Fluctuations of Parabolic Equations with Large Random Potentials

Yu Gu* Guillaume Bal*

September 29, 2014

Abstract

In this paper, we present a fluctuation analysis of a type of parabolic equations with large, highly oscillatory, random potentials around the homogenization limit. With a Feynman-Kac representation, the Kipnis-Varadhan's method, and a quantitative martingale central limit theorem, we derive the asymptotic distribution of the rescaled error between heterogeneous and homogenized solutions under different assumptions in dimension $d \geq 3$. The results depend highly on whether a stationary corrector exits.

1 Introduction

Equations with microscopic structure arise naturally in physics and applied science, and homogenization has become important to derive macroscopic models in both periodic and random settings, see [21, 27, 30, 19]. When the underlying random medium is stationary and ergodic, stochastic homogenization replaces it by a deterministic, and properly-averaged constant, which, from a probabilistic point of view, is a law of large numbers type result. Much less is known regarding the random fluctuations though, e.g., the size of the error between heterogeneous and homogenized solutions, and the distribution of the rescaled error. The goal of this paper is to present a systematic analysis of random fluctuations produced by parabolic equations with large random potentials.

Error estimates have been derived for stochastic homogenization in different contexts, including the recent work on discrete and nonlinear setting [30, 8, 12, 13, 22, 10, 11]. However, asymptotic distributions are less well-understood. When the randomness is sufficiently mixing, it is natural to expect the central limit type of results to hold. For a homogenization constant, they are derived in [25, 6]. For a corrector, they are obtained in [24, 23]. For one dimensional case or equations with bounded random potentials, when certain integral representation of the solution is available, asymptotic distributions of the rescaled errors are derived for both short- and long-range-correlated randomness, leading to Gaussian or possible non-Gaussian limit [9, 7, 1, 5, 4, 14].

In this paper, following the framework of [16], we focus on the example of a parabolic equation with large, highly oscillatory, random potentials. A similar type of equations has been analyzed in [2, 3, 28, 29, 18] to obtain either homogenization or convergence to stochastic partial differential equation (SPDE). Asymptotic Gaussian fluctuations are proved in [3] by combinatorial techniques

*Department of Applied Physics & Applied Mathematics, Columbia University, New York, NY 10027 (yg2254@columbia.edu; gb2030@columbia.edu)

for an equation with Gaussian potentials. One of the main goals here is to present an example of non-Gaussian potential for which such a result holds.

The main tool we use is a probabilistic representation and the Kipnis-Varadhan's method [20], which helps to reduce the error between heterogeneous and homogenized solutions to the Wasserstein distance between martingales and Brownian motions, plus residues caused by a corrector function. By a simple modification of the quantitative martingale central limit theorem developed by Mourrat [22], we obtain an accurate quantification of the Wasserstein distance, and are able to derive the asymptotic distribution under different assumptions in dimension $d \geq 3$. A similar approach will be applied to parabolic operators in divergence form in [17].

The results depend highly on the existence of a stationary corrector through the dimension. On one hand, when the stationary corrector does not exist in $d = 3$, we prove a central limit result in Theorem 2.4 for Gaussian and Poissonian potentials. The weak convergence limit can then be appropriately expressed as a stochastic parabolic equation with an additive noise. While the distribution we analyze is written as a conditional expectation by the probabilistic representation, we are able to link it to a parabolic equation with an additive random potential and eventually show that the random potential can be replaced by a white noise. On the other hand, when the stationary corrector exists in $d \geq 5$, for a large class of strongly mixing potentials, we show in Theorem 2.8 that the random fluctuation converges to the stationary corrector in distribution. The limit is not necessarily Gaussian, and the error decomposition there is consistent with a formal two-scale expansion. For the critical dimension $d = 4$ in which the stationary corrector does not exist, we present a decomposition of the error in Theorem 2.6 for equations with constant initial conditions.

The rest of the paper is organized as follows. We state the main results in Section 2. We then review some estimates obtained in [16] and prove a quantitative martingale central limit theorem in Section 3. In Section 4, 5 and 6, we prove Theorem 2.4 for $d = 3$. In Section 7, Theorem 2.8 and 2.6 are proved for $d \geq 5$ and $d = 4$ respectively. Technical Lemmas are left in the Appendix.

Here are notations used throughout the paper. We use $\mathbb{E}$ to denote the expectation with respect to the random environment, and $\mathbb{E}_B, \mathbb{E}_W$ the expectations with respect to independent Browian motions B_t, W_t, respectively. We denote the normal distribution with mean μ and variance σ^2 by $N(\mu, \sigma^2)$, and $q_t(x)$ is the density function of $N(0, t)$. Let $G_\lambda(x)$ be the Green's function of $\lambda - \frac{1}{2}\Delta$. Let $f^\lambda(x) = \int_{\mathbb{R}^d} \varphi(x-y)G_\lambda(y)dy$, $f_k^\lambda(x) = \int_{\mathbb{R}^d} \varphi(x-y)\partial_{x_k}G_\lambda(y)dy$, where φ is the shape function of the Poissonian potential defined in Assumption 2.3 below. The Fourier transform is denoted as $\mathcal{F}\{f\}(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-i\xi\cdot x}dx$. The convolution is denoted as $(f \star g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy$. When we write $a \lesssim b$, it means $a \leq Cb$ for some $C > 0$ independent of ε. Let $a \wedge b = \min(a, b)$, and $a \vee b = \max(a, b)$. For multidimensional integrations, $\prod_i dx_i$ is abbreviated as dx. Throughout the paper we assume the dimension $d \geq 3$.

2 Problem setup and main results

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a random medium associated with a group of measure-preserving, ergodic transformations $\{\tau_x, x \in \mathbb{R}^d\}$, and $\mathbb{E}$ denote the expectation. Let $\mathbb{V} \in L^2(\Omega)$ such that $\int_\Omega \mathbb{V}(\omega)\mathbb{P}(d\omega) = 0$. Define the stationary random field $V(x, \omega) = \mathbb{V}(\tau_x\omega)$ and consider the following equation when

$d \geq 3$:

$$\partial_t u_\varepsilon(t,x,\omega) = \frac{1}{2}\Delta u_\varepsilon(t,x,\omega) + i\frac{1}{\varepsilon}V(\frac{x}{\varepsilon},\omega)u_\varepsilon(t,x,\omega) \tag{2.1}$$

with initial condition $u_\varepsilon(0,x,\omega) = f(x)$ for $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. We will omit the dependence on the particular realization ω and write $u_\varepsilon(t,x)$ and $V(x)$ from now on.

Let $\{D_k, k=1,\ldots,d\}$ be the $L^2(\Omega)$ generator of T_x, which is defined as $T_x f(\omega) = f(\tau_x\omega)$, and the Laplacian operator $L = \frac{1}{2}\sum_{k=1}^d D_k^2$. We use $\langle .,. \rangle$ to denote the inner product in $L^2(\Omega)$ and $\|.\|$ the $L^2(\Omega)$ norm. Assuming T_x is strongly continuous in $L^2(\Omega)$, we obtain the spectral resolution

$$T_x = \int_{\mathbb{R}^d} e^{i\xi\cdot x}U(d\xi), \tag{2.2}$$

where $U(d\xi)$ is the associated projection valued measure. We assume there is a non-negative power spectrum $\hat{R}(\xi)$ associated with $\mathbb{V}$, i.e., $\hat{R}(\xi)d\xi = (2\pi)^d\langle U(d\xi)\mathbb{V},\mathbb{V}\rangle$. Clearly

$$R(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\hat{R}(\xi)e^{i\xi\cdot x}d\xi \tag{2.3}$$

is the covariance function of V.

[16, Theorem 2.2] shows that if $\hat{R}(\xi)|\xi|^{-2}$ is integrable, then

$$u_\varepsilon(t,x) \to u_{hom}(t,x)$$

in probability with u_{hom} solving the homogenized equation

$$\partial_t u_{hom}(t,x) = \frac{1}{2}\Delta u_{hom}(t,x) - \frac{1}{2}\sigma^2 u_{hom}(t,x) \tag{2.4}$$

with the same initial condition $u_{hom}(0,x) = f(x)$ and the homogenization constant

$$\sigma^2 = \frac{4}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{\hat{R}(\xi)}{|\xi|^2}d\xi.$$

Remark 2.1. For the singularity $|\xi|^{-2}$ to be integrable around the origin, $d \geq 3$ is necessary.

If an additional strongly mixing condition of V is satisfied [16, Assumption 2.4], [16, Theorem 2.6] proves an error estimate:

$$\mathbb{E}\{|u_\varepsilon(t,x) - u_{hom}(t,x)|\} \lesssim \begin{cases} \varepsilon^{\frac{1}{2}} & d=3, \\ \varepsilon|\log\varepsilon|^{\frac{1}{2}} & d=4, \\ \varepsilon & d\geq 5. \end{cases} \tag{2.5}$$

Remark 2.2. For the initial condition f, we actually only need the integrability of $\hat{f}(\xi)(1+|\xi|)$. If $f \equiv const$, since $\int_{\mathbb{R}^d}\delta(\xi)(1+|\xi|)d\xi = 1$, heuristically we still have the integrability of $\hat{f}(\xi)(1+|\xi|)$. It can be checked that the estimate still holds.

The goal of this paper is to go beyond the error estimate and analyze the rescaled fluctuation. In the following, we state the main results under different assumptions on the random potentials.

2.1 Central limit theorem: $d = 3$

Assumption 2.3. *V is assumed to be Gaussian or Poissonian satisfying $\hat{R}(0) > 0$, and*

- *when V is Gaussian, for any $\alpha > 0$, there exists $C_\alpha > 0$ such that the covariance function satisfies $|R(x)| \leq C_\alpha(1 \wedge |x|^{-\alpha})$.*
- *when V is Poissonian, $V(x) = \int_{\mathbb{R}^d} \varphi(x-y)\omega(dy) - c_\varphi$, where the shape function φ is continuous, compactly supported and satisfies $\int_{\mathbb{R}^d} \varphi(x)dx = c_\varphi$, and $\omega(dy)$ is the Poissonian point process with Lebesgue measure dy as its intensity. Then $R(x) = \int_{\mathbb{R}^d} \varphi(x+y)\varphi(y)dy$ is compactly supported, and $\hat{R}(\xi) = |\hat{\varphi}(\xi)|^2$.*

In particular, for the Poissonian case, $\hat{R}(0) > 0$ implies $c_\varphi = \int_{\mathbb{R}^d} \varphi(x)dx \neq 0$, since $\hat{R}(0) = c_\varphi^2$. The following is the main result.

Theorem 2.4 ($d = 3$)**.** *Under Assumption 2.3, we have*

$$\frac{u_\varepsilon(t,x) - u_{hom}(t,x)}{\varepsilon^{\frac{1}{2}}} \Rightarrow v(t,x) \tag{2.6}$$

weakly with $v(t,x)$ solving the following SPDE with additive spatial white noise and zero initial condition:

$$\partial_t v(t,x) = \frac{1}{2}\Delta v(t,x) - \frac{1}{2}\sigma^2 v(t,x) + i\sqrt{\hat{R}(0)}u_{hom}(t,x)\dot{W}(x). \tag{2.7}$$

The weak convergence is in the following sense:

1. *As a process in $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, the finite dimensional distributions of $\varepsilon^{-\frac{1}{2}}(u_\varepsilon(t,x) - u_{hom}(t,x)) \Rightarrow v(t,x)$ weakly.*
2. *The distribution of $\varepsilon^{-\frac{1}{2}} \int_{\mathbb{R}^d}(u_\varepsilon(t,x) - u_{hom}(t,x))g(x)dx \Rightarrow \int_{\mathbb{R}^d} v(t,x)g(x)dx$ weakly for any fixed t and test function $g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$.*

It is clear that $v(t,x)$ is a Gaussian process, so Theorem 2.4 can be regarded as a central limit result.

2.2 Error decomposition by a corrector: $d \geq 4$

For the Laplacian operater $L = \frac{1}{2}\sum_{k=1}^d D_k^2$, a regularized corrector Φ_λ is defined by

$$(\lambda - L)\Phi_\lambda = \mathbb{V} \tag{2.8}$$

for $\lambda > 0$. In Lemma 7.1 below, we will show that the $L^2(\Omega)$ limit of Φ_λ exists iff $\hat{R}(\xi)|\xi|^{-4}$ is integrable. When the potential V is short-range-correlated and $d \geq 5$, we can define the corrector $\Phi = \lim_{\lambda \to 0} \Phi_\lambda$ in $L^2(\Omega)$ and it is the solution of

$$-L\Phi = \mathbb{V}. \tag{2.9}$$

The following mixing assumption is the same as [16, Assumption 2.4].

Assumption 2.5 (Strongly mixing assumption). *$\mathbb{E}\{V^6(x)\} < \infty$ and there exists a mixing coefficient $\rho(r)$ decreasing in $r \in [0, \infty)$ such that for any $\beta > 0$, $\rho(r) \leq C_\beta(1 \wedge r^{-\beta})$ for some $C_\beta > 0$ and the following bound holds:*

$$\mathbb{E}\{\phi_1(V)\phi_2(V)\} \leq \rho(r)\sqrt{\mathbb{E}\{\phi_1^2(V)\}\mathbb{E}\{\phi_2^2(V)\}} \tag{2.10}$$

for any two compact sets K_1, K_2 with $d(K_1, K_2) = \inf_{x_1 \in K_1, x_2 \in K_2}\{|x_1 - x_2|\} \geq r$ and any random variables $\phi_1(V), \phi_2(V)$ with $\phi_i(V)$ being $\mathcal{F}_{K_i}-$measurable and $\mathbb{E}\{\phi_i(V)\} = 0$.

Theorem 2.6 ($d = 4$). *Under Assumption 2.5, if $f \equiv const$, we have for fixed (t, x) that*

$$u_\varepsilon(t,x) = u_{hom}(t,x) + i\varepsilon u_{hom}(t,x)\Phi_{\varepsilon^2}(\tau_{\frac{x}{\varepsilon}}\omega) + o(\varepsilon|\log\varepsilon|^{\frac{1}{2}}), \tag{2.11}$$

where $\frac{o(\varepsilon|\log\varepsilon|^{\frac{1}{2}})}{\varepsilon|\log\varepsilon|^{\frac{1}{2}}} \to 0$ in $L^1(\Omega)$.

We will see below that $\mathbb{E}\{|\varepsilon\Phi_{\varepsilon^2}(\tau_{\frac{x}{\varepsilon}}\omega)|\} \lesssim \varepsilon|\log\varepsilon|^{\frac{1}{2}}$, so Theorem 2.6 implies for fixed (t, x) that

$$\frac{u_\varepsilon(t,x) - u_{hom}(t,x)}{\varepsilon|\log\varepsilon|^{\frac{1}{2}}} \sim iu_{hom}(t,x)\frac{\Phi_{\varepsilon^2}(\tau_{\frac{x}{\varepsilon}}\omega)}{|\log\varepsilon|^{\frac{1}{2}}}. \tag{2.12}$$

It turns out that $|\log\varepsilon|^{-\frac{1}{2}}\Phi_{\varepsilon^2}(\tau_x\omega)$ does not convergence in $L^2(\Omega)$, but we have the convergence in distribution.

Corollary 2.7. *Under the assumption of Theorem 2.6, if we further assume V is Gaussian or Poissonian as in Assumption 2.3, then*

$$\frac{\Phi_{\varepsilon^2}(\tau_x\omega)}{|\log\varepsilon|^{\frac{1}{2}}} \Rightarrow N(0, \frac{4\hat{R}(0)}{(2\pi)^d}) \tag{2.13}$$

in distribution, which implies

$$\frac{u_\varepsilon(t,x) - u_{hom}(t,x)}{\varepsilon|\log\varepsilon|^{\frac{1}{2}}} \Rightarrow iu_{hom}(t,x)N(0, \frac{4\hat{R}(0)}{(2\pi)^d}) \tag{2.14}$$

in distribution.

Theorem 2.8 ($d \geq 5$). *Under Assumption 2.5, we have for fixed (t, x) that*

$$u_\varepsilon(t,x) = u_{hom}(t,x) + i\varepsilon u_{hom}(t,x)\Phi(\tau_{\frac{x}{\varepsilon}}\omega) + i\varepsilon u_{hom}(t,x)C_{\mathbb{V}}t + o(\varepsilon), \tag{2.15}$$

where $C_{\mathbb{V}}$ is some deterministic constant that can be computed explicitly, and $\frac{o(\varepsilon)}{\varepsilon} \to 0$ in $L^1(\Omega)$.

$C_{\mathbb{V}}$ is given by (7.33). If we assume some symmetry property of the distribution of $V(x)$, e.g., $\mathbb{E}\{V(x_1)V(x_2)V(x_3)\} = 0, \forall x_1, x_2, x_3 \in \mathbb{R}^d$ as in the Gaussian case, we have $C_{\mathbb{V}} = 0$, i.e., the bias vanishes.

Since $\Phi(\tau_x\omega)$ is a stationary process, Theorem 2.8 implies that for fixed (t, x), we have

$$\frac{u_\varepsilon(t,x) - \mathbb{E}\{u_\varepsilon(t,x)\}}{\varepsilon} \Rightarrow iu_{hom}(t,x)\Phi(\tau_x\omega) \tag{2.16}$$

in distribution as $\varepsilon \to 0$. The limit is not necessarily Gaussian.

2.3 Remarks on the results

We first point out an important difference between the results in Theorem 2.4 and Theorem 2.6, 2.8. When $d = 3$, we obtain both the weak convergence for fixed (t, x) and the weak convergence weakly in space. When $d \geq 4$, our approach only leads to the weak convergence for fixed (t, x). Take $d \geq 5$ for example, Theorem 2.8 shows the random fluctuation

$$u_\varepsilon(t,x) - \mathbb{E}\{u_\varepsilon(t,x)\} = i\varepsilon u_{hom}(t,x)\Phi(\tau_{\frac{x}{\varepsilon}}\omega) + o(\varepsilon). \tag{2.17}$$

When considered weakly in space, it is actually much smaller than ε. In general, for random variables of the form $\int_{\mathbb{R}^d} V(x/\varepsilon)g(x)dx$ with $g \in \mathcal{C}_c^\infty$, we get an order of $\varepsilon^{\frac{d}{2}}$. In our case, since the power spectrum of $\Phi(\tau_x\omega)$ blows up at the origin, we actually obtain $\int_{\mathbb{R}^d} i\varepsilon u_{hom}(t,x)\Phi(\tau_{\frac{x}{\varepsilon}}\omega)g(x)dx \sim \varepsilon^{\frac{d-2}{2}} \ll \varepsilon$. The size of the error is consistent with the result obtained by the second author for Gaussian potentials [3, Theorem 2], where it is shown that

$$\int_{\mathbb{R}^d} \frac{u_\varepsilon(t,x) - \mathbb{E}\{u_\varepsilon(t,x)\}}{\varepsilon^{\frac{d-2}{2}}} g(x)dx \Rightarrow \int_{\mathbb{R}^d} v(t,x)g(x)dx \tag{2.18}$$

in distribution. $v(t, x)$ in (2.18) is the formal solution to the SPDE (2.7) obtained in Theorem 2.4 when $d = 3$. Note that (2.7) is only well-posed when $d \leq 3$, but $\int_{\mathbb{R}^d} v(t,x)g(x)dx$ is well-defined in any dimension if we plug the formal Wiener integral expression of $v(t, x)$. However, it is straightforward to check that

$$\int_{\mathbb{R}^d} \frac{i\varepsilon u_{hom}(t,x)\Phi(\tau_{\frac{x}{\varepsilon}}\omega)}{\varepsilon^{\frac{d-2}{2}}} g(x)dx \not\Rightarrow \int_{\mathbb{R}^d} v(t,x)g(x)dx \tag{2.19}$$

in distribution. On one hand, it indicates that (2.17) only holds for fixed (t, x) and is not true weakly in space, i.e., the $o(\varepsilon)$ term actually contributes weakly in space. On the other hand, we note that Theorem 2.4 is consistent with (2.18) when $d = 3$.

Now we discuss the different assumptions we made on the random potentials.

When $d = 3$, we assume a Gaussian or Poissonian potential to obtain the following limiting SPDE after some explicit calculations:

$$\partial_t v(t,x) = \frac{1}{2}\Delta v(t,x) - \frac{1}{2}\sigma^2 v(t,x) + i\sqrt{\hat{R}(0)}u_{hom}(t,x)\dot{W}(x).$$

From the above equation, the homogenization constant σ^2 shows up as a potential, and it comes from the averaging of $\varepsilon^{-1}V(x/\varepsilon)$. There is also the spatial white noise $\dot{W}(x)$ coming from the rescaled potential $\varepsilon^{-\frac{3}{2}}V(x/\varepsilon)$. At a certain step, we need to get rid of the interaction between those two terms, and this is precisely the role of Proposition 4.1. Some explicit calculations facilitate our analysis.

For $d \geq 4$, we assume the strongly mixing property, also known as $\rho-$mixing, which is only used in an estimation of fourth-order moments. For the critical case $d = 4$ with the logarithm scaling, we further assume the initial condition is constant to get rid of the interaction between $\varepsilon\int_0^{t/\varepsilon^2} V(B_s)ds$ and $\varepsilon B_{t/\varepsilon^2}$ appeared in the Feynman-Kac representation (3.1) below, and the martingale part does not contribute to the rescaled error in the end. Otherwise, we have the same term coming from the martingale part to deal with as $d = 3$, see (3.28). Proving the central limit result when $d = 4$ reduces to the weak convergence of $|\log \lambda|^{-\frac{1}{2}} \int_{\mathbb{R}^d} G_\lambda(x-y)V(y)dy$, where G_λ is the Green's function of $\lambda - \frac{1}{2}\Delta$, and here we assume again a Gaussian or Poissonian potential.

In the end, we point out that the expansion obtained in Theorem 2.8 is consistent with a formal two-scale expansion. Let us assume that $u_\varepsilon(t,x) = u_{hom}(t,x) + \varepsilon u_1(t,x,y) + \ldots$ with a fast variable $y = x/\varepsilon$, then by collecting terms of order ε^{-1} in (2.1), we have the equation satisfied by u_1:

$$\frac{1}{\varepsilon}\left(\frac{1}{2}\Delta_y u_1(t,x,y) + iV(y)u_{hom}(t,x)\right) = 0. \tag{2.20}$$

The solution u_1 can be formally written as

$$u_1(t,x,y) = iu_{hom}(t,x)\int_{\mathbb{R}^d} G_0(y-z)V(z)dz, \tag{2.21}$$

where G_0 is the Green's function of $-\frac{1}{2}\Delta$. The integral is not defined realization-wise since G_0 is not integrable, but if we pass to the limit from the Green's function of $\lambda - \frac{1}{2}\Delta$, we derive

$$u_1(t,x,y) = \lim_{\lambda\to 0} iu_{hom}(t,x)\int_{\mathbb{R}^d} G_\lambda(y-z)V(z)dz = iu_{hom}(t,x)\Phi(\tau_{\frac{x}{\varepsilon}}\omega), \tag{2.22}$$

then the formal expansion gives $u_\varepsilon(t,x) = u_{hom}(t,x) + i\varepsilon u_{hom}(t,x)\Phi(\tau_{\frac{x}{\varepsilon}}\omega) + \ldots$, which is consistent with Theorem 2.8. This indicates that when a stationary corrector exists, it is possible to obtain the random fluctuation by a formal two-scale expansion.

3 Refining the error

In this section, we review some key estimates in [16], prove a quantitative martingale central limit theorem, and derive a compact form of the properly-rescaled error in (3.29), (3.30) and (3.32) for $d = 3, 4$ and $d \geq 5$ respectively.

