s. as $\epsilon \to 0$ and (A.3), we know that the random variable inside the above expectation converges to zero in probability. The uniform integrability is guaranteed by (A.1) and (A.2). Thus, the r.h.s. of the above display goes to zero as $\epsilon \to 0$.

*Proof of Lemma 2.3.* First, we claim that $\tilde{f}_{\epsilon,n}(\cdot;t,x)$ is a Cauchy sequence in $L^2(\mathbb{R}^2)^{\otimes n}$. It suffices to prove the convergence of

$$\lim_{\epsilon_1,\epsilon_2\to 0}\langle\tilde{f}_{\epsilon_1,n}(\cdot;t,x), \tilde{f}_{\epsilon_2,n}(\cdot;t,x)\rangle_{L^2(\mathbb{R}^2)^{\otimes n}}. \tag{2.8}$$

By applying Lemma A.1, we have

$$\mathbb{E}_{B^1,B^2}\left[\prod_{j=1}^{2}u_0(x+B^j_t)e^{\gamma(t,B^j)}\left(\int_0^t\int_0^t R_{\epsilon_1,\epsilon_2}(B^1_s - B^2_u)dsdu\right)^n\right]$$

converges as $\epsilon_1, \epsilon_2 \to 0$, where $R_{\epsilon_1,\epsilon_2} := \varphi_{\epsilon_1} \star \varphi_{\epsilon_2}$. This proves (2.8).

Next, we show that $\tilde{f}_{\epsilon,n}(\cdot; t, x) \to f_n(\cdot; t, x)$ in $L^1(\mathbb{R}^2)^{\otimes n}$ which implies that $f_n(\cdot; t, x) \in L^2(\mathbb{R}^2)^{\otimes n}$ and completes the proof. We have

$$
\begin{aligned}
&\tilde{f}_{\epsilon,n}(y_1, \ldots, y_n; t, x) \\
&= \frac{e^{t(\mu_1+\mu_2 \log t)+r_\epsilon}}{n!} \mathbb{E}_B \left[ u_0(x + B_t) e^{\gamma(t,B)} \prod_{k=1}^{n} \Phi^{\epsilon}_{t,x,B}(y_k) \right] \\
&= \frac{e^{t(\mu_1+\mu_2 \log t)+F_t+r_\epsilon}}{n!} \hat{\mathbb{E}}_{t,B} \left[ u_0(x + B_t) \prod_{k=1}^{n} \Phi^{\epsilon}_{t,x,B}(y_k) \right] \\
&= e^{t(\mu_1+\mu_2 \log t)+F_t+r_\epsilon} \varphi_\epsilon^{\otimes n} \star \mathscr{G}(y_1 - x, \ldots, y_n - x),
\end{aligned}
\tag{2.9}
$$

with

$$
\mathscr{G}(z_1, \ldots, z_n) := \int_{\mathbb{R}^2} \int_{[0,t]^n_<} u_0(x+z_{n+1}) \mathscr{F}_{s_1,\ldots,s_n,t}(z_1, \ldots, z_n, z_{n+1}) ds dz_{n+1} \in L^1(\mathbb{R}^2)^{\otimes n}.
$$

Since $\varphi_\epsilon^{\otimes n}$ is an approximation to identity, by the classical convolution theorem,

$$
\varphi_\epsilon^{\otimes n} \star \mathscr{G} \to \mathscr{G} \text{ in } L^1(\mathbb{R}^2)^{\otimes n}.
$$

Thus,

$$
\begin{aligned}
\tilde{f}_{\epsilon,n}(y_1, \ldots, y_n; t, x) \to& e^{t(\mu_1+\mu_2 \log t)+F_t} \mathscr{G}(y_1 - x, \ldots, y_n - x) \\
&= f_n(y_1, \ldots, y_n; t, x)
\end{aligned}
$$

in $L^1(\mathbb{R}^2)^{\otimes n}$.

## Appendix A. Technical lemmas

A.1. **Measurability of $\mathscr{F}$.** We show that $\mathscr{F}_{s_1,\ldots,s_n}(x_1, \ldots, x_n)$ is jointly measurable in the $(s, x)$ variable. Fix any $0 < s_1 < \ldots < s_n \le t$, consider

$$
\begin{aligned}
\mathscr{F}^{\epsilon}_{s_1,\ldots,s_n}(x_1, \ldots, x_n) &:= \hat{\mathbb{E}}_{t,B} \left[ \prod_{j=1}^{n} \varphi_\epsilon(B_{s_j} - x_j) \right] \\
&= \int_{\mathbb{R}^{2n}} \prod_{j=1}^{n} \varphi_\epsilon(y_j - x_j) \mathscr{F}_{s_1,\ldots,s_n}(y_1, \ldots, y_n) dy.
\end{aligned}
$$

The last integral converges in $L^1(\mathbb{R}^{2n})$ to $\mathscr{F}_{s_1,\ldots,s_n}$. It is clear that $\mathscr{F}^{\epsilon}_{s_1,\ldots,s_n}(x_1, \ldots, x_n)$ is continuous in both $s$ and $x$ variable, hence it is measurable. If we can show $\mathscr{F}^{\epsilon}_{s_1,\ldots,s_n}(x_1, \ldots, x_n)$ converges in $L^1([0, t]^n_< \times \mathbb{R}^{2n})$ to some $g_{s_1,\ldots,s_n}(x_1, \ldots, x_n)$, then $g = \mathscr{F}$ almost everywhere in $[0, t]^n_< \times \mathbb{R}^{2n}$, which implies $\mathscr{F}$ is measurable.