3.1 Error estimates

By the Feynman-Kac representation and the scaling property of Brownian motion, the solution to (2.1) is written as

$$u_\varepsilon(t,x) = \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(i\varepsilon\int_0^{t/\varepsilon^2} V(\frac{x}{\varepsilon}+B_s)ds)\}. \tag{3.1}$$

Define $y_s := \tau_{\frac{x}{\varepsilon}+B_s}\omega$ as the environmental process taking values in Ω, and the regularized corrector Φ_λ solve the corrector equation$(\lambda - L)\Phi_\lambda = \mathbb{V}$. We choose $\lambda = \varepsilon^2$ from now on. By Itô's formula, the process $X_t^\varepsilon := \varepsilon\int_0^{t/\varepsilon^2}\mathbb{V}(y_s)ds$ can be decomposed as $X_t^\varepsilon = R_t^\varepsilon + M_t^\varepsilon$ with

$$R_t^\varepsilon : = \varepsilon\int_0^{t/\varepsilon^2}\lambda\Phi_\lambda(y_s)ds - \varepsilon\Phi_\lambda(y_{t/\varepsilon^2}) + \varepsilon\Phi_\lambda(y_0), \tag{3.2}$$

$$M_t^\varepsilon : = \varepsilon\int_0^{t/\varepsilon^2}\sum_{k=1}^d D_k\Phi_\lambda(y_s)dB_s^k. \tag{3.3}$$

By [16, Proposition 3.1], we have

$$\mathbb{E}\mathbb{E}_B\{|R_t^\varepsilon|^2\} \lesssim \lambda\langle\Phi_\lambda,\Phi_\lambda\rangle \lesssim \sqrt{\lambda}1_{d=3} + \lambda|\log\lambda|1_{d=4} + \lambda 1_{d\geq 5}. \tag{3.4}$$

If we define $\sigma_\lambda^2 = \sum_{k=1}^d \|D_k\Phi_\lambda\|^2$, then by [16, Proposition 3.2],

$$|\sigma_\lambda^2 - \sigma^2| \lesssim \sqrt{\lambda}1_{d=3} + \lambda|\log\lambda|1_{d=4} + \lambda 1_{d\geq 5}. \tag{3.5}$$

The error is then decomposed into three parts, $u_\varepsilon(t,x) - u_{hom}(t,x) = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3$ with

$$\begin{aligned}
\mathcal{E}_1 &= \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})e^{iX_t^\varepsilon}\} - \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})e^{iM_t^\varepsilon}\}, &(3.6)\\
\mathcal{E}_2 &= \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})e^{iM_t^\varepsilon}\} - \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})e^{-\frac{1}{2}\sigma_\lambda^2 t}\}, &(3.7)\\
\mathcal{E}_3 &= \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})e^{-\frac{1}{2}\sigma_\lambda^2 t}\} - \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})e^{-\frac{1}{2}\sigma^2 t}\}, &(3.8)
\end{aligned}$$

so we have

$$\begin{aligned}
\mathbb{E}\{|\mathcal{E}_1|\} &\lesssim \mathbb{E}\mathbb{E}_B\{|R_t^\varepsilon|\} \lesssim \sqrt{\lambda\langle\Phi_\lambda,\Phi_\lambda\rangle} \lesssim \varepsilon^{\frac{1}{2}}1_{d=3} + \varepsilon|\log\varepsilon|^{\frac{1}{2}}1_{d=4} + \varepsilon 1_{d\geq 5}, &(3.9)\\
|\mathcal{E}_3| &\lesssim |\sigma_\lambda^2 - \sigma^2| \lesssim \varepsilon 1_{d=3} + \varepsilon^2|\log\varepsilon|1_{d=4} + \varepsilon^2 1_{d\geq 5}. &(3.10)
\end{aligned}$$

Clearly, $\mathcal{E}_1$ is of the right order given by (2.5) and $\mathcal{E}_3 \to 0$ after being properly rescaled. $\mathcal{E}_2$ is analyzed through a quantitative martingale central limit theorem. First it is written in the Fourier domain as

$$\mathcal{E}_2 = \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d}\hat{f}(\xi)e^{i\xi\cdot x}\mathbb{E}_B\{e^{i\varepsilon\xi\cdot B_{t/\varepsilon^2}+iM_t^\varepsilon} - e^{-\frac{1}{2}(|\xi|^2+\sigma_\lambda^2)t}\}d\xi. \tag{3.11}$$

Define $\tilde{M}_t^\varepsilon := \varepsilon\xi\cdot B_{t/\varepsilon^2} + M_t^\varepsilon$, then [22, Theorem 3.2] implies

$$\mathbb{E}\{|\mathcal{E}_2|\} \lesssim \int_{\mathbb{R}^d} |\hat{f}(\xi)|\mathbb{E}\mathbb{E}_B\{|\langle\tilde{M}^\varepsilon\rangle_t - (|\xi|^2+\sigma_\lambda^2)t|\}d\xi. \tag{3.12}$$

Since $\langle\tilde{M}^\varepsilon\rangle_t - (|\xi|^2+\sigma_\lambda^2)t = \varepsilon^2\int_0^{t/\varepsilon^2}\left(\sum_{k=1}^d D_k\Phi_\lambda(y_s)^2 - \sigma_\lambda^2\right)ds + 2\varepsilon^2\int_0^{t/\varepsilon^2}\sum_{k=1}^d \xi_k D_k\Phi_\lambda(y_s)ds$, by a second moment estimate in [16, Proposition 3.5, Lemma 3.6], we obtain

$$\begin{aligned}
\mathbb{E}\mathbb{E}_B\{|\varepsilon^2\int_0^{t/\varepsilon^2}\left(\sum_{k=1}^d D_k\Phi_\lambda(y_s)^2 - \sigma_\lambda^2\right)ds|^2\} &\lesssim \varepsilon^2|\log\varepsilon|1_{d=3} + \varepsilon^2 1_{d\geq 4}, &(3.13)\\
\mathbb{E}\mathbb{E}_B\{|2\varepsilon^2\int_0^{t/\varepsilon^2}\sum_{k=1}^d \xi_k D_k\Phi_\lambda(y_s)ds|^2\} &\lesssim \varepsilon 1_{d=3} + \varepsilon^2|\log\varepsilon|1_{d=4} + \varepsilon^2 1_{d\geq 5}, &(3.14)
\end{aligned}$$

so

$$\mathbb{E}\mathbb{E}_B\{|\langle\tilde{M}^\varepsilon\rangle_t - (|\xi|^2+\sigma_\lambda^2)t|^2\} \lesssim \varepsilon 1_{d=3} + \varepsilon^2|\log\varepsilon|1_{d=4} + \varepsilon^2 1_{d\geq 5}, \tag{3.15}$$

which implies

$$\mathbb{E}\{|\mathcal{E}_2|\} \lesssim \varepsilon^{\frac{1}{2}}1_{d=3} + \varepsilon|\log\varepsilon|^{\frac{1}{2}}1_{d=4} + \varepsilon 1_{d\geq 5}. \tag{3.16}$$

Therefore, to analyze the asymptotic distribution of $u_\varepsilon(t,x) - u_{hom}(t,x)$ after proper rescaling, we need to refine $\mathcal{E}_1$ and $\mathcal{E}_2$ to separate those terms of the right order.

Remark 3.1. When applying a refined quantitative martingale central limit theorem to analyze $\mathcal{E}_2$, we will use (3.15) frequently.

3.2 Quantitative martingale central limit theorem

For $\mathcal{E}_1$, using the fact that $|e^{ix}-1-ix|\lesssim |x|^2$, we have that

$$\begin{aligned}&\mathbb{E}\{|\mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})e^{iX_t^\varepsilon}\}-\mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})(1+iR_t^\varepsilon)e^{iM_t^\varepsilon}\}|\}\\ \lesssim&\varepsilon 1_{d=3}+\varepsilon^2|\log\varepsilon|1_{d=4}+\varepsilon^2 1_{d\geq 5}.\end{aligned}\tag{3.17}$$

so we have $\mathbb{E}\{|\mathcal{E}_1-v_{1,\varepsilon}|\}\ll \varepsilon^{\frac{1}{2}}1_{d=3}+\varepsilon|\log\varepsilon|^{\frac{1}{2}}1_{d=4}+\varepsilon 1_{d\geq 5}$ with

$$v_{1,\varepsilon}:=\mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})iR_t^\varepsilon e^{iM_t^\varepsilon}\}.\tag{3.18}$$

Now we analyze $\mathcal{E}_2$. By the expression in (3.11), the goal is reduced to an estimation of $\mathbb{E}_B\{e^{i\varepsilon\xi\cdot B_{t/\varepsilon^2}+iM_t^\varepsilon}-e^{-\frac{1}{2}(|\xi|^2+\sigma_\lambda^2)t}\}$ and separating the terms of the right order. The following is a simply modified quantitative martingale central limit theorem we need.

Proposition 3.2. *Let M_t be a continuous martingale with a right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$ and W_t a standard Brownian motion, then for any $f\in\mathcal{C}_b(\mathbb{R})$ with up to third order bounded and continuous derivatives, we have*

$$|\mathbb{E}\{f(M_1)-f(W_1)-\frac{1}{2}f''(M_\tau)(\langle M\rangle_1-1)\}|\leq C\mathbb{E}\{|\langle M\rangle_1-1|^{\frac{3}{2}}\},\tag{3.19}$$

where $\tau=\sup\{s\in[0,1]|\langle M\rangle_s\leq 1\}$ and the constant C only depends on the bound of f'''.

Proof. The proof follows a special case of [22, Theorem 3.2].

Since M_t is continuous, the quadratic variation process $\langle M\rangle_t$ is continuous as well. It is clear that τ is a stopping time, and we construct $\tilde{M}_t$ on $[0,2]$ as

$$\tilde{M}_t=\begin{cases}M_t & t\in[0,\tau],\\ M_\tau & t\in(\tau,1],\\ M_\tau+b_{t-1} & t\in(1,2-\langle M\rangle_\tau],\\ M_\tau+b_{1-\langle M\rangle_\tau} & t\in(2-\langle M\rangle_\tau,2],\end{cases}\tag{3.20}$$

where b is an independent Brownian motion starting from the origin with a right continuous filtration $(\mathcal{F}_t^b)_{t\geq 0}$.

Clearly $\tilde{M}_t$ is a continuous martingale with the new filtration $\tilde{\mathcal{F}}_t=\sigma(\mathcal{F}_t\cup\mathcal{F}_0^b)$ when $t\leq 1$ and $\tilde{\mathcal{F}}_t=\sigma(\mathcal{F}_t\cup\mathcal{F}_{t-1}^b)$ when $t>1$. $\langle\tilde{M}\rangle_2=1$, so $\tilde{M}_2\sim N(0,1)$, which implies $\mathbb{E}\{f(M_1)-f(W_1)\}=\mathbb{E}\{f(M_1)-f(\tilde{M}_2)\}$. We write

$$f(M_1)-f(\tilde{M}_2)=f(M_1)-f(M_\tau)-(f(\tilde{M}_2)-f(M_\tau)).\tag{3.21}$$

For the first term, we have

$$\begin{aligned}&|\mathbb{E}\{f(M_1)-f(M_\tau)-(M_1-M_\tau)f'(M_\tau)-\frac{1}{2}(M_1-M_\tau)^2f''(M_\tau)\}|\\ =&|\mathbb{E}\{f(M_1)-f(M_\tau)-\frac{1}{2}(\langle M\rangle_1-\langle M\rangle_\tau)f''(M_\tau)\}|\leq C\mathbb{E}\{|M_1-M_\tau|^3\}.\end{aligned}\tag{3.22}$$

For the second term, we have $\tilde{M}_2 = M_\tau + b_{1-\langle M\rangle_\tau}$, so

$$\begin{aligned}&|\mathbb{E}\{f(\tilde{M}_2) - f(M_\tau) - b_{1-\langle M\rangle_\tau} f'(M_\tau) - \frac{1}{2} b^2_{1-\langle M\rangle_\tau} f''(M_\tau)\}|\\ =&|\mathbb{E}\{f(\tilde{M}_2) - f(M_\tau) - \frac{1}{2}(1-\langle M\rangle_\tau) f''(M_\tau)\}| \le C\mathbb{E}\{|b_{1-\langle M\rangle_\tau}|^3\} \le C\mathbb{E}\{(1-\langle M\rangle_\tau)^{\frac{3}{2}}\}.\end{aligned} \tag{3.23}$$

Note that $\mathbb{E}\{|M_1 - M_\tau|^3\} \le C\mathbb{E}\{(\langle M\rangle_1 - \langle M\rangle_\tau)^{\frac{3}{2}}\} \le C\mathbb{E}\{|\langle M\rangle_1 - 1|^{\frac{3}{2}}\}$ and the same estimate holds for $\mathbb{E}\{(1-\langle M\rangle_\tau)^{\frac{3}{2}}\}$. The proof is complete. $\square$

For almost every $\omega \in \Omega$, $\tilde{M}^\varepsilon_t = \varepsilon\xi \cdot B_{t/\varepsilon^2} + M^\varepsilon_t$ is a continuous, square-integrable martingale, we apply Proposition 3.2 with $f = e^{ix}$ and obtain for almost every ω that

$$\begin{aligned}&|\mathbb{E}_B\{e^{i\tilde{M}^\varepsilon_t} - e^{-\frac{1}{2}(|\xi|^2+\sigma^2_\lambda)t} + \frac{1}{2} e^{i\tilde{M}^\varepsilon_\tau}(\langle \tilde{M}^\varepsilon\rangle_t - (|\xi|^2+\sigma^2_\lambda)t)\}|\\ \lesssim&\mathbb{E}_B\{|\langle \tilde{M}^\varepsilon\rangle_t - (|\xi|^2+\sigma^2_\lambda)t|^{\frac{3}{2}}\}\end{aligned} \tag{3.24}$$

where $\tau := \sup\{s \in [0,t]| : \varepsilon^2 \int_0^{s/\varepsilon^2} \sum_{k=1}^d (\xi_k + D_k\Phi_\lambda(y_s))^2 ds \le (|\xi|^2+\sigma^2_\lambda)t\}$.

First we have

$$\begin{aligned}&\mathbb{E}\{|\mathcal{E}_2 - \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d}\hat{f}(\xi)e^{i\xi\cdot x}\mathbb{E}_B\{-\frac{1}{2}e^{i\tilde{M}^\varepsilon_\tau}(\langle \tilde{M}^\varepsilon\rangle_t - (|\xi|^2+\sigma^2_\lambda)t)\}d\xi|\}\\ \lesssim& \int_{\mathbb{R}^d} |\hat{f}(\xi)|\mathbb{E}\mathbb{E}_B\{|\langle \tilde{M}^\varepsilon\rangle_t - (|\xi|^2+\sigma^2_\lambda)t|^{\frac{3}{2}}\}d\xi\\ \lesssim& \int_{\mathbb{R}^d} |\hat{f}(\xi)| \left(\mathbb{E}\mathbb{E}_B\{|\langle \tilde{M}^\varepsilon\rangle_t - (|\xi|^2+\sigma^2_\lambda)t|^2\}\right)^{\frac{3}{4}} d\xi \ll \varepsilon^{\frac{1}{2}} 1_{d=3} + \varepsilon|\log\varepsilon|^{\frac{1}{2}} 1_{d=4} + \varepsilon 1_{d\ge 5}\end{aligned} \tag{3.25}$$

by recalling (3.15).

Next, we consider

$$\begin{aligned}&\mathbb{E}\{|\int_{\mathbb{R}^d} \frac{1}{(2\pi)^d}\hat{f}(\xi)e^{i\xi\cdot x}\mathbb{E}_B\{\frac{1}{2}\left(e^{i\tilde{M}^\varepsilon_\tau} - e^{i\tilde{M}^\varepsilon_t}\right)(\langle \tilde{M}^\varepsilon\rangle_t - (|\xi|^2+\sigma^2_\lambda)t)\}d\xi|\}\\ \le& \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d}|\hat{f}(\xi)|\frac{1}{2}\sqrt{\mathbb{E}\mathbb{E}_B\{|\tilde{M}^\varepsilon_\tau - \tilde{M}^\varepsilon_t|^2\}}\sqrt{\mathbb{E}\mathbb{E}_B\{|\langle \tilde{M}^\varepsilon\rangle_t - (|\xi|^2+\sigma^2_\lambda)t|^2\}}d\xi\\ \le& \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d}|\hat{f}(\xi)|\frac{1}{2}\sqrt{\mathbb{E}\mathbb{E}_B\{\langle \tilde{M}^\varepsilon\rangle_t - \langle \tilde{M}^\varepsilon\rangle_\tau\}}\sqrt{\mathbb{E}\mathbb{E}_B\{|\langle \tilde{M}^\varepsilon\rangle_t - (|\xi|^2+\sigma^2_\lambda)t|^2\}}d\xi\\ \le& \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d}|\hat{f}(\xi)|\frac{1}{2}\sqrt{\mathbb{E}\mathbb{E}_B\{|\langle \tilde{M}^\varepsilon\rangle_t - (|\xi|^2+\sigma^2_\lambda)t|\}}\sqrt{\mathbb{E}\mathbb{E}_B\{|\langle \tilde{M}^\varepsilon\rangle_t - (|\xi|^2+\sigma^2_\lambda)t|^2\}}d\xi\\ \ll&\varepsilon^{\frac{1}{2}} 1_{d=3} + \varepsilon|\log\varepsilon|^{\frac{1}{2}} 1_{d=4} + \varepsilon 1_{d\ge 5}\end{aligned} \tag{3.26}$$

again by using (3.15).

In the end, since

$$\langle \tilde{M}^\varepsilon\rangle_t - (|\xi|^2+\sigma^2_\lambda)t = 2\varepsilon^2\int_0^{t/\varepsilon^2}\sum_{k=1}^d \xi_k D_k\Phi_\lambda(y_s)ds + \varepsilon^2\int_0^{t/\varepsilon^2}(\sum_{k=1}^d D_k\Phi_\lambda(y_s)^2 - \sigma^2_\lambda)ds, \tag{3.27}$$

we obtain the following results by (3.13) and (3.14). For $\approx$, it means the difference goes to zero in $L^1(\Omega)$ as $\varepsilon \to 0$.

When $d=3$, $\mathbb{E}\{|\mathcal{E}_2 - v_{2,\varepsilon}|\} \ll \varepsilon^{\frac{1}{2}}$, where

$$v_{2,\varepsilon} = -\int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} \hat{f}(\xi) e^{i\xi\cdot x} \mathbb{E}_B\{e^{i\tilde{M}_t^\varepsilon} \varepsilon^2 \int_0^{t/\varepsilon^2} \sum_{k=1}^d \xi_k D_k \Phi_\lambda(y_s) ds\} d\xi. \tag{3.28}$$

By writing $v_{1,\varepsilon}$ in Fourier domain as well, we have proved that

$$\begin{aligned} \frac{u_\varepsilon(t,x) - u_{hom}(t,x)}{\varepsilon^{\frac{1}{2}}} \approx & \frac{v_{1,\varepsilon} + v_{2,\varepsilon}}{\varepsilon^{\frac{1}{2}}} \\ = & \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} \hat{f}(\xi) e^{i\xi\cdot x} \varepsilon^{-\frac{1}{2}} \mathbb{E}_B\{e^{i\tilde{M}_t^\varepsilon} \left(iR_t^\varepsilon - \varepsilon^2 \int_0^{t/\varepsilon^2} \sum_{k=1}^d \xi_k D_k \Phi_\lambda(y_s) ds \right)\} d\xi. \end{aligned} \tag{3.29}$$

When $d=4$, if $f(x) \equiv const$, without loss of generality let $f(x) \equiv 1$, then $\hat{f}(\xi) = \delta(\xi)$, and in the Fourier domain the integration only charges $\xi = 0$, so only the bound in (3.13) matters for $\mathcal{E}_2$ and we have $\mathbb{E}\{|\mathcal{E}_2|\} \lesssim \varepsilon \ll \varepsilon|\log\varepsilon|^{\frac{1}{2}}$. Therefore, we obtain

$$\frac{u_\varepsilon(t,x) - u_{hom}(t,x)}{\varepsilon|\log\varepsilon|^{\frac{1}{2}}} \approx \frac{v_{1,\varepsilon}}{\varepsilon|\log\varepsilon|^{\frac{1}{2}}} = \frac{\mathbb{E}_B\{iR_t^\varepsilon e^{iM_t^\varepsilon}\}}{\varepsilon|\log\varepsilon|^{\frac{1}{2}}} \tag{3.30}$$

When $d \geq 5$, $\mathbb{E}\{|\mathcal{E}_2 - v_{2,\varepsilon}|\} \ll \varepsilon$, where

$$v_{2,\varepsilon} = -\frac{1}{2} \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} \hat{f}(\xi) e^{i\xi\cdot x} \mathbb{E}_B\{e^{i\tilde{M}_t^\varepsilon} (\langle \tilde{M}^\varepsilon \rangle_t - (|\xi|^2 + \sigma_\lambda^2)t)\} d\xi, \tag{3.31}$$

so

$$\begin{aligned} \frac{u_\varepsilon(t,x) - u_{hom}(t,x)}{\varepsilon} \approx & \frac{v_{1,\varepsilon} + v_{2,\varepsilon}}{\varepsilon} \\ = & \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} \hat{f}(\xi) e^{i\xi\cdot x} \varepsilon^{-1} \mathbb{E}_B\{e^{i\tilde{M}_t^\varepsilon} \left(iR_t^\varepsilon - \frac{1}{2}(\langle \tilde{M}^\varepsilon \rangle_t - (|\xi|^2 + \sigma_\lambda^2)t) \right)\} d\xi. \end{aligned} \tag{3.32}$$

4 Proof of the main theorem: $d=3$

Now we are ready to prove the main theorem. Recall that $\tilde{M}_t^\varepsilon = \varepsilon\xi \cdot B_{t/\varepsilon^2} + M_t^\varepsilon$, and $X_t^\varepsilon = R_t^\varepsilon + M_t^\varepsilon$, so

$$\tilde{M}_t^\varepsilon = \varepsilon\xi \cdot B_{t/\varepsilon^2} + X_t^\varepsilon - R_t^\varepsilon. \tag{4.1}$$

By (3.4) and (3.14), $\mathbb{E}\mathbb{E}_B\{(\varepsilon^{-\frac{1}{2}} R_t^\varepsilon)^2\}$ and $\mathbb{E}\mathbb{E}_B\{(\varepsilon^{\frac{3}{2}} \int_0^{t/\varepsilon^2} D_k \Phi_\lambda(y_s) ds)^2\}$ are both bounded, so since R_t^ε is small as in (3.4), we can replace $\tilde{M}_t^\varepsilon$ by $\varepsilon\xi \cdot B_{t/\varepsilon^2} + X_t^\varepsilon$ in (3.29) and obtain

$$\begin{aligned} & \frac{u_\varepsilon(t,x) - u_{hom}(t,x)}{\varepsilon^{\frac{1}{2}}} \\ \approx & \mathbb{E}_B\{\int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} \hat{f}(\xi) e^{i\xi\cdot x + i\varepsilon\xi\cdot B_{t/\varepsilon^2}} e^{iX_t^\varepsilon} \left(\varepsilon^{-\frac{1}{2}} iR_t^\varepsilon - \varepsilon^{\frac{3}{2}} \int_0^{t/\varepsilon^2} \sum_{k=1}^d \xi_k D_k \Phi_\lambda(y_s) ds \right) d\xi\}. \end{aligned} \tag{4.2}$$

Let

$$Y_t^\varepsilon := i\varepsilon^{-\frac{1}{2}} \left(\varepsilon \int_0^{t/\varepsilon^2} \lambda \Phi_\lambda(y_s) ds - \varepsilon \Phi_\lambda(y_{t/\varepsilon^2}) + \varepsilon \Phi_\lambda(y_0) \right) - \varepsilon^{\frac{3}{2}} \int_0^{t/\varepsilon^2} \sum_{k=1}^d \xi_k D_k \Phi_\lambda(y_s) ds, \tag{4.3}$$

so $\mathbb{E}\mathbb{E}_B\{|Y_t^\varepsilon|^2\}$ is uniformly bounded, and we have

$$\frac{u_\varepsilon(t,x) - u_{hom}(t,x)}{\varepsilon^{\frac{1}{2}}} \approx \mathbb{E}_B\{\int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} \hat{f}(\xi) e^{i\xi\cdot x + i\varepsilon\xi\cdot B_{t/\varepsilon^2}} e^{iX_t^\varepsilon} Y_t^\varepsilon d\xi\}. \tag{4.4}$$

We show the interaction between X_t^ε and Y_t^ε goes to zero in the following sense:

Proposition 4.1.

$$\mathbb{E}_B\{\int_{\mathbb{R}^d} \hat{f}(\xi) e^{i\xi\cdot x + i\varepsilon\xi\cdot B_{t/\varepsilon^2}} (e^{iX_t^\varepsilon} - e^{-\frac{1}{2}\sigma^2 t}) Y_t^\varepsilon d\xi\} \to 0 \tag{4.5}$$

in $L^2(\Omega)$ *as* $\varepsilon \to 0$.

By the above Proposition, the rescaled corrector can be written as

$$\frac{u_\varepsilon(t,x) - u_{hom}(t,x)}{\varepsilon^{\frac{1}{2}}} \approx \mathbb{E}_B\{\int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} \hat{f}(\xi) e^{i\xi\cdot x + i\varepsilon\xi\cdot B_{t/\varepsilon^2}} e^{-\frac{1}{2}\sigma^2 t} Y_t^\varepsilon d\xi\}. \tag{4.6}$$

For the last term in Y_t^ε, we can write

$$\begin{aligned}
&\mathbb{E}_B\{\int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} \hat{f}(\xi) e^{i\xi\cdot x + i\varepsilon\xi\cdot B_{t/\varepsilon^2}} e^{-\frac{1}{2}\sigma^2 t} \varepsilon^{\frac{3}{2}} \int_0^{t/\varepsilon^2} \sum_{k=1}^d \xi_k D_k \Phi_\lambda(y_s) ds\} \\
=& - i \sum_{k=1}^d \mathbb{E}_B\{\partial_{x_k} f(x + \varepsilon B_{t/\varepsilon^2}) e^{-\frac{1}{2}\sigma^2 t} \varepsilon^{\frac{3}{2}} \int_0^{t/\varepsilon^2} D_k \Phi_\lambda(y_s) ds\} \\
=& - i\mathbb{E}_B\{f(x + \varepsilon B_{t/\varepsilon^2}) e^{-\frac{1}{2}\sigma^2 t} \varepsilon^{\frac{1}{2}} \sum_{k=1}^d \int_0^{t/\varepsilon^2} D_k \Phi_\lambda(y_s) dB_s^k\},
\end{aligned} \tag{4.7}$$

where the last equality comes from a simple application of the duality relation in Malliavin calculus [26]. For the sake of convenience, we present some standard facts about Malliavin calculus in Appendix C.

To summarize, we have

$$\begin{aligned}
&\frac{u_\varepsilon(t,x) - u_{hom}(t,x)}{\varepsilon^{\frac{1}{2}}} \\
\approx& \mathbb{E}_B\{f(x + \varepsilon B_{t/\varepsilon^2}) e^{-\frac{1}{2}\sigma^2 t} i\varepsilon^{-\frac{1}{2}} \left(\varepsilon \int_0^{t/\varepsilon^2} \lambda \Phi_\lambda(y_s) ds - \varepsilon \Phi_\lambda(y_{t/\varepsilon^2}) + \varepsilon \Phi_\lambda(y_0) \right)\} \\
&+ \mathbb{E}_B\{f(x + \varepsilon B_{t/\varepsilon^2}) e^{-\frac{1}{2}\sigma^2 t} i\varepsilon^{\frac{1}{2}} \sum_{k=1}^d \int_0^{t/\varepsilon^2} D_k \Phi_\lambda(y_s) dB_s^k\} \\
=& \mathbb{E}_B\{f(x + \varepsilon B_{t/\varepsilon^2}) e^{-\frac{1}{2}\sigma^2 t} i\varepsilon^{-\frac{1}{2}} \varepsilon \int_0^{t/\varepsilon^2} \mathbb{V}(y_s) ds\} \\
=& \mathbb{E}_B\{f(x + B_t) e^{-\frac{1}{2}\sigma^2 t} i \frac{1}{\varepsilon^{\frac{3}{2}}} \int_0^t V(\frac{x + B_s}{\varepsilon}) ds\},
\end{aligned} \tag{4.8}$$

which combines with the following Proposition to complete the proof of Theorem 2.4.

Proposition 4.2. $\mathbb{E}_B\{f(x+B_t)e^{-\frac{1}{2}\sigma^2 t}i\varepsilon^{-\frac{3}{2}}\int_0^t V(\frac{x+B_s}{\varepsilon})ds\} \Rightarrow v(t,x)$ *in the sense of Theorem 2.4, and $v(t,x)$ solves the following SPDE with additive white noise $\dot{W}(x)$ and zero initial condition:*

$$\partial_t v(t,x) = \frac{1}{2}\Delta v(t,x) - \frac{1}{2}\sigma^2 v(t,x) + i\sqrt{\hat{R}(0)}u_{hom}(t,x)\dot{W}(x). \tag{4.9}$$

Remark 4.3. To combine (4.8) and Proposition 4.2 to prove Theorem 2.4, we need to note that the statistical error caused in (4.8) is $x-$independent, i.e.,

$$\mathbb{E}\{|\frac{u_\varepsilon(t,x) - u_{hom}(t,x)}{\varepsilon^{\frac{1}{2}}} - \mathbb{E}_B\{f(x+B_t)e^{-\frac{1}{2}\sigma^2 t}i\frac{1}{\varepsilon^{\frac{3}{2}}}\int_0^t V(\frac{x+B_s}{\varepsilon})ds\}|\} \le C_\varepsilon \tag{4.10}$$

for some $x-$independent constant $C_\varepsilon \to 0$ as $\varepsilon \to 0$.

5 Asymptotic independence, proof of Proposition 4.1: $d = 3$

Our goal is to prove that $\mathbb{E}_B\{\int_{\mathbb{R}^d}\hat{f}(\xi)e^{i\xi\cdot x+i\varepsilon\xi\cdot B_{t/\varepsilon^2}}(e^{iX_t^\varepsilon} - e^{-\frac{1}{2}\sigma^2 t})Y_t^\varepsilon d\xi\} \to 0$ in probability, and since $X_t^\varepsilon, Y_t^\varepsilon$ both depend on the Brownian motion B_t, we write them as $X_t^\varepsilon(B), Y_t^\varepsilon(B)$ and calculate the second moment

$$\begin{aligned}
&\mathbb{E}\{|\mathbb{E}_B\{\int_{\mathbb{R}^d}\hat{f}(\xi)e^{i\xi\cdot x+i\varepsilon\xi\cdot B_{t/\varepsilon^2}}(e^{iX_t^\varepsilon} - e^{-\frac{1}{2}\sigma^2 t})Y_t^\varepsilon d\xi\}|^2\} \\
=&\mathbb{E}\mathbb{E}_B\mathbb{E}_W\int_{\mathbb{R}^{2d}}\hat{f}(\xi)e^{i\xi\cdot x+i\varepsilon\xi\cdot B_{t/\varepsilon^2}}(e^{iX_t^\varepsilon(B)} - e^{-\frac{1}{2}\sigma^2 t})Y_t^\varepsilon(B) \\
&\qquad\overline{\hat{f}(\eta)e^{i\eta\cdot x+i\varepsilon\eta\cdot W_{t/\varepsilon^2}}(e^{iX_t^\varepsilon(W)} - e^{-\frac{1}{2}\sigma^2 t})Y_t^\varepsilon(W)}d\xi d\eta,
\end{aligned} \tag{5.1}$$

where B, W are independent Brownian motions. We claim that

$$\mathbb{E}_B\mathbb{E}_W|\mathbb{E}\{(e^{iX_t^\varepsilon(B)} - e^{-\frac{1}{2}\sigma^2 t})Y_t^\varepsilon(B)\overline{(e^{iX_t^\varepsilon(W)} - e^{-\frac{1}{2}\sigma^2 t})Y_t^\varepsilon(W)}\}| \to 0 \tag{5.2}$$

as $\varepsilon \to 0$. If the claim is true, then

$$\begin{aligned}
&\mathbb{E}\{|\mathbb{E}_B\{\int_{\mathbb{R}^d}\hat{f}(\xi)e^{i\xi\cdot x+i\varepsilon\xi\cdot B_{t/\varepsilon^2}}(e^{iX_t^\varepsilon} - e^{-\frac{1}{2}\sigma^2 t})Y_t^\varepsilon d\xi\}|^2\} \\
\le&\int_{\mathbb{R}^{2d}}|\hat{f}(\xi)\hat{f}(\eta)|\mathbb{E}_B\mathbb{E}_W|\mathbb{E}\{(e^{iX_t^\varepsilon(B)} - e^{-\frac{1}{2}\sigma^2 t})Y_t^\varepsilon(B)\overline{(e^{iX_t^\varepsilon(W)} - e^{-\frac{1}{2}\sigma^2 t})Y_t^\varepsilon(W)}\}|d\xi d\eta \to 0
\end{aligned} \tag{5.3}$$

and Proposition 4.1 is proved.

Remark 5.1. From the expressions of $X_t^\varepsilon, Y_t^\varepsilon$, it is clear that the dependence of

$$\mathbb{E}_B\mathbb{E}_W|\mathbb{E}\{(e^{iX_t^\varepsilon(B)} - e^{-\frac{1}{2}\sigma^2 t})Y_t^\varepsilon(B)\overline{(e^{iX_t^\varepsilon(W)} - e^{-\frac{1}{2}\sigma^2 t})Y_t^\varepsilon(W)}\}|$$

on ξ, η is only a factor of ξ_k, η_k. Since $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, by the dominated convergence theorem, we obtain Proposition 4.1.

Therefore we only need to prove (5.2) holds. Clearly, when freezing B and W, $X_t^\varepsilon, Y_t^\varepsilon$ are Gaussian if V is Gaussian, and are Poissonian if V is Poissonian, which makes the explicit calculation feasible.

5.1 Poissonian case

Recall that

$$Y_t^\varepsilon(B) = i\varepsilon^{-\frac{1}{2}}\left(\varepsilon\int_0^{t/\varepsilon^2}\lambda\Phi_\lambda(y_s)ds - \varepsilon\Phi_\lambda(y_{t/\varepsilon^2}) + \varepsilon\Phi_\lambda(y_0)\right) - \varepsilon^{\frac{3}{2}}\int_0^{t/\varepsilon^2}\sum_{k=1}^d \xi_k D_k\Phi_\lambda(y_s)ds,$$

and since $V(x) = \int_{\mathbb{R}^d}\varphi(x-y)\omega(dy) - c_\varphi$ in the Poissonian case, we have

$$\begin{aligned} Y_t^\varepsilon(B) =& i\int_{\mathbb{R}^d}\varepsilon^{\frac{5}{2}}\int_0^{t/\varepsilon^2} f^\lambda(\frac{x}{\varepsilon}+B_s-y)ds\omega(dy) - i\varepsilon^{\frac{1}{2}}\int_{\mathbb{R}^d} f^\lambda(\frac{x}{\varepsilon}+B_{t/\varepsilon^2}-y)\omega(dy) \\ &+ i\varepsilon^{\frac{1}{2}}\int_{\mathbb{R}^d} f^\lambda(\frac{x}{\varepsilon}-y)\omega(dy) - \sum_{k=1}^d \xi_k\int_{\mathbb{R}^d}\varepsilon^{\frac{3}{2}}\int_0^{t/\varepsilon^2} f_k^\lambda(\frac{x}{\varepsilon}+B_s-y)ds\omega(dy) - C, \end{aligned} \tag{5.4}$$

for some constant C, where $f^\lambda(x) = \int_{\mathbb{R}^d}\varphi(x-y)G_\lambda(y)$, $f_k^\lambda(x) = \int_{\mathbb{R}^d}\varphi(x-y)\partial_{x_k}G_\lambda(y)dy$, and C is chosen so that $\mathbb{E}\{Y_t^\varepsilon(B)\} = 0$. Therefore, $Y_t^\varepsilon(B) = \int_{\mathbb{R}^d} h_B(y)\omega(dy) - \int_{\mathbb{R}^d} h_B(y)dy$ for some h_B depending on the Brownian path $B_s, s \in [0, t/\varepsilon^2]$.

Similarly, $X_t^\varepsilon(B) = \varepsilon\int_0^{t/\varepsilon^2} V(\frac{x}{\varepsilon}+B_s)ds = \int_{\mathbb{R}^d}\varepsilon\int_0^{t/\varepsilon^2}\varphi(\frac{x}{\varepsilon}+B_s-y)ds\omega(dy) - c_\varphi t/\varepsilon$, and we denote $X_t^\varepsilon(B) = \int_{\mathbb{R}^d} g_B(y)\omega(dy) - \int_{\mathbb{R}^d} g_B(y)dy$ for some real g_B depending on the Brownian path $B_s, s \in [0, t/\varepsilon^2]$.

To calculate $\mathbb{E}\{(e^{iX_t^\varepsilon(B)} - e^{-\frac{1}{2}\sigma^2 t})Y_t^\varepsilon(B)\overline{(e^{iX_t^\varepsilon(W)} - e^{-\frac{1}{2}\sigma^2 t})Y_t^\varepsilon(W)}\}$, since $X_t^\varepsilon(B)$, $Y_t^\varepsilon(B)$, $X_t^\varepsilon(W)$, $Y_t^\varepsilon(W)$ are all integrals with respect to the Poissonian point process $\omega(dy)$, we apply Lemma A.1 to obtain

$$\begin{aligned} &\mathbb{E}\{(e^{iX_t^\varepsilon(B)} - e^{-\frac{1}{2}\sigma^2 t})Y_t^\varepsilon(B)\overline{(e^{iX_t^\varepsilon(W)} - e^{-\frac{1}{2}\sigma^2 t})Y_t^\varepsilon(W)}\} \\ =& e^{-\sigma^2 t}\int_{\mathbb{R}^d} h_B\overline{h_W}dy \\ &- e^{-\frac{1}{2}\sigma^2 t}e^{\int_{\mathbb{R}^d}(e^{ig_B}-1-ig_B)dy}\left(\int_{\mathbb{R}^d} e^{ig_B}h_B\overline{h_W}dy + \int_{\mathbb{R}^d}(e^{ig_B}-1)h_Bdy\int_{\mathbb{R}^d}(e^{ig_B}-1)\overline{h_W}dy\right) \\ &- e^{-\frac{1}{2}\sigma^2 t}e^{\int_{\mathbb{R}^d}(e^{-ig_W}-1+ig_W)dy}\left(\int_{\mathbb{R}^d} e^{-ig_W}h_B\overline{h_W}dy + \int_{\mathbb{R}^d}(e^{-ig_W}-1)h_Bdy\int_{\mathbb{R}^d}(e^{-ig_W}-1)\overline{h_W}dy\right) \\ &+ e^{\int_{\mathbb{R}^d}(e^{ig_B-ig_W}-1-ig_B+ig_W)dy}\left(\int_{\mathbb{R}^d} e^{ig_B-ig_W}h_B\overline{h_W}dy + \int_{\mathbb{R}^d}(e^{ig_B-ig_W}-1)h_Bdy\int_{\mathbb{R}^d}(e^{ig_B-ig_W}-1)\overline{h_W}dy\right). \end{aligned} \tag{5.5}$$

Let $\mathbb{E}\{(e^{iX_t^\varepsilon(B)} - e^{-\frac{1}{2}\sigma^2 t})Y_t^\varepsilon(B)\overline{(e^{iX_t^\varepsilon(W)} - e^{-\frac{1}{2}\sigma^2 t})Y_t^\varepsilon(W)}\} = P_1 + P_2$, where

$$\begin{aligned} P_1 =& \left(e^{-\sigma^2 t} + e^{\int_{\mathbb{R}^d}(e^{ig_B-ig_W}-1-ig_B+ig_W)dy}\right)\int_{\mathbb{R}^d} h_B\overline{h_W}dy \\ &- \left(e^{-\frac{1}{2}\sigma^2 t}e^{\int_{\mathbb{R}^d}(e^{ig_B}-1-ig_B)dy} + e^{-\frac{1}{2}\sigma^2 t}e^{\int_{\mathbb{R}^d}(e^{-ig_W}-1+ig_W)dy}\right)\int_{\mathbb{R}^d} h_B\overline{h_W}dy, \end{aligned} \tag{5.6}$$

and P_2 is the remainder, we have the following lemma concerning P_1.

Lemma 5.2. $\mathbb{E}_B\mathbb{E}_W\{|P_1|\} \to 0$.

Proof. Firstly, we have

$$\begin{aligned}
&\mathbb{E}_B\mathbb{E}_W\{|P_1|\}\\
\leq&\sqrt{\mathbb{E}_B\mathbb{E}_W\left(e^{-\sigma^2 t}+e^{\int_{\mathbb{R}^d}(e^{ig_B-ig_W}-1-ig_B+ig_W)dy}-e^{-\frac{1}{2}\sigma^2 t}e^{\int_{\mathbb{R}^d}(e^{ig_B}-1-ig_B)dy}-e^{-\frac{1}{2}\sigma^2 t}e^{\int_{\mathbb{R}^d}(e^{-ig_W}-1+ig_W)dy}\right)^2}\\
&\times\sqrt{\mathbb{E}_B\mathbb{E}_W\{|\int_{\mathbb{R}^d}h_B\overline{h_W}dy|^2\}}.
\end{aligned}\tag{5.7}$$

Clearly, $\mathbb{E}\{|Y_t^\varepsilon(B)|^2\}=\int_{\mathbb{R}^d}|h_B|^2dy$ and $\mathbb{E}\{|Y_t^\varepsilon(W)|^2\}=\int_{\mathbb{R}^d}|h_W|^2dy$, thus

$$\mathbb{E}_B\mathbb{E}_W\{|\int_{\mathbb{R}^d}h_B\overline{h_W}dy|^2\}\leq\mathbb{E}\mathbb{E}_B\{|Y_t^\varepsilon(B)|^2\}\mathbb{E}\mathbb{E}_W\{|Y_t^\varepsilon(W)|^2\}$$

is uniformly bounded. Then we only have to apply Lemma A.2 to complete the proof. □

The rest is to prove that $\mathbb{E}_B\mathbb{E}_W\{|P_2|\}\to 0$. Actually, by the fact that $e^{\int_{\mathbb{R}^d}(e^{ig_B}-1-ig_B)dy}$, $e^{\int_{\mathbb{R}^d}(e^{-ig_W}-1+ig_W)dy}$ and $e^{\int_{\mathbb{R}^d}(e^{ig_B-ig_W}-1-ig_B+ig_W)dy}$ are uniformly bounded by 1, it suffices to show that in $L^1(B\times W)$

$$\int_{\mathbb{R}^d}(e^{ig_B}-1)h_Bdy\int_{\mathbb{R}^d}(e^{ig_B}-1)\overline{h_W}dy\to 0,\tag{5.8}$$

$$\int_{\mathbb{R}^d}(e^{-ig_W}-1)h_Bdy\int_{\mathbb{R}^d}(e^{-ig_W}-1)\overline{h_W}dy\to 0,\tag{5.9}$$

$$\int_{\mathbb{R}^d}(e^{ig_B-ig_W}-1)h_Bdy\int_{\mathbb{R}^d}(e^{ig_B-ig_W}-1)\overline{h_W}dy\to 0,\tag{5.10}$$

$$\int_{\mathbb{R}^d}(e^{ig_B}-1)h_B\overline{h_W}dy\to 0,\tag{5.11}$$

$$\int_{\mathbb{R}^d}(e^{-ig_W}-1)h_B\overline{h_W}dy\to 0,\tag{5.12}$$

$$\int_{\mathbb{R}^d}(e^{ig_B-ig_W}-1)h_B\overline{h_W}dy\to 0.\tag{5.13}$$

The methods to prove all the above estimates are similar, i.e., we expand e^{ix} in power series and control each term after standard changes of variables. We will only present a detailed proof of (5.10) since it contains all the ingredients and the other terms are handled in a similar way.

Without loss of generality, we can assume $|\varphi(x)|$ is some bounded, radially symmetric and decreasing function with compact support in the estimation.

Since $Y_t^\varepsilon(B)=\int_{\mathbb{R}^d}h_B(y)\omega(dy)-\int_{\mathbb{R}^d}h_B(y)dy$ with

$$\begin{aligned}
h_B(y)=&i\varepsilon^{\frac{5}{2}}\int_0^{t/\varepsilon^2}f^\lambda(\frac{x}{\varepsilon}+B_s-y)ds-\sum_{k=1}^d\xi_k\varepsilon^{\frac{3}{2}}\int_0^{t/\varepsilon^2}f_k^\lambda(\frac{x}{\varepsilon}+B_s-y)ds\\
&+i\varepsilon^{\frac{1}{2}}f^\lambda(\frac{x}{\varepsilon}-y)-i\varepsilon^{\frac{1}{2}}f^\lambda(\frac{x}{\varepsilon}+B_{t/\varepsilon^2}-y)
\end{aligned}\tag{5.14}$$

we can divide the terms into two groups depending on whether they involve the integration in s, i.e.,

$$\begin{aligned} A_1(B) &= \{\varepsilon^{\frac{5}{2}} \int_0^{t/\varepsilon^2} f^\lambda(\frac{x}{\varepsilon} + B_s - y)ds, \varepsilon^{\frac{3}{2}} \int_0^{t/\varepsilon^2} f_k^\lambda(\frac{x}{\varepsilon} + B_s - y)ds\}, \\ A_2(B) &= \{\varepsilon^{\frac{1}{2}} f^\lambda(\frac{x}{\varepsilon} - y), \varepsilon^{\frac{1}{2}} f^\lambda(\frac{x}{\varepsilon} + B_{t/\varepsilon^2} - y)\}. \end{aligned}$$

A similar decomposition holds for $h_B(W)$. To prove $\mathbb{E}_B\mathbb{E}_W\{|\int_{\mathbb{R}^d}(e^{ig_B - ig_W} - 1)h_B dy \int_{\mathbb{R}^d}(e^{ig_B - ig_W} - 1)\overline{h_W} dy|\} \to 0$, there are four groups of terms concerning h_B, h_W to deal with, i.e., $(A_1(B), A_1(W))$, $(A_1(B), A_2(W))$, $(A_2(B), A_1(W))$, and $(A_2(B), A_2(W))$. In the following, we will analyze them separately.

5.1.1 $(A_1(B), A_1(W))$

Lemma 5.3.

$$\mathbb{E}_B\mathbb{E}_W\{|\int_{\mathbb{R}^d}(e^{ig_B - ig_W} - 1)h_B dy \int_{\mathbb{R}^d}(e^{ig_B - ig_W} - 1)\overline{h_W} dy|\} \to 0$$

as $\varepsilon \to 0$ *for* $h_B = \int_0^{t/\varepsilon^2} g_1(\frac{x}{\varepsilon} + B_s - y)ds$ *and* $\overline{h_W} = \int_0^{t/\varepsilon^2} g_2(\frac{x}{\varepsilon} + W_s - y)ds$, *where* $g_1, g_2 \in \{\varepsilon^{\frac{5}{2}} f^\lambda, \varepsilon^{\frac{3}{2}} f_k^\lambda\}$.

Firstly, we write

$$\begin{aligned} &\int_{\mathbb{R}^d}(e^{ig_B - ig_W} - 1)h_B dy \int_{\mathbb{R}^d}(e^{ig_B - ig_W} - 1)\overline{h_W} dy \\ =& \sum_{m_1,m_2,m_3,m_4 \geq 0} \frac{i^{m_1 - m_2 + m_3 - m_4}}{m_1!m_2!m_3!m_4!} \int_{\mathbb{R}^{2d}} g_B(y)^{m_1} g_W(y)^{m_2} g_B(z)^{m_3} g_W(z)^{m_4} h_B(y)\overline{h_W}(z) dydz, \end{aligned} \tag{5.15}$$

and clearly the indexes satisfy $m_1 + m_2 \geq 1$ and $m_3 + m_4 \geq 1$. For each term, let $N(m_i) = \sum_{i=1}^4 m_i$, and we have

$$\begin{aligned} &|\int_{\mathbb{R}^{2d}} g_B(y)^{m_1} g_W(y)^{m_2} g_B(z)^{m_3} g_W(z)^{m_4} h_B(y)\overline{h_W}(z) dydz| \\ \lesssim& \varepsilon^{N(m_i)} \int_{\mathbb{R}^{2d}} \int_{[0,t/\varepsilon^2]^{N(m_i)+2}} \prod_{i=1}^{m_1} |\varphi|(B_{s_i} - y) \prod_{i=m_1+1}^{m_1+m_3} |\varphi|(B_{s_i} - z)|g_1|(B_{\tilde{s}} - y) \\ &\prod_{i=1}^{m_2} |\varphi|(W_{u_i} - y) \prod_{i=m_2+1}^{m_2+m_4} |\varphi|(W_{u_i} - z)|g_2|(W_{\tilde{u}} - z) dsdud\tilde{s}d\tilde{u}dydz, \end{aligned} \tag{5.16}$$

where we have changed variables $y \to y + \frac{x}{\varepsilon}$ and $z \to z + \frac{x}{\varepsilon}$.

Since $m_1 + m_2 \geq 1$ and $m_3 + m_4 \geq 1$, there are four cases.

1. $m_1 m_3 \neq 0$.
2. $m_2 m_4 \neq 0$.