For fixed $s_1, \ldots, s_n$, we have

$$
\int_{\mathbb{R}^{2n}} |\mathscr{F}^{\epsilon}_{s_1,\ldots,s_n}(x_1, \ldots, x_n) - \mathscr{F}^{\delta}_{s_1,\ldots,s_n}(x_1, \ldots, x_n)| dx \to 0
$$

as $\epsilon, \delta \to 0$. In addition,

$$\int_{\mathbb{R}^{2n}} |\mathscr{F}^{\epsilon}_{s_1,\dots,s_n}(x_1,\dots,x_n) - \mathscr{F}^{\delta}_{s_1,\dots,s_n}(x_1,\dots,x_n)|dx \\ \leq \int_{\mathbb{R}^{2n}} \left(\mathscr{F}^{\epsilon}_{s_1,\dots,s_n}(x_1,\dots,x_n) + \mathscr{F}^{\delta}_{s_1,\dots,s_n}(x_1,\dots,x_n)\right) dx = 2.$$

Thus, by the dominated convergence theorem, we have

$$\int_{[0,t]^n_<} \int_{\mathbb{R}^{2n}} |\mathscr{F}^{\epsilon}_{s_1,\dots,s_n}(x_1,\dots,x_n) - \mathscr{F}^{\delta}_{s_1,\dots,s_n}(x_1,\dots,x_n)|dxds \to 0$$

as $\epsilon, \delta \to 0$. This completes the proof.

A.2. **Estimates on intersection local time.** We collect some standard estimates on the intersection local time of planar Brownian motion. Recall that $R_{\epsilon_1,\epsilon_2} = \varphi_{\epsilon_1} \star \varphi_{\epsilon_2}$, and assume that the Brownian motion is built on the probability space $(\Sigma, \mathcal{A}, \mathbb{P}_B)$.

**Lemma A.1.** *For any $\lambda > 0$, there exist constants $C, t_0 > 0$ such that*

$$\sup_{\epsilon\in(0,1],t\in[0,t_0]} \mathbb{E}_B[e^{\lambda\gamma_\epsilon(t,B)}] \leq C\,, \tag{A.1}$$

*and for all $n \in \mathbb{N}$,*

$$\sup_{\epsilon_1,\epsilon_2\in(0,1]} \mathbb{E}_{B^1,B^2}\left[\left(\int_{[0,t]^2} R_{\epsilon_1,\epsilon_2}(B^1_s - B^2_u)dsdu\right)^n\right] \leq n!(Ct)^n\,. \tag{A.2}$$

*In addition,*

$$\int_{[0,t]^2} R_{\epsilon_1,\epsilon_2}(B^1_s - B^2_u)dsdu \to \int_{[0,t]^2} \delta(B^1_s - B^2_u)dsdu \tag{A.3}$$

*in $L^2(\Sigma)$ as $\epsilon_1, \epsilon_2 \to 0$, where the r.h.s. is the so-called mutual intersection local time of planar Brownian motions.*

*Proof* The uniform exponential integrability (A.1) is shown in [6, Lemma A.1]. It also contains a moment estimate of the form

$$\sup_{\epsilon\in(0,1]} \mathbb{E}_{B^1,B^2}\left[\left(\int_{[0,t]^2} R_{\epsilon}(B^1_s - B^2_u)dsdu\right)^n\right] \\ \leq \mathbb{E}_{B^1,B^2}\left[\left(\int_{[0,t]^2} \delta(B^1_s - B^2_u)dsdu\right)^n\right] \leq n!(Ct)^n.$$

The same proof leads to (A.2).

Since

$$\int_{[0,t]^2} R_{\epsilon}(B^1_s - B^2_u)dsdu \to \int_{[0,t]^2} \delta(B^1_s - B^2_u)dsdu$$

in $L^2(\Sigma)$, to prove (A.3), it suffices to show that as $\epsilon_1, \epsilon_2 \to 0$,

$$\int_{[0,t]^2} R_{\epsilon_1,\epsilon_2}(B^1_s - B^2_u)dsdu - \int_{[0,t]^2} R_{\epsilon_1}(B^1_s - B^2_u)dsdu \to 0$$

in $L^2(\Sigma)$, which reduces to the convergence of

$$\mathbb{E}_{B^1,B^2}\left[\int_{[0,t]^4} R_{\epsilon_1,\epsilon_2}(B^1_{s_1}-B^2_{u_1})R_{\epsilon_3,\epsilon_4}(B^1_{s_2}-B^2_{u_2})dsdu\right]$$

as $\epsilon_j \to 0, j=1,2,3,4$. We write $R_{\epsilon_i,\epsilon_j}$ in the Fourier domain so that the above expectation equals to

$$\frac{1}{(2\pi)^4}\int_{[0,t]^4}\int_{\mathbb{R}^4}\hat{\varphi}(\epsilon_1\xi)\hat{\varphi}(\epsilon_2\xi)\hat{\varphi}(\epsilon_3\eta)\hat{\varphi}(\epsilon_4\eta)\mathbb{E}_{B^1,B^2}[e^{i\xi\cdot(B^1_{s_1}-B^2_{u_1})}e^{i\eta\cdot(B^1_{s_2}-B^2_{u_2})}]d\xi d\eta dsdu\,.$$

It suffices to use the bound

$$\int_{[0,t]^4}\int_{\mathbb{R}^4}\mathbb{E}_{B^1,B^2}[e^{i\xi\cdot(B^1_{s_1}-B^2_{u_1})}e^{i\eta\cdot(B^1_{s_2}-B^2_{u_2})}]d\xi d\eta dsdu<\infty$$

and the dominated convergence theorem to complete the proof.

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