3. $m_2 = m_3 = 0$.

4. $m_1 = m_4 = 0$.

Lemma 5.4. *In all four cases, we have*

$$\begin{aligned}
&\varepsilon^{N(m_i)}\mathbb{E}_B\mathbb{E}_W\int_{\mathbb{R}^{2d}}\int_{[0,t/\varepsilon^2]^{N(m_i)+2}}\prod_{i=1}^{m_1}|\varphi|(B_{s_i}-y)\prod_{i=m_1+1}^{m_1+m_3}|\varphi|(B_{s_i}-z)|g_1|(B_{\tilde{s}}-y)\\
&\qquad\prod_{i=1}^{m_2}|\varphi|(W_{u_i}-y)\prod_{i=m_2+1}^{m_2+m_4}|\varphi|(W_{u_i}-z)|g_2|(W_{\tilde{u}}-z)dsdud\tilde{s}d\tilde{u}dydz\\
\leq&\varepsilon^{N(m_i)-1}|\log\varepsilon|^2C^{N(m_i)+2}(m_1+m_3+2)!(m_2+m_4+2)!
\end{aligned}\tag{5.17}$$

for some constant $C > 0$.

The proof of Lemma 5.4 is left in the Appendix.

Now we only have to note that

$$\sum_{m_1+m_2\geq 1, m_3+m_4\geq 1}\frac{\varepsilon^{N(m_i)-1}|\log\varepsilon|^2}{m_1!m_2!m_3!m_4!}C^{N(m_i)+2}(m_1+m_3+2)!(m_2+m_4+2)!\to 0 \tag{5.18}$$

as $\varepsilon \to 0$ to complete the proof of Lemma 5.3.

5.1.2 $(A_2(B), A_2(W))$

Lemma 5.5.

$$\mathbb{E}_B\mathbb{E}_W\{|\int_{\mathbb{R}^d}(e^{ig_B-ig_W}-1)h_Bdy\int_{\mathbb{R}^d}(e^{ig_B-ig_W}-1)\overline{h_W}dy|\}\to 0$$

as $\varepsilon \to 0$ for $h_B = \varepsilon^{\frac{1}{2}}f^\lambda(\frac{x}{\varepsilon}+\tilde{B}-y)$ with $\tilde{B}\in\{0, B_{t/\varepsilon^2}\}$, and $\overline{h_W} = \varepsilon^{\frac{1}{2}}f^\lambda(\frac{x}{\varepsilon}+\tilde{W}-y)$ with $\tilde{W}\in\{0, W_{t/\varepsilon^2}\}$.

Again, we expand e^{ix} in power series to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d}(e^{ig_B-ig_W}-1)h_Bdy\int_{\mathbb{R}^d}(e^{ig_B-ig_W}-1)\overline{h_W}dy\\
=&\sum_{m_1,m_2,m_3,m_4\geq 0}\frac{i^{m_1-m_2+m_3-m_4}}{m_1!m_2!m_3!m_4!}\int_{\mathbb{R}^{2d}}g_B(y)^{m_1}g_W(y)^{m_2}g_B(z)^{m_3}g_W(z)^{m_4}h_B(y)\overline{h_W}(z)dydz,
\end{aligned}\tag{5.19}$$

with $m_1+m_2\geq 1, m_3+m_4\geq 1$. For each term, let $N(m_i)=\sum_{i=1}^4 m_i$, we have

$$\begin{aligned}
&|\int_{\mathbb{R}^{2d}}g_B(y)^{m_1}g_W(y)^{m_2}g_B(z)^{m_3}g_W(z)^{m_4}h_B(y)\overline{h_W}(z)dydz|\\
\lesssim&\varepsilon^{N(m_i)+1}\int_{\mathbb{R}^{2d}}\int_{[0,t/\varepsilon^2]^{N(m_i)}}\prod_{i=1}^{m_1}|\varphi|(B_{s_i}-y)\prod_{i=m_1+1}^{m_1+m_3}|\varphi|(B_{s_i}-z)|f^\lambda|(\tilde{B}-y)\\
&\qquad\prod_{i=1}^{m_2}|\varphi|(W_{u_i}-y)\prod_{i=m_2+1}^{m_2+m_4}|\varphi|(W_{u_i}-z)|f^\lambda|(\tilde{W}-z)dsdudydz.
\end{aligned}\tag{5.20}$$

Applying the following lemmas, the proof of Lemma 5.5 is complete.

Lemma 5.6. *When $N(m_i) \geq 4$, we have*

$$\begin{aligned}
&\varepsilon^{N(m_i)+1}\mathbb{E}_B\mathbb{E}_W \int_{\mathbb{R}^{2d}} \int_{[0,t/\varepsilon^2]^{N(m_i)}} \prod_{i=1}^{m_1} |\varphi|(B_{s_i}-y) \prod_{i=m_1+1}^{m_1+m_3} |\varphi|(B_{s_i}-z)|f^\lambda|(\tilde{B}-y) \\
&\qquad \prod_{i=1}^{m_2} |\varphi|(W_{u_i}-y) \prod_{i=m_2+1}^{m_2+m_4} |\varphi|(W_{u_i}-z)|f^\lambda|(\tilde{W}-z) ds du dy dz \\
&\leq (m_1+m_3)!(m_2+m_4)! C^{N(m_i)} \varepsilon^{N(m_i)-3}.
\end{aligned} \tag{5.21}$$

Lemma 5.7. *When $N(m_i) = 2, 3$, we have*

$$\mathbb{E}_B\mathbb{E}_W \Big| \int_{\mathbb{R}^{2d}} g_B(y)^{m_1} g_W(y)^{m_2} g_B(z)^{m_3} g_W(z)^{m_4} h_B(y) \overline{h_W}(z) dy dz \Big| \lesssim \varepsilon. \tag{5.22}$$

The proofs of Lemma 5.6 and 5.7 are left in the Appendix.

5.1.3 $(A_1(B), A_2(W))$ and $(A_2(B), A_1(W))$

By symmetry, we only analyze $(A_1(B), A_2(W))$, i.e.,

Lemma 5.8.

$$\mathbb{E}_B\mathbb{E}_W \{ | \int_{\mathbb{R}^d} (e^{ig_B - ig_W} - 1) h_B dy \int_{\mathbb{R}^d} (e^{ig_B - ig_W} - 1) \overline{h_W} dy | \} \to 0$$

as $\varepsilon \to 0$ for $h_B = \int_0^{t/\varepsilon^2} g(\frac{x}{\varepsilon} + B_s - y) ds$ and $\overline{h_W} = \varepsilon^{\frac{1}{2}} f^\lambda(\frac{x}{\varepsilon} + \tilde{W} - y)$, where $g \in \{\varepsilon^{\frac{5}{2}} f^\lambda, \varepsilon^{\frac{3}{2}} f_k^\lambda\}$, and $\tilde{W} \in \{0, W_{t/\varepsilon^2}\}$.

Similarly, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} (e^{ig_B - ig_W} - 1) h_B dy \int_{\mathbb{R}^d} (e^{ig_B - ig_W} - 1) \overline{h_W} dy \\
=& \sum_{m_1+m_2 \geq 1, m_3+m_4 \geq 1} \frac{i^{m_1 - m_2 + m_3 - m_4}}{m_1! m_2! m_3! m_4!} \int_{\mathbb{R}^{2d}} g_B(y)^{m_1} g_W(y)^{m_2} g_B(z)^{m_3} g_W(z)^{m_4} h_B(y) \overline{h_W}(z) dy dz,
\end{aligned} \tag{5.23}$$

and let $N(m_i) = \sum_{i=1}^4 m_i$, the following two lemmas suffice to show Lemma 5.8.

Lemma 5.9. *If $N(m_i) \geq 3$, then*

$$\begin{aligned}
&\varepsilon^{N(m_i)+\frac{1}{2}}\mathbb{E}_B\mathbb{E}_W \int_{\mathbb{R}^{2d}} \int_{[0,t/\varepsilon^2]^{N(m_i)+1}} \prod_{i=1}^{m_1} |\varphi|(B_{s_i}-y) \prod_{i=m_1+1}^{m_1+m_3} |\varphi|(B_{s_i}-z)|g|(B_s-y) \\
&\qquad \prod_{i=1}^{m_2} |\varphi|(W_{u_i}-y) \prod_{i=m_2+1}^{m_2+m_4} |\varphi|(W_{u_i}-z)|f^\lambda|(\tilde{W}-z) ds du dy dz \\
&\leq \varepsilon^{N(m_i)-2} |\log \varepsilon| (m_2+m_4+1)!(m_1+m_3+1)!.
\end{aligned} \tag{5.24}$$

Lemma 5.10. *If $N(m_i)=2$, then*

$$\mathbb{E}_B\mathbb{E}_W\Big|\int_{\mathbb{R}^{2d}} g_B(y)^{m_1}g_W(y)^{m_2}g_B(z)^{m_3}g_W(z)^{m_4}h_B(y)\overline{h_W}(z)dydz\Big| \lesssim \varepsilon|\log\varepsilon|. \tag{5.25}$$

The proofs of Lemma 5.9 and 5.10 are left in the Appendix.

5.2 Gaussian case

When V is Gaussian, $X_t(B), Y_t(B), X_t(W), Y_t(W)$ are all Gaussian when freezing B, W, and our goal is to prove (5.2) by an explicit calculation and estimation of

$$\mathbb{E}\{(e^{iX_t^\varepsilon(B)}-e^{-\frac{1}{2}\sigma^2 t})Y_t^\varepsilon(B)\overline{(e^{iX_t^\varepsilon(W)}-e^{-\frac{1}{2}\sigma^2 t})Y_t^\varepsilon(W)}\}.$$

If (N_1,N_2,N_3,N_4) are jointly Gaussian with zero mean and covariance matrix Σ, by explicit calculation, we have

$$\begin{aligned}
&\mathbb{E}\{(e^{iN_1}-e^{-\frac{1}{2}\sigma^2 t})(e^{iN_2}-e^{-\frac{1}{2}\sigma^2 t})N_3N_4\}\\
=&\Sigma_{31}\Sigma_{41}\left(e^{-\frac{1}{2}\sigma^2 t}e^{-\frac{1}{2}\Sigma_{11}}-e^{-\frac{1}{2}\Sigma_{11}-\frac{1}{2}\Sigma_{22}-\Sigma_{12}}\right)\\
+&\Sigma_{32}\Sigma_{42}\left(e^{-\frac{1}{2}\sigma^2 t}e^{-\frac{1}{2}\Sigma_{22}}-e^{-\frac{1}{2}\Sigma_{11}-\frac{1}{2}\Sigma_{22}-\Sigma_{12}}\right)\\
-&\Sigma_{32}\Sigma_{41}e^{-\frac{1}{2}\Sigma_{11}-\frac{1}{2}\Sigma_{22}-\Sigma_{12}}\\
-&\Sigma_{31}\Sigma_{42}e^{-\frac{1}{2}\Sigma_{11}-\frac{1}{2}\Sigma_{22}-\Sigma_{12}}\\
+&\Sigma_{34}\left(e^{-\sigma^2 t}+e^{-\frac{1}{2}\Sigma_{11}-\frac{1}{2}\Sigma_{22}-\Sigma_{12}}-e^{-\frac{1}{2}\sigma^2 t}e^{-\frac{1}{2}\Sigma_{11}}-e^{-\frac{1}{2}\sigma^2 t}e^{-\frac{1}{2}\Sigma_{22}}\right).
\end{aligned} \tag{5.26}$$

Let Σ be the covariance matrix of $(X_t^\varepsilon(B), -X_t^\varepsilon(W), Y_t^\varepsilon(B), \overline{Y_t^\varepsilon(W)})$, and

$$\mathbb{E}\{(e^{iX_t^\varepsilon(B)}-e^{-\frac{1}{2}\sigma^2 t})Y_t^\varepsilon(B)\overline{(e^{iX_t^\varepsilon(W)}-e^{-\frac{1}{2}\sigma^2 t})Y_t^\varepsilon(W)}\}=P_1+P_2,$$

where

$$P_1=\Sigma_{34}\left(e^{-\sigma^2 t}+e^{-\frac{1}{2}\Sigma_{11}-\frac{1}{2}\Sigma_{22}-\Sigma_{12}}-e^{-\frac{1}{2}\sigma^2 t}e^{-\frac{1}{2}\Sigma_{11}}-e^{-\frac{1}{2}\sigma^2 t}e^{-\frac{1}{2}\Sigma_{22}}\right), \tag{5.27}$$

and P_2 is the remainder.

Lemma 5.11. $\mathbb{E}_B\mathbb{E}_W\{|P_1|\}\to 0$.

Proof. First, we have

$$\begin{aligned}
&\mathbb{E}_B\mathbb{E}_W\{|P_1|\}\\
\leq&\sqrt{\mathbb{E}_B\mathbb{E}_W\left(e^{-\sigma^2 t}+e^{-\frac{1}{2}\Sigma_{11}-\frac{1}{2}\Sigma_{22}-\Sigma_{12}}-e^{-\frac{1}{2}\sigma^2 t}e^{-\frac{1}{2}\Sigma_{11}}-e^{-\frac{1}{2}\sigma^2 t}e^{-\frac{1}{2}\Sigma_{22}}\right)^2}\\
&\times\sqrt{\mathbb{E}_B\mathbb{E}_W|\Sigma_{34}|^2}.
\end{aligned} \tag{5.28}$$

Clearly, $\Sigma_{34}=\mathbb{E}\{Y_t^\varepsilon(B)\overline{Y_t^\varepsilon(W)}\}$, so $\mathbb{E}_B\mathbb{E}_W|\Sigma_{34}|^2\leq\mathbb{E}\mathbb{E}_B\{|Y_t^\varepsilon(B)|^2\}\mathbb{E}\mathbb{E}_W\{|Y_t^\varepsilon(W)|^2\}$ is uniformly bounded. Then we only need to apply Lemma A.3 to complete the proof. ☐

The following lemma suffices to prove $\mathbb{E}_B\mathbb{E}_W\{|P_2|\}\to 0$.

Lemma 5.12. $\mathbb{E}_B\mathbb{E}_W\{|\Sigma_{ij}|^2\} \to 0$ *with* $i \in \{3,4\}, j \in \{1,2\}$.

Proof. By symmetry, we only need to prove that $\mathbb{E}_B\mathbb{E}_W\{I_n^2\} \to 0$ for

$$I_1 = \varepsilon^{\frac{7}{2}} \int_{[0,t/\varepsilon^2]^2} (R \star G_\lambda)(x_s - B_u) ds du, \tag{5.29}$$

$$I_2 = \varepsilon^{\frac{3}{2}} \int_{[0,t/\varepsilon^2]} (R \star G_\lambda)(x_{t/\varepsilon^2} - B_u) du, \tag{5.30}$$

$$I_3 = \varepsilon^{\frac{3}{2}} \int_{[0,t/\varepsilon^2]} (R \star G_\lambda)(B_u) du, \tag{5.31}$$

$$I_4 = \varepsilon^{\frac{5}{2}} \int_{[0,t/\varepsilon^2]^2} (R \star \partial_{x_k} G_\lambda)(x_s - B_u) ds du, \tag{5.32}$$

where $x \in \{B, W\}$. All cases are contained in Lemma A.6, A.8 if we replace R by some bounded, integrable, positive, and radially symmetric and decreasing function, G_λ by $e^{-c\sqrt{\lambda}|x|}|x|^{2-d}$, and $\partial_{x_k} G_\lambda$ by $e^{-c\sqrt{\lambda}|x|}|x|^{1-d}$ for some constant $c > 0$. In the end, we only need to apply Lemma A.5 to conclude the proof. $\square$

6 Gaussian Limit, proof of Proposition 4.2: $d = 3$

Let $v_\varepsilon(t,x) := \mathbb{E}_B\{f(x+B_t)e^{-\frac{1}{2}\sigma^2 t} i\varepsilon^{-\frac{3}{2}} \int_0^t V(\frac{x+B_s}{\varepsilon})ds\}$, we show that it is a solution to a parabolic equation with an additive potential.

Lemma 6.1. $v_\varepsilon(t,x)$ *solves the following equation*

$$\partial_t v_\varepsilon(t,x) = \frac{1}{2}\Delta v_\varepsilon(t,x) - \frac{1}{2}\sigma^2 v_\varepsilon(t,x) + i u_{hom}(t,x) \frac{1}{\varepsilon^{\frac{3}{2}}} V(\frac{x}{\varepsilon}) \tag{6.1}$$

with zero initial condition.

Proof. By Feynman-Kac formula, we can write the solution to (6.1) as

$$v_\varepsilon(t,x) = \mathbb{E}_B\{\int_0^t e^{-\frac{1}{2}\sigma^2 s} i u_{hom}(t-s, x+B_s) \frac{1}{\varepsilon^{\frac{3}{2}}} V(\frac{x+B_s}{\varepsilon}) ds\}. \tag{6.2}$$

Since u_{hom} solves the homogenized equation (2.4), $u_{hom}(t,x) = \mathbb{E}_W\{f(x+W_t)e^{-\frac{1}{2}\sigma^2 t}\}$, so we have

$$\begin{aligned} v_\varepsilon(t,x) =& \mathbb{E}_B\mathbb{E}_W\{\int_0^t e^{-\frac{1}{2}\sigma^2 s} i f(x+B_s+W_{t-s}) e^{-\frac{1}{2}\sigma^2(t-s)} \frac{1}{\varepsilon^{\frac{3}{2}}} V(\frac{x+B_s}{\varepsilon}) ds\} \\ =& \mathbb{E}_B\mathbb{E}_W\{\int_0^t f(x+B_s+W_{t-s}) e^{-\frac{1}{2}\sigma^2 t} i \frac{1}{\varepsilon^{\frac{3}{2}}} V(\frac{x+B_s}{\varepsilon}) ds\} \\ =& \mathbb{E}_B\{f(x+B_t) e^{-\frac{1}{2}\sigma^2 t} i \frac{1}{\varepsilon^{\frac{3}{2}}} \int_0^t V(\frac{x+B_s}{\varepsilon}) ds\}. \end{aligned} \tag{6.3}$$

$\square$

Since v_ε solves (6.1) with zero initial condition, the solution may be written as

$$v_\varepsilon(t,x) = i\int_0^t \int_{\mathbb{R}^d} \mathcal{G}_{t-s}(x-y)u_{hom}(s,y)\frac{1}{\varepsilon^{\frac{3}{2}}}V(\frac{y}{\varepsilon})dyds, \tag{6.4}$$

where $\mathcal{G}_{t-s}(x-y) = e^{-\frac{1}{2}\sigma^2(t-s)}q_{t-s}(x-y)$.

We first show for fixed (t,x), $v_\varepsilon(t,x) \Rightarrow v(t,x)$ in distribution.

The solution to the limiting SPDE (4.9) can be written as

$$v(t,x) = i\int_0^t \int_{\mathbb{R}^d} \mathcal{G}_{t-s}(x-y)\sqrt{\hat{R}(0)}u_{hom}(s,y)W(dy)ds, \tag{6.5}$$

with $W(dy)$ the Wiener integral.

Let

$$\begin{aligned} var_\varepsilon : &= \int_0^t\int_0^t\int_{\mathbb{R}^{2d}} \mathcal{G}_{t-s}(x-y)\mathcal{G}_{t-u}(x-z)u_{hom}(s,y)u_{hom}(u,z)\frac{1}{\varepsilon^3}R(\frac{y-z}{\varepsilon})dydzdsdu, \\ var : &= \hat{R}(0)\int_0^t\int_0^t\int_{\mathbb{R}^d} \mathcal{G}_{t-s}(x-z)\mathcal{G}_{t-u}(x-z)u_{hom}(s,z)u_{hom}(u,z)dzdsdu. \end{aligned}$$

Lemma 6.2. *$var_\varepsilon \to var$.*

Proof. By change of variables, we have

$$var_\varepsilon = \int_0^t\int_0^t\int_{\mathbb{R}^{2d}} \mathcal{G}_{t-s}(x-z-\varepsilon w)\mathcal{G}_{t-u}(x-z)u_{hom}(s,z+\varepsilon w)u_{hom}(u,z)R(w)dwdzdsdu. \tag{6.6}$$

For fixed $s,u \in (0,t)$,

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} \mathcal{G}_{t-s}(x-z-\varepsilon w)\mathcal{G}_{t-u}(x-z)u_{hom}(s,z+\varepsilon w)u_{hom}(u,z)R(w)dwdz \\ \to&\hat{R}(0)\int_{\mathbb{R}^d} \mathcal{G}_{t-s}(x-z)\mathcal{G}_{t-u}(x-z)u_{hom}(s,z)u_{hom}(u,z)dz \end{aligned} \tag{6.7}$$

by the dominated convergence theorem. Since u_{hom} is bounded, we have

$$|\int_{\mathbb{R}^{2d}} \mathcal{G}_{t-s}(x-z-\varepsilon w)\mathcal{G}_{t-u}(x-z)u_{hom}(s,z+\varepsilon w)u_{hom}(u,z)R(w)dwdz| \lesssim \frac{1}{(2t-s-u)^{\frac{d}{2}}}, \tag{6.8}$$

which is integrable in $[0,t]^2$ since $d=3$. Thus again by the dominated convergence theorem, the proof is complete. $\square$

If V is Gaussian, then $v_\varepsilon(t,x)$ is Gaussian. Since both the mean and variance converge, we have $v_\varepsilon(t,x) \Rightarrow v(t,x)$ in distribution. For the convergence of finite dimensional distributions, we only need to show the convergence of $\mathbb{E}\{v_\varepsilon(t_1,x_1)v_\varepsilon(t_2,x_2)\}$, but the proof is the same as in Lemma 6.2.

If V is Poissonian $V(x) = \int_{\mathbb{R}^d}\varphi(x-y)\omega(dy) - c_\varphi$, then

$$\begin{aligned} v_\varepsilon(t,x) =&i\int_{\mathbb{R}^d}\left(\int_0^t\int_{\mathbb{R}^d} \mathcal{G}_{t-s}(x-y)u_{hom}(s,y)\frac{1}{\varepsilon^{\frac{3}{2}}}\varphi(\frac{y}{\varepsilon}-z)dyds\right)\omega(dz) \\ &-i\int_0^t\int_{\mathbb{R}^d} \mathcal{G}_{t-s}(x-y)u_{hom}(s,y)\frac{1}{\varepsilon^{\frac{3}{2}}}c_\varphi dyds \end{aligned} \tag{6.9}$$

is Poissonian as well, and we have the following lemma.

Lemma 6.3. *For any $\theta \in \mathbb{R}$,*

$$\mathbb{E}\{\exp(\theta v_\varepsilon(t,x))\} \to \mathbb{E}\{\exp(\theta v(t,x))\} \tag{6.10}$$

as $\varepsilon \to 0$.

Proof. Let $f_\varepsilon(z) = \int_0^t \int_{\mathbb{R}^d} \mathcal{G}_{t-s}(x-y) u_{hom}(s,y) \varepsilon^{-\frac{3}{2}} \varphi(\frac{y}{\varepsilon} - z) dy ds$, then

$$\begin{aligned}
&\mathbb{E}\{\exp(i\theta \int_{\mathbb{R}^d} f_\varepsilon(z)\omega(dz) - i\theta \int_0^t \int_{\mathbb{R}^d} \mathcal{G}_{t-s}(x-y) u_{hom}(s,y) \varepsilon^{-\frac{3}{2}} c_\varphi dy ds)\} \\
=& \exp(\int_{\mathbb{R}^d} (e^{i\theta f_\varepsilon(z)} - 1) dz - i\theta \int_0^t \int_{\mathbb{R}^d} \mathcal{G}_{t-s}(x-y) u_{hom}(s,y) \varepsilon^{-\frac{3}{2}} c_\varphi dy ds) \\
=& \exp(\int_{\mathbb{R}^d} \sum_{k=2}^\infty \frac{1}{k!} (i\theta)^k f_\varepsilon(z)^k dz),
\end{aligned} \tag{6.11}$$

since $\int_{\mathbb{R}^d} \varphi(z) dz = c_\varphi$.

When $k=2$, $\int_{\mathbb{R}^d} f_\varepsilon(z)^2 dz = var_\varepsilon$, so by Lemma 6.2, $\int_{\mathbb{R}^d} f_\varepsilon(z)^2 dz \to var$.

When $k \geq 3$, note that $\mathcal{G}_{t-s}(x-y) \leq q_{t-s}(x-y)$ and u_{hom} is bounded, so we have $|f_\varepsilon(z)| \lesssim \int_0^t \int_{\mathbb{R}^d} q_s(x-y) \frac{1}{\varepsilon^{\frac{3}{2}}} |\varphi|(\frac{y}{\varepsilon} - z) dy ds$, which implies

$$\int_{\mathbb{R}^d} |f_\varepsilon(z)|^k dz \lesssim \frac{1}{\varepsilon^{\frac{3k}{2}}} \int_{\mathbb{R}^d} \int_{[0,t]^k} \int_{\mathbb{R}^{kd}} \prod_{i=1}^k q_{s_i}(x-y_i) |\varphi|(\frac{y_i}{\varepsilon} - z) dy ds dz. \tag{6.12}$$

In the Fourier domain, by change of variables and integration in z, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} |f_\varepsilon(z)|^k dz \lesssim& \frac{1}{\varepsilon^{\frac{3k}{2}}} \int_{[0,t]^k} \int_{\mathbb{R}^{kd}} \prod_{i=1}^k |\mathcal{F}\{|\varphi|\}(\xi_i)| e^{-\frac{|\xi_i|^2}{2\varepsilon^2} s_i} \delta(\sum_{i=1}^k \xi_i) d\xi ds \\
=& \frac{1}{\varepsilon^{\frac{3k}{2}}} \int_{[0,t]^k} \int_{\mathbb{R}^{(k-1)d}} |\mathcal{F}\{|\varphi|\}(-\sum_{i=2}^k \xi_i)| e^{-\frac{|\sum_{i=2}^k \xi_i|^2}{2\varepsilon^2} s_1} \prod_{i=2}^k |\mathcal{F}\{|\varphi|\}(\xi_i)| e^{-\frac{|\xi_i|^2}{2\varepsilon^2} s_i} d\xi ds.
\end{aligned} \tag{6.13}$$

Changing variables $\xi_2 \to \varepsilon \xi_2$, $s_i \to \varepsilon^2 s_i, i \geq 3$, and since $|\mathcal{F}\{|\varphi|\}|$ is uniformly bounded, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} |f_\varepsilon(z)|^k dz \lesssim \varepsilon^{\frac{k}{2}-1} \int_{[0,t]^2} \int_{[0,t/\varepsilon^2]^{k-2}} \int_{\mathbb{R}^{(k-1)d}} & e^{-\frac{1}{2}|\xi_2 + \frac{1}{\varepsilon}\sum_{i=3}^k \xi_i|^2 s_1} e^{-\frac{1}{2}|\xi_2|^2 s_2} \\
& \prod_{i=3}^k |\mathcal{F}\{|\varphi|\}(\xi_i)| e^{-\frac{|\xi_i|^2}{2} s_i} d\xi ds.
\end{aligned} \tag{6.14}$$

Clearly $\int_{\mathbb{R}^d} e^{-\frac{1}{2}|\xi_2 + \frac{1}{\varepsilon}\sum_{i=3}^k \xi_i|^2 s_1} e^{-\frac{1}{2}|\xi_2|^2 s_2} d\xi_2 \lesssim |s_1 + s_2|^{-\frac{d}{2}}$, which is integrable in $[0,t]^2$ when $d=3$. Now we only have to integrate in $s_i, i \geq 3$ and use the fact that $\mathcal{F}\{|\varphi|\}(\xi)|\xi|^{-2}$ is integrable to conclude that $\int_{\mathbb{R}^d} |f_\varepsilon(z)|^k dz \leq C^k \varepsilon^{\frac{k}{2}-1}$, so

$$\sum_{k \geq 3} \frac{1}{k!} |\theta|^k \int_{\mathbb{R}^d} |f_\varepsilon(z)|^k dz \to 0 \tag{6.15}$$

as $\varepsilon \to 0$. The proof is complete. $\Box$

To prove the convergence of finite dimensional distributions in the Poissonian case, we only need to apply the results for Gaussian when $k = 2$, and use the fact that $|\sum_{i=1}^N a_i|^k \le N^{k-1} \sum_{i=1}^N |a_i|^k$ when $k \ge 3$ in the proof of Lemma 6.3.

For the convergence of

$$\int_{\mathbb{R}^d} v_\varepsilon(t,x) g(x) dx \Rightarrow \int_{\mathbb{R}^d} v(t,x) g(x) dx \tag{6.16}$$

weakly for $g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, the discussion is the same as in Lemma 6.2 and 6.3.

7 Proof of the main theorem: $d \ge 4$

We first consider the case $d \ge 5$ when the stationary corrector exists. For constant initial condition, we will see that the discussion of the critical case $d = 4$ is similar to $d \ge 5$.

The following lemma confirms that the existence of a stationary corrector is equivalent with the integrability of $\hat{R}(\xi)|\xi|^{-4}$.

Lemma 7.1. *The equation $-L\Phi = \mathbb{V}$ has a solution in $L^2(\Omega)$ only when $\hat{R}(\xi)|\xi|^{-4}$ is integrable, and we have the regularized corrector $\Phi_\lambda \to \Phi$ in $L^2(\Omega)$.*

Proof. By spectral representation, the solution should be written as

$$\Phi = \int_{\mathbb{R}^d} \frac{2}{|\xi|^2} U(d\xi)\mathbb{V}, \tag{7.1}$$

and for it to be well-defined, we need

$$\langle \int_{\mathbb{R}^d} \frac{2}{|\xi|^2} U(d\xi)\mathbb{V}, \int_{\mathbb{R}^d} \frac{2}{|\xi|^2} U(d\xi)\mathbb{V}\rangle = \frac{4}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\hat{R}(\xi)}{|\xi|^4} d\xi < \infty. \tag{7.2}$$

If the integrability condition holds, we have

$$\langle \Phi_\lambda - \Phi, \Phi_\lambda - \Phi \rangle \lesssim \int_{\mathbb{R}^d} \frac{\lambda^2 \hat{R}(\xi)}{(2\lambda + |\xi|^2)^2 |\xi|^4} d\xi \to 0 \tag{7.3}$$

by the dominated convergence theorem. $\Box$

Under Assumption 2.5, $R(x)$ decays sufficiently fast, so $\hat{R}(\xi)$ is bounded, and the stationary corrector exists when $d \ge 5$. We recall (3.32) that

$$\begin{aligned} \frac{u_\varepsilon(t,x) - u_{hom}(t,x)}{\varepsilon} \approx & \frac{v_{1,\varepsilon} + v_{2,\varepsilon}}{\varepsilon} \\ = & \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} \hat{f}(\xi) e^{i\xi\cdot x} \varepsilon^{-1} \mathbb{E}_B\{e^{i\tilde{M}_t^\varepsilon} \left(iR_t^\varepsilon - \frac{1}{2}(\langle \tilde{M}^\varepsilon \rangle_t - (|\xi|^2 + \sigma_\lambda^2)t) \right)\} d\xi. \end{aligned} \tag{7.4}$$

$v_{1,\varepsilon}$ corresponds to the contribution from the remainder R_t^ε, and $v_{2,\varepsilon}$ corresponds to the contribution from the martingales, i.e., by the quantitative martingale central limit theorem, it reduces to the difference between quadratic variations $\langle \tilde{M}^\varepsilon \rangle_t - (|\xi|^2 + \sigma_\lambda^2)t$.

We will analyze $v_{1,\varepsilon}/\varepsilon$ and $v_{2,\varepsilon}/\varepsilon$ separately, and it turns out that the remainder contributes to the random corrector while the martingale part contributes to the deterministic error.

7.1 Analysis of $v_{1,\varepsilon}$: $d \geq 5$

Recall that

$$\frac{v_{1,\varepsilon}}{\varepsilon} = \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} \hat{f}(\xi) e^{i\xi\cdot x} \varepsilon^{-1} \mathbb{E}_B\{e^{i\tilde{M}_t^\varepsilon} i R_t^\varepsilon\} d\xi, \tag{7.5}$$

where $R_t^\varepsilon = \varepsilon \int_0^{t/\varepsilon^2} \lambda \Phi_\lambda(y_s) ds - \varepsilon \Phi_\lambda(y_{t/\varepsilon^2}) + \varepsilon \Phi_\lambda(y_0)$ with the environmental process $y_s = \tau_{\frac{x}{\varepsilon}+B_s}\omega$, we discuss the three terms respectively.

Lemma 7.2. *$\int_{\mathbb{R}^d} (2\pi)^{-d} \hat{f}(\xi) e^{i\xi\cdot x} \mathbb{E}_B\{e^{i\tilde{M}_t^\varepsilon} i\Phi_\lambda(y_0)\} d\xi - iu_{hom}(t,x)\Phi(\tau_{\frac{x}{\varepsilon}}\omega) \to 0$ in $L^1(\Omega)$.*

Proof. First of all, we show that

$$\int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} \hat{f}(\xi) e^{i\xi\cdot x} \left(\mathbb{E}_B\{e^{i\tilde{M}_t^\varepsilon}\} - e^{-\frac{1}{2}(|\xi|^2+\sigma_\lambda^2)t} \right) i\Phi_\lambda(y_0) d\xi \to 0 \tag{7.6}$$

in $L^1(\Omega)$. The result comes from an application of the quantitative martingale central limit theorem, together with the fact that $\mathbb{E}\mathbb{E}_B\{|\langle \tilde{M}^\varepsilon \rangle_t - (|\xi|^2 + \sigma_\lambda^2)t|^2\} \lesssim \varepsilon^2$ and $\mathbb{E}\{\Phi_\lambda^2\}$ is uniformly bounded when $d \geq 5$. Since $|\sigma_\lambda^2 - \sigma^2| \lesssim \varepsilon^2$ and $\Phi_\lambda \to \Phi$ by Lemma 7.1, we obtain that

$$\int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} \hat{f}(\xi) e^{i\xi\cdot x} \mathbb{E}_B\{e^{i\tilde{M}_t^\varepsilon} i\Phi_\lambda(y_0)\} d\xi - \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} \hat{f}(\xi) e^{i\xi\cdot x} e^{-\frac{1}{2}(|\xi|^2+\sigma^2)t} i\Phi(y_0) d\xi \to 0 \tag{7.7}$$

in $L^1(\Omega)$. The proof is complete. □

Lemma 7.3. *$\int_{\mathbb{R}^d} (2\pi)^{-d} \hat{f}(\xi) e^{i\xi\cdot x} \mathbb{E}_B\{e^{i\tilde{M}_t^\varepsilon} \Phi_\lambda(y_{t/\varepsilon^2})\} d\xi \to 0$ in $L^1(\Omega)$.*

Proof. We only need to show that $\mathbb{E}_B\{e^{i\tilde{M}_t^\varepsilon} \Phi_\lambda(y_{t/\varepsilon^2})\} \to 0$ in $L^1(\Omega)$. Recall that $\tilde{M}_t^\varepsilon = \sum_{k=1}^d \varepsilon \int_0^{t/\varepsilon^2} (\xi_k + D_k \Phi_\lambda(y_s)) dB_s^k$. For any $u \in (0, t/\varepsilon^2)$ that may depend on ε, we consider

$$\mathbb{E}_B\{e^{i\sum_{k=1}^d \varepsilon \int_0^u (\xi_k + D_k\Phi_\lambda(y_s))dB_s^k} \Phi_\lambda(y_{t/\varepsilon^2})\} = \mathbb{E}_B\{\mathbb{E}_B\{\Phi_\lambda(y_{t/\varepsilon^2})|\mathcal{F}_u\} e^{i\sum_{k=1}^d \varepsilon \int_0^u (\xi_k + D_k\Phi_\lambda(y_s))dB_s^k}\} \tag{7.8}$$

with $\mathcal{F}_s$ the natural filtration associated with B. The r.h.s. of the last display can be bounded by $\mathbb{E}_B\{|\mathbb{E}_B\{\Phi_\lambda(y_{t/\varepsilon^2})|\mathcal{F}_u\}|\}$, and since y_s is invariant with respect to $\mathbb{P}$, we have

$$\mathbb{E}\mathbb{E}_B\{|\mathbb{E}_B\{\Phi_\lambda(y_{t/\varepsilon^2})|\mathcal{F}_u\}|\} = \mathbb{E}\{|\mathbb{E}_B\{\Phi_\lambda(y_{t/\varepsilon^2-u})\}|\}. \tag{7.9}$$

By an explicit calculation, we have

$$\mathbb{E}\{|\mathbb{E}_B\{\Phi_\lambda(y_s)\}|^2\} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{R_{\Phi_\lambda}}(\xi) e^{-|\xi|^2 s} d\xi \to 0 \tag{7.10}$$

as $s \to \infty$, where R_{Φ_λ} is the covariance function of Φ_λ and satisfies $\widehat{R_{\Phi_\lambda}}(\xi) \lesssim \hat{R}(\xi)|\xi|^{-4}$. Now we have

$$\begin{aligned} \mathbb{E}_B\{e^{i\tilde{M}_t^\varepsilon} \Phi_\lambda(y_{t/\varepsilon^2})\} =& \mathbb{E}_B\{(e^{i\tilde{M}_t^\varepsilon} - e^{i\sum_{k=1}^d \varepsilon \int_0^u (\xi_k + D_k\Phi_\lambda(y_s))dB_s^k}) \Phi_\lambda(y_{t/\varepsilon^2})\} \\ &+ \mathbb{E}_B\{e^{i\sum_{k=1}^d \varepsilon \int_0^u (\xi_k + D_k\Phi_\lambda(y_s))dB_s^k} \Phi_\lambda(y_{t/\varepsilon^2})\} \end{aligned} \tag{7.11}$$

for any $u \in (0, t/\varepsilon^2)$. The second term goes to zero in $L^1(\Omega)$ if we choose u so that $t/\varepsilon^2 - u \to \infty$ as $\varepsilon \to 0$ by the above discussion. For the first term, its L^1 norm is bounded by $\sqrt{\varepsilon^2(t/\varepsilon^2 - u)}$ since Φ_λ is bounded in L^2. Therefore, we only need to choose u so that $t/\varepsilon^2 - u \to \infty$ and $\varepsilon^2(t/\varepsilon^2 - u) \to 0$, e.g., when $t/\varepsilon^2 - u = 1/\varepsilon$ to complete the proof. □

Lemma 7.4. $\int_{\mathbb{R}^d}(2\pi)^{-d}\hat{f}(\xi)e^{i\xi\cdot x}\mathbb{E}_B\{e^{i\tilde{M}_t^\varepsilon}\int_0^{t/\varepsilon^2}\lambda\Phi_\lambda(y_s)ds\}d\xi \to 0$ *in* $L^2(\Omega)$.

Proof. We only need to show $\mathbb{E}\mathbb{E}_B\{|\int_0^{t/\varepsilon^2}\lambda\Phi_\lambda(y_s)ds|^2\} \to 0$. By an explicitly calculation, we have that

$$\begin{aligned}\mathbb{E}\mathbb{E}_B\{|\int_0^{t/\varepsilon^2}\lambda\Phi_\lambda(y_s)ds|^2\} =& \int_0^{t/\varepsilon^2}\int_0^{t/\varepsilon^2}\lambda^2\mathbb{E}_B\{R_{\Phi_\lambda}(B_s - B_u)\}dsdu \\ =& \frac{1}{(2\pi)^d}\int_0^t\int_0^t\int_{\mathbb{R}^d}\widehat{R_{\Phi_\lambda}}(\xi)e^{-\frac{1}{2}|\xi|^2\frac{|s-u|}{\varepsilon^2}}dsdu,\end{aligned} \tag{7.12}$$

where $R_{\Phi_\lambda}(x)$ is the covariance function of Φ_λ. Clearly $\widehat{R_{\Phi_\lambda}}(\xi) \lesssim \hat{R}(\xi)|\xi|^{-4}$, so by the dominated convergence theorem, the proof is complete. □

Combining Lemma 7.2, 7.3 and 7.4, we conclude that

$$\frac{v_{1,\varepsilon}}{\varepsilon} - iu_{hom}(t,x)\Phi(\tau_{\frac{x}{\varepsilon}}\omega) \to 0 \tag{7.13}$$

in $L^1(\Omega)$.

7.2 Analysis of $v_{2,\varepsilon}$: $d \geq 5$

Let

$$Z_{\lambda,\xi} := 2\sum_{k=1}^d \xi_k D_k\Phi_\lambda + \sum_{k=1}^d (D_k\Phi_\lambda)^2 - \sigma_\lambda^2,$$

we have

$$\frac{v_{2,\varepsilon}}{\varepsilon} = -\frac{1}{2}\int_{\mathbb{R}^d}\frac{1}{(2\pi)^d}\hat{f}(\xi)e^{i\xi\cdot x}\mathbb{E}_B\{e^{i\tilde{M}_t^\varepsilon}\varepsilon\int_0^{t/\varepsilon^2}Z_{\lambda,\xi}(y_s)ds\}d\xi. \tag{7.14}$$

We will show that $Z_{\lambda,\xi}$ can be replaced by

$$Z_\xi := 2\sum_{k=1}^d \xi_k D_k\Phi + \sum_{k=1}^d (D_k\Phi)^2 - \sigma^2,$$

so the term $\varepsilon\int_0^{t/\varepsilon^2}Z_\xi(y_s)ds$ is again of the form of Brownian motion in random scenery, to which we will apply Kipnis-Varadhan's method again.

Lemma 7.5. $\mathbb{E}\mathbb{E}_B\{|\varepsilon\int_0^{t/\varepsilon^2}(Z_{\lambda,\xi}(y_s) - Z_\xi(y_s))ds|\} \to 0$ *as* $\varepsilon \to 0$.

Proof. We first have $\mathbb{E}\{|Z_{\lambda,\xi} - Z_\xi|\} \lesssim \sum_{k=1}^d \|D_k\Phi_\lambda - D_k\Phi\| + |\sigma_\lambda^2 - \sigma^2|$. By a straightforward calculation, we have $r.h.s. \lesssim \varepsilon^{\frac{d}{2}-1}1_{d\leq 5} + \varepsilon^2|\log\varepsilon|^{\frac{1}{2}}1_{d=6} + \varepsilon^2 1_{d\geq 7}$, so when $d \geq 5$,

$$\mathbb{E}\mathbb{E}_B\{|\varepsilon\int_0^{t/\varepsilon^2}(Z_{\lambda,\xi}(y_s) - Z_\xi(y_s))ds|\} \lesssim \frac{1}{\varepsilon}\mathbb{E}\{|Z_{\lambda,\xi} - Z_\xi|\} \to 0 \tag{7.15}$$

as $\varepsilon \to 0$. □

The above proof shows that $Z_{\lambda,\xi} \to Z_\xi$ in $L^1(\Omega)$. We claim that the convergence is actually in $L^2(\Omega)$ by proving $\{Z_{\lambda,\xi}\}$ is a Cauchy sequence in $L^2(\Omega)$.

Proposition 7.6. *$Z_{\lambda,\xi} \to Z_\xi$ in $L^2(\Omega)$ and $\mathbb{E}\mathbb{E}_B\{(\varepsilon \int_0^{t/\varepsilon^2} Z_\xi(y_s)ds)^2\}$ is bounded uniformly in ε.*

Proof. Since $D_k\Phi_\lambda \to D_k\Phi$ in $L^2(\Omega)$ by [16, Proposition 3.2], we will show the convergence in $L^2(\Omega)$ of $\sum_{k=1}^d (D_k\Phi_\lambda)^2 - \sigma_\lambda^2$. It already converges in $L^1(\Omega)$ by the proof of Lemma 7.5, so we only need to show it is a Cauchy sequence in $L^2(\Omega)$ by proving $\langle \sum_{k=1}^d (D_k\Phi_{\lambda_1})^2 - \sigma_{\lambda_1}^2, \sum_{k=1}^d (D_k\Phi_{\lambda_2})^2 - \sigma_{\lambda_2}^2\rangle$ converges as $\lambda_1, \lambda_2 \to 0$. By a direct calculation, we obtain that

$$\langle \sum_{k=1}^d (D_k\Phi_{\lambda_1})^2 - \sigma_{\lambda_1}^2, \sum_{k=1}^d (D_k\Phi_{\lambda_2})^2 - \sigma_{\lambda_2}^2\rangle = \sum_{m,n=1}^d I_{mn}(\lambda_1,\lambda_2) - \sigma_{\lambda_1}^2\sigma_{\lambda_2}^2, \tag{7.16}$$

where

$$\begin{aligned} &I_{mn}(\lambda_1,\lambda_2) = \langle (D_m\Phi_{\lambda_2})^2, (D_n\Phi_{\lambda_2})^2\rangle \\ =& \int_{\mathbb{R}^{4d}} \partial_{x_m}G_{\lambda_1}(y_1)\partial_{x_m}G_{\lambda_1}(z_1)\partial_{x_n}G_{\lambda_2}(y_2)\partial_{x_n}G_{\lambda_2}(z_2)\mathbb{E}\{V(y_1)V(y_2)V(z_1)V(z_2)\}dy_1dy_2dz_1dz_2. \end{aligned} \tag{7.17}$$

Clearly $\partial_{x_k}G_\lambda(y) \to \partial_{x_k}G_0(y)$ almost everywhere as $\lambda \to 0$ with G_0 the Green's function of $-\frac{1}{2}\Delta$. We also have the bound $|\nabla G_\lambda(y)| \lesssim |y|^{1-d}$. Moreover, by the strongly mixing property in Assumption 2.5 and [18, Lemma 2.3], we have

$$|\mathbb{E}\{V(y_1)V(y_2)V(z_1)V(z_2)\}| \lesssim \Psi(y_1-y_2)\Psi(z_1-z_2) + \Psi(y_1-z_2)\Psi(z_1-y_2) + \Psi(y_1-z_1)\Psi(y_2-z_2) \tag{7.18}$$

for some Ψ satisfying $|\Psi(x)| \lesssim 1 \wedge |x|^{-\beta}$ for any $\beta > 0$. By the dominated convergence theorem and the convergence of $\sigma_\lambda^2 \to \sigma^2$, we have the convergence of $\sum_{m,n=1}^d I_{mn}(\lambda_1,\lambda_2) - \sigma_{\lambda_1}^2\sigma_{\lambda_2}^2$. So $\sum_{k=1}^d (D_k\Phi_\lambda)^2 - \sigma_\lambda^2 \to \sum_{k=1}^d (D_k\Phi)^2 - \sigma^2$ in $L^2(\Omega)$.

For the uniform boundedness of $\varepsilon\int_0^{t/\varepsilon^2} Z_\xi(y_s)ds$, by [16, Lemma 3.4], we only need to show the integrability of $R_{1,\xi}(x)|x|^{2-d}$ and $R_{2,\xi}(x)|x|^{2-d}$, with $R_{1,\xi}, R_{2,\xi}$ the covariance function of $2\sum_{k=1}^d \xi_k D_k\Phi$ and $\sum_{k=1}^d (D_k\Phi)^2 - \sigma^2$ respectively. By the convergence in $L^2(\Omega)$, $R_{i,\xi}(x) = \lim_{\lambda\to 0} R_{i,\lambda,\xi}(x), i = 1,2$, where $R_{1,\lambda,\xi}, R_{2,\lambda,\xi}$ are the covariance function of $2\sum_{k=1}^d \xi_k D_k\Phi_\lambda$ and $\sum_{k=1}^d (D_k\Phi_\lambda)^2 - \sigma_\lambda^2$. [16, Proposition 3.5] shows that

$$|R_{1,\lambda,\xi}(x)| + |R_{2,\lambda,\xi}(x)| \lesssim (1+|\xi|)^2\left(\lambda^{\frac{d}{2}-1}e^{-c\sqrt{\lambda}|x|} + 1 \wedge \frac{e^{-c\sqrt{\lambda}|x|}}{|x|^{d-2}} + 1 \wedge \frac{1}{|x|^\beta}\right) \tag{7.19}$$

for some $c > 0$ and $\beta > 0$ sufficiently large. By taking the limit $\lambda \to 0$, we obtain

$$|R_{1,\xi}(x)| + |R_{2,\xi}(x)| \lesssim (1+|\xi|)^2\left(1 \wedge \frac{1}{|x|^{d-2}} + 1 \wedge \frac{1}{|x|^\beta}\right), \tag{7.20}$$

and clearly it implies the integrability of $(|R_{1,\xi}(x)| + |R_{2,\xi}(x)|)|x|^{2-d}$ since $d \geq 5$. The proof is complete. □

Now we show that for $\varepsilon\int_0^{t/\varepsilon^2} Z_\xi(y_s)ds$, we can apply Kipnis-Varadhan's result. Since we are in the probability space Ω with the measure-preserving and ergodic transformations $\{\tau_x, x \in \mathbb{R}^d\}$, the only assumption we need to verify is $\langle Z_\xi, -L^{-1}Z_\xi\rangle < \infty$, see [16, Assumption 2.1] and the proof of [16, Theorem 2.2]. By Kipnis-Varadhan, it is equivalent with the finiteness of the asymptotic variance, i.e., $\mathbb{E}\mathbb{E}_B\{(\varepsilon\int_0^{t/\varepsilon^2} Z_\xi(y_s)ds)^2\}$ is bounded uniformly in ε in our context. For the sake of convenience, we present the proof in the following lemma.

Lemma 7.7. *For any $\mathbb{V} \in L^2(\Omega)$, $\langle \mathbb{V}, -L^{-1}\mathbb{V}\rangle < \infty$ is equivalent with the fact that $\mathbb{E}\mathbb{E}_B\{(\frac{1}{\sqrt{t}}\int_0^t \mathbb{V}(\tau_{B_s}\omega)ds)^2\}$ is bounded uniformly in t.*

Proof. First, let $\hat{R}(\xi)$ be the power spectrum associated with $\mathbb{V}$, then $\langle \mathbb{V}, -L^{-1}\mathbb{V}\rangle < \infty$ is equivalent with the integrability of $\hat{R}(\xi)|\xi|^{-2}$. Now by an explicit calculation, we have

$$\mathbb{E}\mathbb{E}_B\{(\frac{1}{\sqrt{t}}\int_0^t \mathbb{V}(\tau_{B_s}\omega)ds)^2\} = \frac{2}{t}\int_0^t \left(\int_0^s \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d}\hat{R}(\xi)e^{-\frac{1}{2}|\xi|^2 u}d\xi du\right) ds. \tag{7.21}$$

As a function of s, $\int_0^s \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d}\hat{R}(\xi)e^{-\frac{1}{2}|\xi|^2 u}d\xi du$ is positive and increasing, so for the r.h.s. of the above display to be uniformly bounded in t, an equivalent condition is

$$\lim_{s\to\infty}\int_0^s \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d}\hat{R}(\xi)e^{-\frac{1}{2}|\xi|^2 u}d\xi du < \infty,$$

i.e., the integrability of $\hat{R}(\xi)|\xi|^{-2}$. The proof is complete. □

By Lemma 7.5, we know that

$$\frac{v_{2,\varepsilon}}{\varepsilon} \approx -\frac{1}{2}\int_{\mathbb{R}^d} \frac{1}{(2\pi)^d}\hat{f}(\xi)e^{i\xi\cdot x}\mathbb{E}_B\{e^{i\tilde{M}_t^\varepsilon}\varepsilon\int_0^{t/\varepsilon^2} Z_\xi(y_s)ds\}d\xi, \tag{7.22}$$

where $\approx$ means the error goes to zero in $L^1(\Omega)$. Since $\tilde{M}_t^\varepsilon = \sum_{k=1}^d \varepsilon\int_0^{t/\varepsilon^2}(\xi_k + D_k\Phi_\lambda(y_s))dB_s^k$, by the convergence $D_k\Phi_\lambda \to D_k\Phi$ in $L^2(\Omega)$, we further obtain

$$\frac{v_{2,\varepsilon}}{\varepsilon} \approx -\frac{1}{2}\int_{\mathbb{R}^d} \frac{1}{(2\pi)^d}\hat{f}(\xi)e^{i\xi\cdot x}\mathbb{E}_B\{e^{i\sum_{k=1}^d \varepsilon\int_0^{t/\varepsilon^2}(\xi_k + D_k\Phi(y_s))dB_s^k}\varepsilon\int_0^{t/\varepsilon^2} Z_\xi(y_s)ds\}d\xi. \tag{7.23}$$

By Kipnis-Varadhan's method and Proposition 7.6,

$$\mathbb{E}\mathbb{E}_B\{\left(\varepsilon\int_0^{t/\varepsilon^2} Z_\xi(y_s)ds - \sum_{k=1}^d \varepsilon\int_0^{t/\varepsilon^2} D_k\tilde{\Phi}(y_s)dB_s^k\right)^2\} \to 0, \tag{7.24}$$

where $\tilde{\Phi}$ is the corrector corresponding to Z_ξ. This leads to

$$\frac{v_{2,\varepsilon}}{\varepsilon} \approx -\frac{1}{2}\int_{\mathbb{R}^d} \frac{1}{(2\pi)^d}\hat{f}(\xi)e^{i\xi\cdot x}\mathbb{E}_B\{e^{i\sum_{k=1}^d \varepsilon\int_0^{t/\varepsilon^2}(\xi_k + D_k\Phi(y_s))dB_s^k}\sum_{k=1}^d\varepsilon\int_0^{t/\varepsilon^2} D_k\tilde{\Phi}(y_s)dB_s^k\}d\xi. \tag{7.25}$$

Since we have two martingales here, we apply martingale central limit theorem and ergodic theorem to show that $\mathbb{E}_B\{e^{i\sum_{k=1}^d \varepsilon\int_0^{t/\varepsilon^2}(\xi_k + D_k\Phi(y_s))dB_s^k}\sum_{k=1}^d\varepsilon\int_0^{t/\varepsilon^2} D_k\tilde{\Phi}(y_s)dB_s^k\}$ converges to some constant in $L^1(\Omega)$.

Proposition 7.8. *Define $\Sigma_{11} = |\xi|^2 + \sum_{k=1}^d \|D_k\Phi\|^2$ and $\Sigma_{12} = \sum_{k=1}^d \langle D_k\Phi, D_k\tilde{\Phi}\rangle$, we have*

$$\mathbb{E}_B\{e^{i\sum_{k=1}^d \varepsilon\int_0^{t/\varepsilon^2}(\xi_k + D_k\Phi(y_s))dB_s^k}\sum_{k=1}^d\varepsilon\int_0^{t/\varepsilon^2} D_k\tilde{\Phi}(y_s)dB_s^k\} \to ie^{-\frac{1}{2}\Sigma_{11}t}\Sigma_{12}t \tag{7.26}$$

in $L^1(\Omega)$.

Proof. By stationarity, we can replace y_s by $\tau_{B_s}\omega$. To simplify the notation, we let $N_1^\varepsilon = \sum_{k=1}^d \varepsilon\int_0^{t/\varepsilon^2}(\xi_k + D_k\Phi(y_s))dB_s^k$ and $N_2^\varepsilon = \sum_{k=1}^d \varepsilon\int_0^{t/\varepsilon^2} D_k\tilde{\Phi}(y_s)dB_s^k$. By martingale central limit theorem and ergodic theorem, we obtain that for almost every $\omega \in \Omega$,

$$(N_1^\varepsilon, N_2^\varepsilon) \Rightarrow (N_1, N_2), \tag{7.27}$$

where N_1, N_2 are joint Gaussian with mean zero and covariance matrix $t\Sigma$, and $\Sigma_{11} = |\xi|^2 + \sum_{k=1}^d \|D_k\Phi\|^2$, $\Sigma_{12} = \sum_{k=1}^d \langle D_k\Phi, D_k\tilde{\Phi}\rangle$, and $\Sigma_{22} = \sum_{k=1}^d \|D_k\tilde{\Phi}\|^2$. Let $g_K(x) = (x\wedge K)\vee(-K)$ be a continuous and bounded cutoff function for $K > 0$, and $h_K(x) = x - g_K(x)$ we have

$$\mathbb{E}_B\{e^{iN_1^\varepsilon}N_2^\varepsilon\} = \mathbb{E}_B\{e^{iN_1^\varepsilon}g_K(N_2^\varepsilon)\} + \mathbb{E}_B\{e^{iN_1^\varepsilon}h_K(N_2^\varepsilon)\} \tag{7.28}$$

For the second term, we have

$$\mathbb{E}\mathbb{E}_B\{|h_K(N_2^\varepsilon)|\} \le \mathbb{E}\mathbb{E}_B\{|N_2^\varepsilon|1_{|N_2^\varepsilon|\ge K}\} \le \frac{1}{K}\mathbb{E}\mathbb{E}_B\{|N_2^\varepsilon|^2\} \lesssim \frac{1}{K}. \tag{7.29}$$

Therefore,

$$\begin{aligned}\limsup_{\varepsilon\to 0}\mathbb{E}\{|\mathbb{E}_B\{e^{iN_1^\varepsilon}N_2^\varepsilon\} - \mathbb{E}\{e^{iN_1}N_2\}|\} \lesssim &\lim_{\varepsilon\to 0}\mathbb{E}\{|\mathbb{E}_B\{e^{iN_1^\varepsilon}g_K(N_2^\varepsilon)\} - \mathbb{E}\{e^{iN_1}g_K(N_2)\}|\}\\ &+ |\mathbb{E}\{e^{iN_1}N_2\} - \mathbb{E}\{e^{iN_1}g_K(N_2)\}| + \frac{1}{K}\\ =&|\mathbb{E}\{e^{iN_1}N_2\} - \mathbb{E}\{e^{iN_1}g_K(N_2)\}| + \frac{1}{K}\end{aligned} \tag{7.30}$$

by the weak convergence and dominated convergence theorem. Now we let $K \to \infty$ and calculate $\mathbb{E}\{e^{iN_1}N_2\} = ie^{-\frac{1}{2}\Sigma_{11}t}\Sigma_{12}t$ to complete the proof. □

The above proposition implies that

$$\begin{aligned}\frac{v_{2,\varepsilon}}{\varepsilon} \to &-\frac{1}{2}\int_{\mathbb{R}^d}\frac{1}{(2\pi)^d}\hat{f}(\xi)e^{i\xi\cdot x}ie^{-\frac{1}{2}(|\xi|^2+\sigma^2)t}d\xi\sum_{k=1}^d\langle D_k\Phi, D_k\tilde{\Phi}\rangle t\\ =&-\frac{1}{2}iu_{hom}(t,x)\sum_{k=1}^d\langle D_k\Phi, D_k\tilde{\Phi}\rangle t\end{aligned} \tag{7.31}$$

in $L^1(\Omega)$. By combining (3.32), (7.13) and (7.31), we conclude that

$$u_\varepsilon(t,x) - u_{hom}(t,x) = v_{1,\varepsilon} + v_{2,\varepsilon} + o(\varepsilon) = iu_{hom}(t,x)\Phi(\tau_{\frac{x}{\varepsilon}}\omega) + iu_{hom}(t,x)C_{\mathbb{V}}t + o(\varepsilon) \tag{7.32}$$

if we define

$$C_{\mathbb{V}} := -\frac{1}{2}\sum_{k=1}^d\langle D_k\Phi, D_k\tilde{\Phi}\rangle. \tag{7.33}$$

The proof of Theorem 2.8 is complete.

If we assume the symmetry condition of the distribution of $V(x) = \mathbb{V}(\tau_x\omega)$, $\mathbb{E}\{V(x_1)V(x_2)V(x_3)\} = 0, \forall x_1, x_2, x_3 \in \mathbb{R}^d$ as in the Gaussian case, by a direct calculation we can obtain $C_{\mathbb{V}} = 0$, i.e., the bias vanishes.

7.3 Analysis of $v_{1,\varepsilon}$: $d = 4$

When $d = 4$, by assuming the initial condition $f \equiv 1$, the error from the martingale part is negligible. From (3.30), we have

$$\frac{u_\varepsilon(t,x) - u_{hom}(t,x)}{\varepsilon|\log\varepsilon|^{\frac{1}{2}}} \approx \frac{v_{1,\varepsilon}}{\varepsilon|\log\varepsilon|^{\frac{1}{2}}} = \frac{\mathbb{E}_B\{iR_t^\varepsilon e^{iM_t^\varepsilon}\}}{\varepsilon|\log\varepsilon|^{\frac{1}{2}}}, \tag{7.34}$$

with the error going to zero in $L^1(\Omega)$. We recall that $R_t^\varepsilon = \varepsilon\int_0^{t/\varepsilon^2}\lambda\Phi_\lambda(y_s)ds - \varepsilon\Phi_\lambda(y_{t/\varepsilon^2}) + \varepsilon\Phi_\lambda(y_0)$ with $y_s = \tau_{\frac{x}{\varepsilon}+B_s}\omega$. The analysis is very similar with $d \geq 5$, and we present it through the following lemmas in parallel with Lemma 7.2, 7.3, 7.4.

Lemma 7.9. $|\log\varepsilon|^{-\frac{1}{2}}\mathbb{E}_B\{e^{iM_t^\varepsilon}\Phi_\lambda(y_0)\} - e^{-\frac{1}{2}\sigma^2 t}|\log\varepsilon|^{-\frac{1}{2}}\Phi_\lambda(y_0) \to 0$ *in* $L^1(\Omega)$ *as* $\varepsilon \to 0$.

Proof. The proof is the same as Lemma 7.2 if we note that $|\log\varepsilon|^{-1}\langle\Phi_\lambda,\Phi_\lambda\rangle < \infty$ and $\mathbb{E}\mathbb{E}_B\{|\langle M^\varepsilon\rangle_t - \sigma_\lambda^2 t|^2\} + |\sigma_\lambda^2 - \sigma^2| \lesssim \varepsilon^2|\log\varepsilon|$. □

Lemma 7.10. $|\log\varepsilon|^{-\frac{1}{2}}\mathbb{E}_B\{e^{iM_t^\varepsilon}\Phi_\lambda(y_{t/\varepsilon^2})\} \to 0$ *in* $L^1(\Omega)$ *as* $\varepsilon \to 0$.

Proof. The proof is the same as Lemma 7.3 except that we have to show as in (7.10) that

$$\frac{1}{|\log\varepsilon|}\mathbb{E}\{|\mathbb{E}_B\{\Phi_\lambda(y_s)\}|^2\} = \frac{1}{|\log\varepsilon|}\int_{\mathbb{R}^d}\frac{1}{(2\pi)^d}\frac{\hat{R}(\xi)}{(\lambda+\frac{1}{2}|\xi|^2)^2}e^{-|\xi|^2 s} \to 0 \tag{7.35}$$

for $s \in (0, t/\varepsilon^2)$ chosen so that $\varepsilon^2 s \to 0$. By Lemma A.9, we have

$$\frac{1}{|\log\varepsilon|}\int_{\mathbb{R}^d}\frac{1}{(2\pi)^d}\frac{\hat{R}(\xi)}{(\lambda+\frac{1}{2}|\xi|^2)^2}e^{-|\xi|^2 s}d\xi = \frac{1}{|\log\varepsilon|}\int_{\mathbb{R}^d}\frac{1}{(2\pi)^d}\frac{\hat{R}(\sqrt{\lambda}\xi)}{(1+\frac{1}{2}|\xi|^2)^2}e^{-|\xi|^2\lambda s}d\xi \lesssim \frac{1+|\log\lambda s|}{|\log\varepsilon|}. \tag{7.36}$$

Now we choose $s = \varepsilon^{-2}|\log\varepsilon|^{-1}$. In this way $|\log\lambda s| = \log|\log\varepsilon| \ll |\log\varepsilon|$ and $\varepsilon^2 s = |\log\varepsilon|^{-1} \to 0$ as $\varepsilon \to 0$. The proof is complete. □

Lemma 7.11. $|\log\varepsilon|^{-\frac{1}{2}}\mathbb{E}_B\{e^{iM_t^\varepsilon}\lambda\int_0^{t/\varepsilon^2}\Phi_\lambda(y_s)ds\} \to 0$ *in* $L^2(\Omega)$ *as* $\varepsilon \to 0$.

Proof. By an explicit calculation and Lemma A.9,

$$\begin{aligned}
\mathbb{E}\mathbb{E}_B\{|\lambda\int_0^{t/\varepsilon^2}\Phi_\lambda(y_s)ds|^2\} =& \frac{1}{(2\pi)^d}\int_0^{t/\varepsilon^2}\int_0^{t/\varepsilon^2}\int_{\mathbb{R}^d}\lambda^2\frac{\hat{R}(\xi)}{(\lambda+\frac{1}{2}|\xi|^2)^2}e^{-\frac{1}{2}|\xi|^2|s-u|}d\xi ds du \\
\lesssim& \int_0^t\int_0^t\int_{\mathbb{R}^d}\frac{1}{(1+\frac{1}{2}|\xi|^2)^2}e^{-\frac{1}{2}|\xi|^2|s-u|}d\xi ds du \\
\lesssim& \int_0^t\int_0^t(1+\log|s-u|)ds du < \infty,
\end{aligned} \tag{7.37}$$

so the proof is complete. □

Now we can combine (3.30), Lemma 7.9, 7.10 and 7.11 to conclude that

$$\frac{u_\varepsilon(t,x) - u_{hom}(t,x)}{\varepsilon|\log\varepsilon|^{\frac{1}{2}}} - iu_{hom}(t,x)\frac{\Phi_{\varepsilon^2}(\tau_{\frac{x}{\varepsilon}}\omega)}{|\log\varepsilon|^{\frac{1}{2}}} \to 0 \tag{7.38}$$

in $L^1(\Omega)$. The proof of Theorem 2.6 is complete.

7.4 Proof of Corollary 2.7: $d = 4$

The last goal is to prove the convergence in distribution of $|\log\varepsilon|^{-\frac{1}{2}}\Phi_{\varepsilon^2}(\tau_x\omega)$ for Gaussian and Poissonian potentials. To keep the notation simple, we consider $|\log\lambda|^{-\frac{1}{2}}\Phi_\lambda(\tau_x\omega)$.

For the Gaussian case, since $\mathbb{E}\{\Phi_\lambda\} = 0$, we only need to show the convergence of variance, i.e., $|\log\lambda|^{-1}\langle\Phi_\lambda, \Phi_\lambda\rangle$. This is given by Lemma A.10.

For the Poissonian case, we prove the weak convergence again by the characteristic function.

Lemma 7.12. *If V is Poissonian as in Assumption 2.3, then we have $\forall\theta\in\mathbb{R}$:*

$$\mathbb{E}\{\exp(i\theta\frac{\Phi_\lambda(\tau_x\omega)}{|\log\lambda|^{\frac{1}{2}}})\} \to \exp(-\frac{\hat{R}(0)\theta^2}{(2\pi)^d}). \tag{7.39}$$

Proof. The proof is similar to Lemma 6.3.

By stationarity, we can choose $x = 0$. For Poissonian potential $V(x) = \int_{\mathbb{R}^d}\varphi(x-y)\omega(dy) - c_\varphi$, then $\Phi_\lambda(\tau_0\omega) = \int_{\mathbb{R}^d} f^\lambda(-x)\omega(dx) - c_\varphi\int_{\mathbb{R}^d} G_\lambda(x)dx$, where $f^\lambda(x) = \int_{\mathbb{R}^d} G_\lambda(x-y)\varphi(y)dy$.

Now we can write

$$\mathbb{E}\{\exp(i\theta\frac{\Phi_\lambda(\tau_0\omega)}{|\log\lambda|^{\frac{1}{2}}})\} = \exp\left(\int_{\mathbb{R}^d}(e^{i\theta|\log\lambda|^{-\frac{1}{2}}f_\lambda(-x)} - 1)dx - i\theta|\log\lambda|^{-\frac{1}{2}}c_\varphi\int_{\mathbb{R}^d} G_\lambda(x)dx\right). \tag{7.40}$$

Since $\int_{\mathbb{R}^d} f_\lambda(-x)dx = c_\varphi\int_{\mathbb{R}^d} G_\lambda(x)dx$, after a Taylor expansion and change of variables, we have

$$\mathbb{E}\{\exp(i\theta\frac{\Phi_\lambda(\tau_x\omega)}{|\log\lambda|^{\frac{1}{2}}})\} = \exp\left(\int_{\mathbb{R}^d}\sum_{k=2}^\infty\frac{1}{k!}(i\theta|\log\lambda|^{-\frac{1}{2}}f_\lambda(x))^k dx\right). \tag{7.41}$$

First, when $k = 2$, we have $\int_{\mathbb{R}^d} f_\lambda(x)^2dx = \langle\Phi_\lambda, \Phi_\lambda\rangle$, therefore by Lemma A.10,

$$\frac{\int_{\mathbb{R}^d} f_\lambda(x)^2dx}{|\log\lambda|} \to \frac{2\hat{R}(0)}{(2\pi)^d}. \tag{7.42}$$

For $k \geq 3$, we have

$$\begin{aligned}\frac{\int_{\mathbb{R}^d}|f_\lambda(x)|^kdx}{|\log\lambda|^{\frac{k}{2}}} \leq & C^k\frac{1}{|\log\lambda|^{\frac{k}{2}}}\int_{\mathbb{R}^{kd}}\prod_{i=1}^k\frac{|\mathcal{F}\{|\varphi|\}(\xi_i)|}{\lambda+\frac{1}{2}|\xi_i|^2}\delta(\sum_{i=1}^k\xi_i)d\xi \\ = & C^k\frac{1}{|\log\lambda|^{\frac{k}{2}}}\int_{\mathbb{R}^{(k-1)d}}\prod_{i=1}^{k-1}\frac{|\mathcal{F}\{|\varphi|\}(\xi_i)|}{\lambda+\frac{1}{2}|\xi_i|^2}\frac{|\mathcal{F}\{|\varphi|\}(-\xi_1-\ldots-\xi_{k-1})|}{\lambda+\frac{1}{2}|\xi_1+\ldots+\xi_{k-1}|^2}d\xi\end{aligned} \tag{7.43}$$

for some constant C. Since φ is continuous and compactly supported, we can assume here $|\mathcal{F}\{|\varphi|\}|$ to be bounded, fast-decaying, radially symmetric and decreasing. Then for the integration in ξ_{k-1}, by Lemma A.4, we obtain

$$\int_{\mathbb{R}^d}\frac{|\mathcal{F}\{|\varphi|\}(\xi_{k-1})|}{\lambda+\frac{1}{2}|\xi_{k-1}|^2}\frac{|\mathcal{F}\{|\varphi|\}(-\xi_1-\ldots-\xi_{k-1})|}{\lambda+\frac{1}{2}|\xi_1+\ldots+\xi_{k-1}|^2}d\xi_{k-1} \leq \int_{\mathbb{R}^d}\left(\frac{\mathcal{F}\{|\varphi|\}(\xi_{k-1})}{\lambda+\frac{1}{2}|\xi_{k-1}|^2}\right)^2 d\xi_{k-1}. \tag{7.44}$$

The r.h.s. of the above display is of the form $\langle \Phi_\lambda, \Phi_\lambda \rangle$ with $\hat{R}(\xi)$ replaced by $|\mathcal{F}\{|\varphi|\}(\xi_{k-1})|^2$, so by Lemma A.10, we obtain

$$\frac{\int_{\mathbb{R}^d} |f_\lambda(x)|^k dx}{|\log \lambda|^{\frac{k}{2}}} \leq C^k \frac{1}{|\log \lambda|^{\frac{k-2}{2}}} \int_{\mathbb{R}^{(k-2)d}} \prod_{i=1}^{k-2} \frac{\mathcal{F}\{|\varphi|\}(\xi_i)}{\lambda + \frac{1}{2}|\xi_i|^2} d\xi \leq \frac{C^k}{|\log \lambda|^{\frac{k-2}{2}}} \tag{7.45}$$

for some possibly different constant $C > 0$. This leads to

$$\int_{\mathbb{R}^d} \sum_{k \geq 3} \frac{1}{k!} (|\theta| |\log \lambda|^{-\frac{1}{2}} |f_\lambda(x)|)^k dx \leq \sum_{k \geq 3} \frac{C^k |\theta|^k}{k! |\log \lambda|^{\frac{k-2}{2}}} \to 0 \tag{7.46}$$

as $\lambda \to 0$. The proof is complete. $\square$

Acknowledgment

The authors would like to thank the anonymous referees for their careful reading of the manuscript. The first author would like to thank Jean-Christophe Mourrat for many helpful discussions, in particular for sharing the ideas of proving Lemma 7.3 and applying Kipnis-Varadhan's method to the difference between quadratic variations in Section 7.2. This paper was partially funded by AFOSR Grant NSSEFF-FA9550-10-1-0194 and NSF grant DMS-1108608.

A Technical lemmas

Lemma A.1. *If $h_1 \in L^1, h_2, h_3 \in L^1 \cap L^2$, then*

$$\begin{aligned}
&\mathbb{E}\{e^{i\int_{\mathbb{R}^d} h_1(y)\omega(dy)} \int_{\mathbb{R}^d} h_2(y)\omega(dy) \int_{\mathbb{R}^d} h_3(y)\omega(dy)\} \\
=& \exp(\int_{\mathbb{R}^d} (e^{ih_1(y)} - 1)dy) \left(\int_{\mathbb{R}^d} e^{ih_1(y)} h_2(y)h_3(y)dy + \int_{\mathbb{R}^d} e^{ih_1(y)} h_2(y)dy \int_{\mathbb{R}^d} e^{ih_1(y)} h_3(y)dy \right).
\end{aligned} \tag{A.1}$$

Proof. If h_i are all compactly supported, it is a direct calculation. The general case can be proved by approximation. $\square$

Lemma A.2.

$$\exp(\int_{\mathbb{R}^d} (e^{i\varepsilon \int_0^{t/\varepsilon^2} \varphi(B_s - y)ds} - 1 - i\frac{c_\varphi t}{\varepsilon})dy) \quad \to \quad e^{-\frac{1}{2}\sigma^2 t}, \tag{A.2}$$

$$\exp(\int_{\mathbb{R}^d} (e^{i\varepsilon \int_0^{t/\varepsilon^2} (\varphi(B_s - y) - \varphi(W_s - y))ds} - 1)dy) \quad \to \quad e^{-\sigma^2 t} \tag{A.3}$$

in probability as $\varepsilon \to 0$.

Proof. We first point out that (A.2) and (A.3) is related to the convergence of the annealed characteristic function for random variables of the form $\varepsilon\int_0^{t/\varepsilon^2} V(B_s)ds$ and $\varepsilon\int_0^{t/\varepsilon^2}(V(B_s)-V(W_s))ds$, respectively, when V is Poissonian. This is discussed in detail in [15]. By [15, Proposition 3.7, 3.8], we have that as $\varepsilon\to 0$

$$\int_{\mathbb{R}^d}\left(\varepsilon\int_0^{t/\varepsilon^2}\varphi(B_s-y)ds\right)^2 dy\to\sigma^2 t, \tag{A.4}$$

$$\int_{\mathbb{R}^d}\sum_{k\geq 3}\frac{1}{k!}\left(\varepsilon\int_0^{t/\varepsilon^2}|\varphi|(B_s-y)ds\right)^k dy\to 0 \tag{A.5}$$

in probability, which directly leads to (A.2) if we expand e^{ix} in power series and use the fact $\int_{\mathbb{R}^d}\varphi(x)dx=c_\varphi$.

For (A.3), we use the fact that $|a+b|^k\leq 2^{k-1}(|a|^k+|b|^k)$ together with (A.5) to derive that

$$\int_{\mathbb{R}^d}\sum_{k\geq 3}\frac{1}{k!}\left(i\varepsilon\int_0^{t/\varepsilon^2}(\varphi(B_s-y)-\varphi(W_s-y))ds\right)^k dy\to 0$$

in probability. The rest is to show

$$\varepsilon^2\int_0^{t/\varepsilon^2}\int_0^{t/\varepsilon^2}\varphi(B_s-y)\varphi(W_u-y)dsdu=\varepsilon^2\int_0^{t/\varepsilon^2}\int_0^{t/\varepsilon^2}R(B_s-W_u)dsdu\to 0$$

in probability. Assuming R is positive without loss of generality, we have in Fourier domain that

$$\begin{aligned}\mathbb{E}\{\varepsilon^2\int_0^{t/\varepsilon^2}\int_0^{t/\varepsilon^2}R(B_s-W_u)dsdu\}\lesssim&\varepsilon^2\int_0^{t/\varepsilon^2}\int_0^{t/\varepsilon^2}\int_{\mathbb{R}^d}\hat{R}(\xi)e^{-\frac{1}{2}|\xi|^2(s+u)}d\xi dsdu\\ \lesssim&\int_{\mathbb{R}^d}\frac{\hat{R}(\xi)}{|\xi|^2}\frac{\varepsilon^2}{|\xi|^2}(1-e^{-\frac{1}{2}|\xi|^2\frac{t}{\varepsilon^2}})d\xi\to 0\end{aligned}$$

by the dominated convergence theorem, which completes the proof. □

Lemma A.3.

$$\varepsilon^2\int_0^{t/\varepsilon^2}\int_0^{t/\varepsilon^2}R(B_s-B_u)dsdu \;\to\; \sigma^2 t \tag{A.6}$$

$$\varepsilon^2\int_0^{t/\varepsilon^2}\int_0^{t/\varepsilon^2}R(B_s-W_u)dsdu \;\to\; 0 \tag{A.7}$$

in probability as $\varepsilon\to 0$.

Proof. We note that $\varepsilon^2\int_0^{t/\varepsilon^2}\int_0^{t/\varepsilon^2}R(B_s-B_u)dsdu$ is the variance of $\varepsilon\int_0^{t/\varepsilon^2}V(B_s)ds$. By [15, Proposition 3.7], we obtain (A.6). The proof of (A.7) is contained in the proof of Lemma A.2. □

Lemma A.4. *Assume f is positive, radially symmetric and decreasing, and integrable around the origin, g is bounded, integrable, positive, radially symmetric and decreasing, the $f\star g$ is bounded, radially symmetric and decreasing.*

Proof. Clearly, $f \star g$ is bounded and radially symmetric, so we only need to prove it is radially decreasing.

By Fubini theorem and symmetry, it can be reduced to the one-dimensional case. Let $F(x) = \int_{\mathbb{R}} f(x+y)g(y)dy$, and by approximation, we assume f is smooth and bounded. So $F'(x) = \int_{\mathbb{R}} f'(x+y)g(y)dy = \int_{\mathbb{R}} f'(y)g(y-x)dy$, which implies

$$F'(x) = \int_{-\infty}^{0} f'(y)(g(x-y) - g(x+y))dy.$$

When $x > 0, y < 0$, we have $x - y = |x| + |y| \geq |x+y|$, so $g(x-y) \leq g(x+y)$, and since $f'(y) \geq 0$, we have $F'(x) \leq 0$. The proof is complete. □

Lemma A.5.

$$\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f^\lambda|(x+y)}{|x|^{d-2}} dx \lesssim \varepsilon^{-1}, \tag{A.8}$$

$$\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f_k^\lambda|(x+y)}{|x|^{d-2}} dx \lesssim |\log\varepsilon|, \tag{A.9}$$

$$\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(|f^\lambda| \star |\varphi|)(x+y)}{|x|^{d-2}} dx \lesssim \varepsilon^{-1}, \tag{A.10}$$

$$\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(|f_k^\lambda| \star |\varphi|)(x+y)}{|x|^{d-2}} dx \lesssim |\log\varepsilon|. \tag{A.11}$$

Proof. Recall that $f^\lambda = \varphi \star G_\lambda, f_k^\lambda = \varphi \star \partial_{x_k} G_\lambda$, and G_λ is the Green's function of $\lambda - \frac{1}{2}\Delta$, so $|G_\lambda(x)| \lesssim e^{-c\sqrt{\lambda}|x|}|x|^{2-d}$, $|\partial_{x_k} G_\lambda(x)| \lesssim e^{-c\sqrt{\lambda}|x|}|x|^{1-d}$ for some constant $c > 0$. Without loss of generality, we assume $|\varphi|$ is bounded, radially symmetry and decreasing function with compact support, and replace $G_\lambda, \partial_{x_k} G_\lambda$ by the above bounds in the estimates.

We take $\int_{\mathbb{R}^d} |f^\lambda|(x+y)|x|^{2-d}dx$ for example. The proof of the other inequalities is similar.

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{|f^\lambda(x+y)|}{|x|^{d-2}} dx &\lesssim \int_{\mathbb{R}^d} \frac{1}{|x|^{d-2}} \int_{\mathbb{R}^d} |\varphi|(x+y-z) \frac{e^{-c\sqrt{\lambda}|z|}}{|z|^{d-2}} dzdx \\ &\leq \int_{\mathbb{R}^{2d}} \frac{|\varphi|(x-z)}{|x|^{d-2}} \frac{e^{-c\sqrt{\lambda}|z|}}{|z|^{d-2}} dzdx\end{aligned}$$

by Lemma A.4, since $|\varphi| \star (e^{-c\sqrt{\lambda}|x|}|x|^{2-d})$ is a bounded, integrable, radially symmetric and decreasing function again by Lemma A.4. Now we assume $|\varphi|(x) \lesssim 1 \wedge |x|^{-\alpha}$ for some $\alpha > 0$ sufficiently large, and bound the integral in z by

$$\int_{\mathbb{R}^d} \frac{e^{-c\sqrt{\lambda}|z|}}{|z|^{d-2}} \left(1 \wedge \frac{1}{|x-z|^\alpha}\right) dz \lesssim 1 \wedge \left(\frac{1}{|x|^{\alpha-2}} + \frac{e^{-\rho\sqrt{\lambda}|x|}}{|x|^{d-2}}\right)$$

for some constant $\rho > 0$ [16, Lemma A.3]. The rest is a straightforward calculation. □

Lemma A.6. *Assume f, g are bounded, integrable, positive, and radially symmetric and decreasing, then we have*

$$\mathbb{E}_B\mathbb{E}_W \int_{[0,t/\varepsilon^2]^2} f(x_s - \tilde{B})g(y_u - \tilde{W})dsdu \lesssim \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f(x+y)}{|x|^{d-2}}dx \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{g(x+y)}{|x|^{d-2}}dx \tag{A.12}$$

where $x, y \in \{B, W\}$, and $\tilde{B}, \tilde{W} \in \{0, B_{t/\varepsilon^2}, W_{t/\varepsilon^2}\}$.

Proof. The proofs for all choices of $x_s, y_u, \tilde{B}, \tilde{W}$ are similar. We take one example that contains all the ingredients.

Let $x_s = B_s, y_u = B_u, \tilde{B} = B_{t/\varepsilon^2}, \tilde{W} = W_{t/\varepsilon^2}$, and we consider

$$\mathbb{E}_B\mathbb{E}_W \int_{[0,t/\varepsilon^2]^2} f(B_s - B_{t/\varepsilon^2})g(B_u - W_{t/\varepsilon^2})dsdu = (i) + (ii),$$

where

$$\begin{aligned}(i) &= \mathbb{E}_B\mathbb{E}_W \int_{0<s<u<t/\varepsilon^2} f(B_s - B_{t/\varepsilon^2})g(B_u - W_{t/\varepsilon^2})dsdu,\\ (ii) &= \mathbb{E}_B\mathbb{E}_W \int_{0<u<s<t/\varepsilon^2} f(B_s - B_{t/\varepsilon^2})g(B_u - W_{t/\varepsilon^2})dsdu.\end{aligned}$$

For (i), by change of variables, we have

$$(i) = \mathbb{E}_W \int_{\mathbb{R}_+^2} \int_{\mathbb{R}^{3d}} 1_{u_1+u_2<t/\varepsilon^2} f(x+y)g(z+x-W_{t/\varepsilon^2})q_{u_1}(z)q_{u_2}(y)q_{t/\varepsilon^2-u_1-u_2}(x)dxdydzdu_1du_2.$$

For the integrals in y, z, by Lemma A.4, we have

$$\begin{aligned}(i) \leq &\mathbb{E}_W \int_{\mathbb{R}_+^2} \int_{\mathbb{R}^{3d}} 1_{u_1+u_2<t/\varepsilon^2} f(y)g(z)q_{u_1}(z)q_{u_2}(y)q_{t/\varepsilon^2-u_1-u_2}(x)dxdydzdu_1du_2\\ = &\int_{\mathbb{R}_+^2} \int_{\mathbb{R}^{3d}} 1_{u_1+u_2<t/\varepsilon^2} f(y)g(z)q_{u_1}(z)q_{u_2}(y)dydzdu_1du_2.\end{aligned}$$

By change of variables $\lambda_1 = -\frac{|z|^2}{2u_1}, \lambda_2 = -\frac{|y|^2}{2u_2}$, we have

$$(i) \lesssim \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f(x+y)}{|x|^{d-2}}dx \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{g(x+y)}{|x|^{d-2}}dx.$$

For (ii), by change of variables, we have

$$(ii) = \mathbb{E}_W \int_{\mathbb{R}_+^2} \int_{\mathbb{R}^{3d}} 1_{u_1+u_2<t/\varepsilon^2} f(y)g(x-W_{t/\varepsilon^2})q_{u_1}(y)q_{u_2}(x)dxdydu_1du_2,$$

and the rest is the same. The proof is complete. □

Lemma A.7. *Assume f, g are bounded, integrable, positive, and radially symmetric and decreasing, then we have*

$$\mathbb{E}_B\mathbb{E}_W \int_{[0,t/\varepsilon^2]^3} f(x_{s_1} - B_{s_2})g(y_{s_3} - \tilde{W})ds \lesssim \frac{1}{\varepsilon^2} \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f(x+y)}{|x|^{d-2}}dx \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{g(x+y)}{|x|^{d-2}}dx, \quad \text{(A.13)}$$

where $x, y \in \{B, W\}, \tilde{W} \in \{0, B_{t/\varepsilon^2}, W_{t/\varepsilon^2}\}$.

Proof. The proof is similar to that of Lemma A.6. We do not present the details here. □

Lemma A.8. *Assume f, g are bounded, integrable, positive, and radially symmetric and decreasing, then we have*

$$\mathbb{E}_B\mathbb{E}_W \int_{[0,t/\varepsilon^2]^4} f(x_{s_1} - B_{u_1})g(x_{s_2} - B_{u_2})dsdu \lesssim \frac{1}{\varepsilon^4} \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f(x+y)}{|x|^{d-2}}dx \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{g(x+y)}{|x|^{d-2}}dx, \quad \text{(A.14)}$$

where $x \in \{B, W\}$.

Proof. The proof is similar to that of Lemma A.6. We do not present the details here. □

Lemma A.9. *When $d = 4$ and $s > 0$,*

$$\int_{\mathbb{R}^d} \frac{e^{-|\xi|^2 s}}{(1+\frac{1}{2}|\xi|^2)^2}d\xi \lesssim 1 + |\log s|. \quad \text{(A.15)}$$

Proof. By a change of coordinate,

$$\int_{\mathbb{R}^d} \frac{e^{-|\xi|^2 s}}{(1+\frac{1}{2}|\xi|^2)^2}d\xi \lesssim \int_0^\infty \frac{e^{-r^2 s}r^3}{(1+r^2)^2}dr \lesssim 1 + \int_1^\infty \frac{e^{-r^2 s}}{r}dr \lesssim 1 + |\log s|. \quad \text{(A.16)}$$

□

Lemma A.10. *When $d = 4$, $|\log\lambda|^{-1}\langle\Phi_\lambda, \Phi_\lambda\rangle \to 2(2\pi)^{-d}\hat{R}(0)$.*

Proof. First we consider the following integral

$$\frac{1}{|\log\lambda|}\int_0^{\frac{1}{\sqrt{\lambda}}} \frac{f(\sqrt{\lambda}r)r^3}{(1+\frac{1}{2}r^2)^2}dr \quad \text{(A.17)}$$

for some smooth and fast-decaying f. By an integration by parts, we have

$$\frac{1}{|\log\lambda|}\int_0^{\frac{1}{\sqrt{\lambda}}} \frac{f(\sqrt{\lambda}r)r^3}{(1+\frac{1}{2}r^2)^2}dr \to 2f(0) \quad \text{(A.18)}$$

as $\lambda \to 0$.

Secondly, we have

$$\begin{aligned}\frac{\langle \Phi_\lambda, \Phi_\lambda \rangle}{|\log \lambda|} =& \frac{1}{(2\pi)^d |\log \lambda|} \int_{\mathbb{R}^d} \frac{\hat{R}(\xi)}{(\lambda + \frac{1}{2}|\xi|^2)^2} d\xi \\ =& \frac{1}{(2\pi)^d |\log \lambda|} \left(\int_{|\xi|>1} + \int_{|\xi|\le 1} \right) \frac{\hat{R}(\xi)}{(\lambda + \frac{1}{2}|\xi|^2)^2} d\xi.\end{aligned} \tag{A.19}$$

Clearly the first part goes to zero. For the second part, by a change of variables, we obtain

$$\frac{1}{(2\pi)^d |\log \lambda|} \int_{|\xi|\le 1} \frac{\hat{R}(\xi)}{(\lambda + \frac{1}{2}|\xi|^2)^2} d\xi = \frac{1}{(2\pi)^d |\log \lambda|} \int_{|\xi|\le 1/\sqrt{\lambda}} \frac{\hat{R}(\sqrt{\lambda}\xi)}{(1 + \frac{1}{2}|\xi|^2)^2} d\xi. \tag{A.20}$$

Since $R(x)$ decays sufficiently fast, $\hat{R}(\xi)$ is smooth, so by (A.18), the proof is complete. □

B Proof of Lemma 5.4, 5.6, 5.7, 5.9, 5.10

Proof of Lemma 5.4. For the indexes satisfying $m_1 + m_2 \ge 1, m_3 + m_4 \ge 1$, there are the following four cases.

1. $m_1 m_3 \neq 0$.
2. $m_2 m_4 \neq 0$.
3. $m_2 = m_3 = 0$.
4. $m_1 = m_4 = 0$.

If $m_1 m_3 \neq 0$, we first consider the expectation in W. For any permutation of $\{u_1, \ldots, u_{m_2+m_4}, \tilde{u}\}$, denoted by $\mathcal{S}$, we have

$$\begin{aligned}&\int_{\mathcal{S}} \mathbb{E}_W \{ \prod_{i=1}^{m_2} |\varphi|(W_{u_i} - y) \prod_{i=m_2+1}^{m_2+m_4} |\varphi|(W_{u_i} - z) |g_2|(W_{\tilde{u}} - z) \} du d\tilde{u} \\ =& \int_{0 \le u_1 \ldots \le u_{m_2+m_4+1} \le t/\varepsilon^2} \prod_{i=1}^{m_2+m_4+1} \mathbb{E}_W \{ |\varphi_i|(W_{u_i} - y_i) \} du\end{aligned} \tag{B.1}$$

for φ_i chosen as φ, g_2, and y_i chosen as y, z depending on the permutation. By a standard change of variables, we have

$$
\begin{aligned}
&\int_{0\le u_1\ldots\le u_{m_2+m_4+1}\le t/\varepsilon^2} \prod_{i=1}^{m_2+m_4+1} \mathbb{E}_W\{|\varphi_i|(W_{u_i}-y_i)\}du \\
=&\int_{\mathbb{R}_+^{m_2+m_4+1}}\int_{\mathbb{R}^{(m_2+m_4+1)d}} 1_{\sum_{i=1}^{m_2+m_4+1} u_i\le \frac{t}{\varepsilon^2}} \prod_{i=1}^{m_2+m_4+1} |\varphi_i|(\sum_{j=1}^i x_j - y_i) q_{u_i}(x_i) dxdu \\
\lesssim&\int_{\mathbb{R}_+^{m_2+m_4+1}}\int_{\mathbb{R}^{(m_2+m_4+1)d}} 1_{\sum_{i=1}^{m_2+m_4+1} \frac{|x_i|^2}{2\lambda_i}\le \frac{t}{\varepsilon^2}} \prod_{i=1}^{m_2+m_4+1} |\varphi_i|(\sum_{j=1}^i x_j - y_i) \frac{1}{|x_i|^{d-2}}\lambda_i^{\frac{d}{2}-2}e^{-\lambda_i}dxd\lambda \\
\lesssim&\int_{\mathbb{R}_+^{m_2+m_4+1}}\int_{\mathbb{R}^{(m_2+m_4+1)d}} \prod_{i=1}^{m_2+m_4+1} |\varphi_i|(\sum_{j=1}^i x_j - y_i) \frac{1}{|x_i|^{d-2}}\lambda_i^{\frac{d}{2}-2}e^{-\lambda_i}dxd\lambda.
\end{aligned}
\tag{B.2}
$$

First integrate in $\lambda_i, i=1,\ldots,m_2+m_4+1$, then in $x_{m_2+m_4+1},\ldots,x_1$, since $\int_{\mathbb{R}^d}|\varphi|(x+y)|x|^{2-d}dx$ is uniformly bounded in y, we have

$$
\int_{\mathcal{S}} \mathbb{E}_W\{\prod_{i=1}^{m_2}|\varphi|(W_{u_i}-y)\prod_{i=m_2+1}^{m_2+m_4}|\varphi|(W_{u_i}-z)|g_2|(W_{\tilde u}-z)\}dud\tilde u \lesssim \sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|g_2|(x+y)|}{|x|^{d-2}}dx, \tag{B.3}
$$

which leads to

$$
\begin{aligned}
&\int_{[0,t/\varepsilon^2]^{m_2+m_4+1}} \mathbb{E}_W\{\prod_{i=1}^{m_2}|\varphi|(W_{u_i}-y)\prod_{i=m_2+1}^{m_2+m_4}|\varphi|(W_{u_i}-z)|g_2|(W_{\tilde u}-z)\}dud\tilde u \\
\le& C^{m_2+m_4+1}(m_2+m_4+1)!\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|g_2|(x+y)|}{|x|^{d-2}}dx
\end{aligned}
\tag{B.4}
$$

for some constant C.

Next, we consider the expectation in B. The analyze is similar except that we have to deal with integration in y, z. Again for any permutation of $\{s_1,\ldots,s_{m_1+m_3},\tilde s\}$, denoted by $\mathcal{S}$, we consider

$$
\begin{aligned}
&\int_{\mathbb{R}^{2d}}\int_{\mathcal{S}}\prod_{i=1}^{m_1}|\varphi|(B_{s_i}-y)\prod_{i=m_1+1}^{m_1+m_3}|\varphi|(B_{s_i}-z)|g_1|(B_{\tilde s}-y)dsd\tilde s dydz \\
=&\int_{\mathbb{R}^{2d}}\int_{0\le s_1\ldots\le s_{m_1+m_3+1}\le t/\varepsilon^2}\prod_{i=1}^{m_1+m_3+1}\mathbb{E}_B\{|\varphi_i|(B_{s_i}-y_i)\}dsdydz \\
=&\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}_+^{m_1+m_3+1}}\int_{\mathbb{R}^{(m_1+m_3+1)d}} 1_{\sum_{i=1}^{m_1+m_3+1} u_i\le\frac{t}{\varepsilon^2}}\prod_{i=1}^{m_1+m_3+1}|\varphi_i|(\sum_{j=1}^i x_j-y_i)q_{u_i}(x_i)dxdudydz,
\end{aligned}
\tag{B.5}
$$

where φ_i is either φ or g_1 and y_i is either y or z depending on the permutation. Let i_y, i_z be the smallest indexes such that $y_{i_y}=y$ and $y_{i_z}=z$. By the same change of variables $\lambda_i=\frac{|x_i|^2}{2u_i}$ for

$i \neq i_y, i_z$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^{2d}} \int_{\mathcal{S}} \prod_{i=1}^{m_1} |\varphi|(B_{s_i} - y) \prod_{i=m_1+1}^{m_1+m_3} |\varphi|(B_{s_i} - z) |g_1|(B_{\tilde{s}} - y) ds d\tilde{s} dy dz \\
\lesssim &\int_{\mathbb{R}^{2d}} \int_{\mathbb{R}_+^{m_1+m_3+1}} \int_{\mathbb{R}^{(m_1+m_3+1)d}} 1_{u_{i_y}+u_{i_z} \leq \frac{t}{\varepsilon^2}} \left(\prod_{i=1}^{m_1+m_3+1} |\varphi_i|(\sum_{j=1}^{i} x_j - y_i) \right) \\
&\left(\prod_{i \neq i_y, i_z} \frac{1}{|x_i|^{d-2}} \lambda_i^{\frac{d}{2}-2} e^{-\lambda_i} \right) q_{u_{i_y}}(x_{i_y}) q_{u_{i_z}}(x_{i_z}) dx du d\lambda dy dz.
\end{aligned}
\tag{B.6}
$$

Let $\tilde{i}_y$ be the second smallest index such that $y_{\tilde{i}_y} = y$. The following is the order in which we integrate with respect to x, u, λ, y, z in (B.6). It ensures that the integral of $|g_1|$ always contains a factor of $1/|x|^{d-2}$.

First integrate in λ_i, then integrate in $x_{m_1+m+3+1}, \ldots, x_{\max(\tilde{i}_y, i_z)+1}$.

If $i_z > \tilde{i}_y$, for $|\varphi_{i_z}|(\sum_{j=1}^{i_z} x_j - z)$, we integrate in z; next, we integrate in $x_{i_z}, \ldots, x_{\tilde{i}_y+1}$. Since $i_z > \tilde{i}_y$, we have $i_y = 1, \tilde{i}_y = 2$. So we are left with $|\varphi_1|(x_1 - y)|\varphi_2|(x_1 + x_2 - y)|x_2|^{2-d} q_{u_1}(x_1)$, integrate in y, x_2, x_1. In the end, we integrate in u_{i_y}, u_{i_z}.

If $i_z < \tilde{i}_y$, for $|\varphi_{i_y}|(\sum_{j=1}^{i_y} x_j - y)|\varphi_{\tilde{i}_y}|(\sum_{j=1}^{\tilde{i}_y} x_j - y)$, integrate in $y, x_{\tilde{i}_y}$, then integrate in $x_{\tilde{i}_y-1}, \ldots, x_{i_z+1}$; for $|\varphi_{i_z}|(\sum_{j=1}^{i_z} x_j - z)$, integrate in z. Since $i_z = 1$ or 2, we integrate in x_1, and in the end, integrate in u_{i_y}, u_{i_z}.

After the above integration, and using the fact that $\int_{\mathbb{R}^d} |\varphi|(x+y)|x|^{2-d} dx$ is uniformly bounded in y, we arrive at the following estimate

$$
\begin{aligned}
&\mathbb{E}_B \int_{\mathbb{R}^{2d}} \int_{\mathcal{S}} \prod_{i=1}^{m_1} |\varphi|(B_{s_i} - y) \prod_{i=m_1+1}^{m_1+m_3} |\varphi|(B_{s_i} - z) |g_1|(B_{\tilde{s}} - y) ds d\tilde{s} dy dz \\
\lesssim &\frac{1}{\varepsilon^4} \max(\sup_y \int_{\mathbb{R}^d} \frac{|g_1|(x+y)}{|x|^{d-2}} dx, \sup_y \int_{\mathbb{R}^d} \frac{(|g_1| \star |\varphi|)(x+y)}{|x|^{d-2}} dx),
\end{aligned}
\tag{B.7}
$$

where the factor of ε^{-4} comes from integration in u_{i_y}, u_{i_z}.

Therefore,

$$
\begin{aligned}
&\mathbb{E}_B \int_{\mathbb{R}^{2d}} \int_{[0,t/\varepsilon^2]^{m_1+m_3+1}} \prod_{i=1}^{m_1} |\varphi|(B_{s_i} - y) \prod_{i=m_1+1}^{m_1+m_3} |\varphi|(B_{s_i} - z) |g_1|(B_{\tilde{s}} - y) ds d\tilde{s} dy dz \\
\leq &C^{m_1+m_3+1} (m_1 + m_3 + 1)! \frac{1}{\varepsilon^4} \max(\sup_y \int_{\mathbb{R}^d} \frac{|g_1|(x+y)}{|x|^{d-2}} dx, \sup_y \int_{\mathbb{R}^d} \frac{(|g_1| \star |\varphi|)(x+y)}{|x|^{d-2}} dx)
\end{aligned}
\tag{B.8}
$$

for some constant C.

Now we only need to combine (B.4) and (B.8) together with Lemma A.5 to complete the proof for the case $m_1 m_3 \neq 0$.

If $m_2 m_4 \neq 0$, the discussion is the same by symmetry.

If $m_2 = m_3 = 0$, the discussion is similar except that when taking $\mathbb{E}_W, \mathbb{E}_B$, we have to deal with the integral in z, y respectively.

In the end, we deal with the case when $m_1 = m_4 = 0$, so $m_2 \geq 1, m_3 \geq 1$.

We first look at the case when $m_2 = m_3 = 1$, i.e.,

$$\begin{aligned}
&\mathbb{E}_B\mathbb{E}_W \int_{\mathbb{R}^{2d}} \int_{[0,t/\varepsilon^2]^4} |\varphi|(B_s - z)|g_1|(B_{\tilde{s}} - y)|\varphi|(W_u - y)|g_2|(W_{\tilde{u}} - z) ds du d\tilde{s} d\tilde{u} dy dz \\
=&\mathbb{E}_B\mathbb{E}_W \int_{[0,t/\varepsilon^2]^4} (|g_1| \star |\varphi|)(B_{\tilde{s}} - W_u)(|g_2| \star |\varphi|)(W_{\tilde{u}} - B_s) ds du d\tilde{s} d\tilde{u} \\
\lesssim&\frac{1}{\varepsilon^4} \sup_y \int_{\mathbb{R}^d} \frac{(|g_1| \star |\varphi|)(x+y)}{|x|^{d-2}} dx \sup_y \int_{\mathbb{R}^d} \frac{(|g_2| \star |\varphi|)(x+y)}{|x|^{d-2}} dx \\
\lesssim&\frac{|\log \varepsilon|^2}{\varepsilon}
\end{aligned} \tag{B.9}$$

by Lemma A.8 and A.5.

Next, we look at the case when $m_2 + m_3 \geq 3$. By symmetry, we assume $m_2 \geq 2$. Consider $\mathbb{E}_B$ and dz, by similar discussion as before, we obtain that

$$\begin{aligned}
&\mathbb{E}_B \int_{\mathbb{R}^d} \int_{[0,t/\varepsilon^2]^{m_3+2}} \prod_{i=1}^{m_3} |\varphi|(B_{s_i} - z)|g_1|(B_{\tilde{s}} - y)|g_2|(W_{\tilde{u}} - z) ds d\tilde{s} d\tilde{u} dz \\
\lesssim&(m_3+1)! \frac{1}{\varepsilon^2} \sup_y \int_{\mathbb{R}^d} \frac{(|g_2| \star |\varphi|)(x+y)}{|x|^{d-2}} dx \sup_y \int_{\mathbb{R}^d} \frac{|g_1|(x+y)}{|x|^{d-2}} dx.
\end{aligned} \tag{B.10}$$

Consider $\mathbb{E}_W$ and dy, we obtain that

$$\mathbb{E}_W \int_{\mathbb{R}^d} \int_{[0,t/\varepsilon^2]^{m_2}} \prod_{i=1}^{m_2} |\varphi|(W_{u_i} - y) du dy \lesssim m_2! \frac{1}{\varepsilon^2}. \tag{B.11}$$

Combining them with Lemma A.5, the proof is complete. □

Proof of Lemma 5.6.

First, we note that f^λ is uniformly bounded, since $\mathcal{F}\{f^\lambda\}(\xi) = \mathcal{F}\{\varphi\}(\xi)(\lambda + \frac{1}{2}|\xi|^2)^{-1}$ is bounded in L^1.

Similarly, there are four cases.

If $m_1 m_3 \neq 0$, we use a constant to bound f^λ, and the rest of the discussion is similar to the proof of Lemma 5.4; i.e., first take $\mathbb{E}_W$, then take $\mathbb{E}_B$ while dealing with integrals in y, z. We get the following estimate

$$\begin{aligned}
&\mathbb{E}_B\mathbb{E}_W \int_{\mathbb{R}^{2d}} \int_{[0,t/\varepsilon^2]^{N(m_i)}} \prod_{i=1}^{m_1} |\varphi|(B_{s_i} - y) \prod_{i=m_1+1}^{m_1+m_3} |\varphi|(B_{s_i} - z)|f^\lambda|(\tilde{B} - y) \\
&\qquad \prod_{i=1}^{m_2} |\varphi|(W_{u_i} - y) \prod_{i=m_2+1}^{m_2+m_4} |\varphi|(W_{u_i} - z)|f^\lambda|(\tilde{W} - z) ds du dy dz \\
\lesssim&(m_1+m_3)!(m_2+m_4)! \frac{1}{\varepsilon^4}
\end{aligned} \tag{B.12}$$

If $m_2m_4 \neq 0$, by symmetry, we get the same estimate as in (B.12).

If $m_2 = m_3 = 0$ or $m_1 = m_4 = 0$, again we bound $|f^\lambda|$ by constant, and when taking $\mathbb{E}_W, \mathbb{E}_B$, deal with the integral in z, y respectively. In the end, we get the same estimate as in (B.12), which completes the proof. □

Proof of Lemma 5.7.

If $\sum_{i=1}^4 m_i = 2$, we have

$$\begin{aligned}&\int_{\mathbb{R}^{2d}} g_B(y)^{m_1} g_W(y)^{m_2} g_B(z)^{m_3} g_W(z)^{m_4} h_B(y)\overline{h_W}(z)dydz\\ =&\varepsilon^3 \int_{[0,t/\varepsilon^2]^2} F^\lambda(x_s - \tilde{B})F^\lambda(y_u - \tilde{W})dsdu,\end{aligned} \tag{B.13}$$

where $F^\lambda(x) = \int_{\mathbb{R}^d} \varphi(x+y)f^\lambda(y)dy$, and $x, y \in \{B, W\}$, $\tilde{B} \in \{0, B_{t/\varepsilon^2}\}$, $\tilde{W} \in \{0, W_{t/\varepsilon^2}\}$. Note that $|F^\lambda|(x) \leq |f^\lambda| \star |\varphi|(-x)$ since $|\varphi|$ is symmetric. In the following, we will always replace $|f^\lambda|$ by $|\varphi| \star (e^{-c\sqrt{\lambda}|x|}|x|^{2-d})$ in the estimates, so we can assume it is radially symmetric. By Lemma A.6, we have that

$$\begin{aligned}\mathbb{E}_B\mathbb{E}_W|\varepsilon^3 \int_{[0,t/\varepsilon^2]^2} F^\lambda(x_s - \tilde{B})F^\lambda(y_u - \tilde{W})dsdu| \lesssim&\varepsilon^3 \left(\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(|f^\lambda| \star |\varphi|)(x+y)}{|x|^{d-2}}dx\right)^2\\ \lesssim&\varepsilon,\end{aligned} \tag{B.14}$$

where the last inequality comes from Lemma A.5.

If $\sum_{i=1}^4 m_i = 3$ and $m_i \neq 2$ for all i, without loss of generality assume $m_1 = 0$, so we have

$$\begin{aligned}&\mathbb{E}_B\mathbb{E}_W\varepsilon^4 \int_{\mathbb{R}^{2d}} \int_{[0,t/\varepsilon^2]^3} |\varphi|(B_{s_1} - z)|f^\lambda|(\tilde{B} - y)|\varphi|(W_{u_1} - y)|\varphi|(W_{u_2} - z)|f^\lambda|(\tilde{W} - z)dsdudydz\\ \lesssim&\mathbb{E}_B\mathbb{E}_W\varepsilon^4 \int_{[0,t/\varepsilon^2]^3} (|\varphi| \star |\varphi|)(B_{s_1} - W_{u_2})(|f^\lambda| \star |\varphi|)(\tilde{B} - W_{u_1})dsdu\\ \lesssim&\varepsilon^4 \frac{1}{\varepsilon^2} \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(|f^\lambda| \star |\varphi|)(x+y)}{|x|^{d-2}}dx \lesssim \varepsilon,\end{aligned} \tag{B.15}$$

where in the first inequality, we bound $|f^\lambda|(\tilde{W} - z)$ by a constant, while in the second and third inequalities, we apply Lemma A.7 and A.5.

If $\sum_{i=1}^4 m_i = 3$ and $m_i = 2$ for some i, there are two cases by symmetry, $m_1 = 1, m_3 = 2$ or $m_1 = 1, m_4 = 2$.

When $m_1 = 1, m_3 = 2$, similarly we have

$$\begin{aligned}&\mathbb{E}_B\mathbb{E}_W\varepsilon^4 \int_{\mathbb{R}^{2d}} \int_{[0,t/\varepsilon^2]^3} |\varphi|(B_{s_1} - y)|\varphi|(B_{s_2} - z)|\varphi|(B_{s_3} - z)|f^\lambda|(\tilde{B} - y)|f^\lambda|(\tilde{W} - z)dsdydz\\ \lesssim&\mathbb{E}_B\varepsilon^4 \int_{[0,t/\varepsilon^2]^3} (|\varphi| \star |\varphi|)(B_{s_2} - B_{s_3})(|f^\lambda| \star |\varphi|)(\tilde{B} - B_{s_1})ds\\ \lesssim&\varepsilon^4 \frac{1}{\varepsilon^2} \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(|f^\lambda| \star |\varphi|)(x+y)}{|x|^{d-2}}dx \lesssim \varepsilon\end{aligned} \tag{B.16}$$

by Lemma A.7.

When $m_1=1, m_4=2$, similarly we have

$$
\begin{aligned}
&\mathbb{E}_B\mathbb{E}_W\varepsilon^4\int_{\mathbb{R}^{2d}}\int_{[0,t/\varepsilon^2]^3}|\varphi|(B_{s_1}-y)|\varphi|(W_{u_1}-z)|\varphi|(W_{u_2}-z)|f^\lambda|(\tilde{B}-y)|f^\lambda|(\tilde{W}-z)dsdudydz\\
\lesssim&\mathbb{E}_B\mathbb{E}_W\varepsilon^4\int_{[0,t/\varepsilon^2]^3}(|\varphi|\star|\varphi|)(W_{u_1}-W_{u_2})(|f^\lambda|\star|\varphi|)(\tilde{B}-B_{s_1})dsdu\\
\lesssim&\varepsilon^4\frac{1}{\varepsilon^2}\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(|f^\lambda|\star|\varphi|)(x+y)}{|x|^{d-2}}dx\lesssim\varepsilon
\end{aligned}
\tag{B.17}
$$

by Lemma A.7. The proof is complete. □

Proof of Lemma 5.9.

The discussion is similar as in the proof of Lemma 5.4, so we do not present all the details.

If $m_1m_3\neq 0$, we first use constant to bound $|f^\lambda|$, then take $\mathbb{E}_W$. Next we take $\mathbb{E}_B$ and deal with the integral in y,z. In the end, we obtain

$$
\begin{aligned}
&\varepsilon^{N(m_i)+\frac{1}{2}}\mathbb{E}_B\mathbb{E}_W\int_{\mathbb{R}^{2d}}\int_{[0,t/\varepsilon^2]^{N(m_i)+1}}\prod_{i=1}^{m_1}|\varphi|(B_{s_i}-y)\prod_{i=m_1+1}^{m_1+m_3}|\varphi|(B_{s_i}-z)|g|(B_s-y)\\
&\qquad\prod_{i=1}^{m_2}|\varphi|(W_{u_i}-y)\prod_{i=m_2+1}^{m_2+m_4}|\varphi|(W_{u_i}-z)|f^\lambda|(\tilde{W}-z)dsdudydz \qquad\qquad (B.18)\\
\lesssim&\varepsilon^{N(m_i)+\frac{1}{2}-4}(m_2+m_4)!(m_1+m_3+1)!\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(|\varphi|\star|g|)(x+y)}{|x|^{d-2}}dx\\
\lesssim&\varepsilon^{N(m_i)-2}|\log\varepsilon|(m_2+m_4)!(m_1+m_3+1)!
\end{aligned}
$$

For other cases, the discussion is similar. The proof is complete. □

Proof of Lemma 5.10. When $\sum_{i=1}^4 m_i=2$, we have

$$
\begin{aligned}
&|\int_{\mathbb{R}^{2d}}g_B(y)^{m_1}g_W(y)^{m_2}g_B(z)^{m_3}g_W(z)^{m_4}h_B(y)\overline{h_W}(z)dydz|\\
\leq&\varepsilon^{2+\frac{1}{2}}\int_{[0,t/\varepsilon^2]^3}R_1(x_{s_1}-B_{s_2})R_2(y_{s_3}-\tilde{W})ds,
\end{aligned}
\tag{B.19}
$$

where $R_1(x)=\int_{\mathbb{R}^d}|\varphi|(x+y)|g|(y)dy$, $R_2(x)=\int_{\mathbb{R}^d}|\varphi|(x+y)|f^\lambda|(y)dy$, and $x,y\in\{B,W\},\tilde{W}\in\{0,W_{t/\varepsilon^2}\}$. If we replace $|g|$ and $|f^\lambda|$ by the corresponding radially symmetric and decreasing bound, then by Lemma A.7, we have

$$
\begin{aligned}
&\mathbb{E}_B\mathbb{E}_W\int_{[0,t/\varepsilon^2]^3}R_1(x_{s_1}-B_{s_2})R_2(y_{s_3}-\tilde{W})ds\\
\leq&\frac{1}{\varepsilon^2}\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{R_1(x+y)}{|x|^{d-2}}dx\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{R_2(x+y)}{|x|^{d-2}}dx,
\end{aligned}
\tag{B.20}
$$

so by Lemma A.5, we finally obtain

$$\begin{aligned}&\mathbb{E}_B\mathbb{E}_W|\int_{\mathbb{R}^{2d}} g_B(y)^{m_1} g_W(y)^{m_2} g_B(z)^{m_3} g_W(z)^{m_4} h_B(y)\overline{h_W}(z)dydz|\\ \leq&\varepsilon^{2+\frac{1}{2}}\frac{1}{\varepsilon^2}\varepsilon^{\frac{3}{2}}|\log\varepsilon|\frac{1}{\varepsilon}=\varepsilon|\log\varepsilon|,\end{aligned} \tag{B.21}$$

which completes the proof. □

C On (4.7): duality relation in Malliavin calculus

Let $H=\oplus^d L^2([0,t])$ and B_t a standard Brownian motion in $\mathbb{R}^d$. Take the isonormal Gaussian space $\{W(h)\}$ on H defined as $W(h)=\sum_{k=1}^d\int_0^t\tilde{h}_k(s)dB_s^k$ when $h=(\tilde{h}_1,\tilde{h}_2,\ldots,\tilde{h}_d)\in H$, then B_t is written as

$$B_t=(W(h_1),W(h_2),\ldots,W(h_d))$$

with $h_i\in\oplus^d L^2([0,t])$, and only its $i-$ component is non-zero and equal to $1_{[0,t]}$.

Let $F=f(W(h_1),W(h_2),\ldots,W(h_d))$ for any test function f, and $G=(g_1(B_s),g_2(B_s),\ldots,g_d(B_s))$. Then the Skorohod integral for G is defined as

$$\delta(G)=\sum_{k=1}^d\int_0^t g_k(B_s)dB_s^k. \tag{C.1}$$

The duality relation reads

$$\mathbb{E}\{F\delta(G)\}=\mathbb{E}\{\langle DF,G\rangle_H\}, \tag{C.2}$$

with D the Malliavin derivative operator.

We have $DF=\sum_{k=1}^d\partial_k f(B_t)h_k$, so

$$\langle DF,G\rangle_H=\sum_{k=1}^d\partial_k f(B_t)\int_0^t g_k(B_s)ds. \tag{C.3}$$

Therefore, (C.2) implies

$$\sum_{k=1}^d\mathbb{E}\{f(B_t)\int_0^t g_k(B_s)dB_s^k\}=\sum_{k=1}^d\mathbb{E}\{\partial_k f(B_t)\int_0^t g_k(B_s)ds\}. \tag{C.4}$$

References

[1] G. Bal, *Central limits and homogenization in random media*, Multiscale Modeling & Simulation, 7 (2008), pp. 677–702. 1

[2] ——, *Convergence to spdes in stratonovich form*, Communications in Mathematical Physics, 292 (2009), pp. 457–477. 1

[3] G. Bal, *Homogenization with large spatial random potential*, Multiscale Modeling & Simulation, 8 (2010), pp. 1484–1510. 1, 6

[4] G. Bal, J. Garnier, Y. Gu, and W. Jing, *Corrector theory for elliptic equations with oscillatory and random potentials with long range correlations*, Asymptot. Anal, 77 (2012), pp. 123–145. 1

[5] G. Bal, J. Garnier, S. Motsch, and V. Perrier, *Random integrals and correctors in homogenization*, Asymptotic Analysis, 59 (2008), pp. 1–26. 1

[6] M. Biskup, M. Salvi, and T. Wolff, *A central limit theorem for the effective conductance: Linear boundary data and small ellipticity contrasts*, Communications in Mathematical Physics, 328 (2014), pp. 701–731. 1

[7] A. Bourgeat and A. Piatnitski, *Estimates in probability of the residual between the random and the homogenized solutions of one-dimensional second-order operator*, Asymptotic Analysis, 21 (1999), pp. 303–315. 1

[8] L. A. Caffarelli and P. E. Souganidis, *Rates of convergence for the homogenization of fully nonlinear uniformly elliptic pde in random media*, Inventiones mathematicae, 180 (2010), pp. 301–360. 1

[9] R. Figari, E. Orlandi, and G. Papanicolaou, *Mean field and gaussian approximation for partial differential equations with random coefficients*, SIAM Journal on Applied Mathematics, 42 (1982), pp. 1069–1077. 1

[10] A. Gloria, S. Neukamm, and F. Otto, *Quantification of ergodicity in stochastic homogenization: optimal bounds via spectral gap on glauber dynamics*, Inventiones mathematicae, (2013), pp. 1–61. 1

[11] ——, *An optimal quantitative two-scale expansion in stochastic homogenization of discrete elliptic equations*, ESAIM: Mathematical Modelling and Numerical Analysis, 48 (2014), pp. 325–346. 1

[12] A. Gloria and F. Otto, *An optimal variance estimate in stochastic homogenization of discrete elliptic equations*, The Annals of Probability, 39 (2011), pp. 779–856. 1

[13] ——, *An optimal error estimate in stochastic homogenization of discrete elliptic equations*, The Annals of Applied Probability, 22 (2012), pp. 1–28. 1

[14] Y. Gu and G. Bal, *Random homogenization and convergence to integrals with respect to the rosenblatt process*, Journal of Differential Equations, 253 (2012), pp. 1069–1087. 1

[15] ——, *An invariance principle for brownian motion in random scenery*, Electron. J. Probab, 19 (2014), pp. 1–19. 32

[16] ——, *Weak convergence approach to a parabolic equation with large random potential*, to appear in Annales de l'Institut Henri Poincaré Probabilités et Statistiques, (2014). 1, 2, 3, 4, 7, 8, 26, 33

[17] Y. Gu and J.-C. Mourrat, *Fluctuations in stochastic homogenization of parabolic equations with random coefficients*, In preparation., (2014). 2

[18] M. Hairer, E. Pardoux, and A. Piatnitski, *Random homogenisation of a highly oscillatory singular potential*, Stochastic Partial Differential Equations: Analysis and Computations, 1 (2013), pp. 571–605. 1, 26

[19] V. V. Jikov, S. M. Kozlov, and O. A. Oleĭnik, *Homogenization of differential operators and integral functionals*, Springer-Verlag, Berlin, 1994. Translated from the Russian by G. A. Yosifian [G. A. Iosif′yan]. 1

[20] C. Kipnis and S. Varadhan, *Central limit theorem for additive functionals of reversible markov processes and applications to simple exclusions*, Communications in Mathematical Physics, 104 (1986), pp. 1–19. 2

[21] S. M. Kozlov, *Averaging of random operators*, Matematicheskii Sbornik, 151 (1979), pp. 188–202. 1

[22] J.-C. Mourrat, *Kantorovich distance in the martingale clt and quantitative homogenization of parabolic equations with random coefficients*, Probability Theory and Related Fields, (2012), pp. 1–36. 1, 2, 8, 9

[23] J.-C. Mourrat and J. Nolen, *A clt for linear functionals of the stationary corrector*, In preparation., (2014). 1

[24] J.-C. Mourrat and F. Otto, *Correlation structure of the corrector in stochastic homogenization*, arXiv preprint arXiv:1402.1924, (2014). 1

[25] J. Nolen, *Normal approximation for a random elliptic equation*, Probability Theory and Related Fields, (2011), pp. 1–40. 1

[26] D. Nualart, *The Malliavin calculus and related topics*, Probability and its Applications (New York), Springer-Verlag, Berlin, second ed., 2006. 12

[27] G. C. Papanicolaou and S. R. S. Varadhan, *Boundary value problems with rapidly oscillating random coefficients*, in Random fields, Vol. I, II (Esztergom, 1979), Colloq. Math. Soc. János Bolyai, 27, North Holland, Amsterdam, New York, 1981, pp. 835–873. 1

[28] E. Pardoux and A. Piatnitski, *Homogenization of a singular random one dimensional pde*, GAKUTO Internat. Ser. Math. Sci. Appl, 24 (2006), pp. 291–303. 1

[29] É. Pardoux and A. Piatnitski, *Homogenization of a singular random one-dimensional pde with time-varying coefficients*, The Annals of Probability, 40 (2012), pp. 1316–1356. 1

[30] V. Yurinskii, *Averaging of symmetric diffusion in random medium*, Siberian Mathematical Journal, 27 (1986), pp. 603–613. 1