Radiative Transport Limit of Dirac Equations with Random Electromagnetic Field

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Abstract

This paper concerns the kinetic limit of the Dirac equation with a random electromagnetic field. We give a detailed mathematical analysis of the radiative transport limit for the phase space energy density of solutions to the Dirac equation. Our derivation is based on a martingale method and a perturbed test function expansion. This requires the electromagnetic field to be a Markovian space-time random field.

The main mathematical tool in the derivation of the kinetic limit is the matrix-valued Wigner transform of the vector-valued Dirac solution. The major novelty compared to the scalar (Schrödinger) case is the proof of the weak convergence of cross-modes to zero. The propagating modes are shown to converge in an appropriate probabilistic sense to their deterministic limit.

1 Introduction

The Dirac equation is the relativistic version of the Schrödinger equation and describes very fast electrons propagating in an electromagnetic field. In this paper, we consider the semiclassical limit of the Dirac equation when the electromagnetic field is random and time-dependent.

The problem falls into the category of high frequency wave propagating in highly heterogeneous media, which has been modeled by radiative transfer equation in many areas, e.g., quantum waves in semiconductors, electromagnetic waves in turbulent atmosphere and plasma, underwater acoustic waves, elastic waves in the Earth's crust. Such kinetic models account for the multiple interactions of wave fields with the fluctuations of the underlying media. In the so-called weak-coupling limit we consider here, waves propagate over distances that are large compared to the typical wavelength in the system, and the fluctuations of the underlying media have a weak amplitude with a correlation length comparable

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to the wavelength. Many derivations of radiative transfer equations are based on formal expansions, e.g., a systematic method to derive kinetic equations from symmetric first-order hyperbolic systems, including systems of acoustic and elastic equations, has been presented in [27] in the weak-coupling limit and extended in various forms in [1, 3, 18, 26].

Mathematically rigorous derivations are notoriously difficult in the setting of spatially varying randomness. Most proofs have been obtained for the Schrödinger equation with a time-independent Gaussian potential, see [10, 14, 28], with an extension to (discrete) wave equations in [22]. They are based on the Neumann series expansion of the solution to the Schrödinger equation and appropriate estimates that allow passage to the limit. Such techniques were also used in [5] to obtain the random mixing of the phase of the Schrödinger solution and in [2, 20, 29, 30] to analyze limits of solutions with low-frequency initial conditions and large, high-frequency, random potentials.

The derivation of kinetic limits is much simplified in the setting of time-dependent random coefficients and does not require the aforementioned infinite Neumann series expansion. Assuming a Markovian structure of the random potential enables us to use a martingale method and a suitable perturbed test functions expansion. A limit theorem for one dimensional waves where such methods are used is given in [23], and more general ones in [8]. The same approach has been applied to Schrödinger equation in different settings, see [6, 7, 15, 17, 24].

The Markovianity of the random coefficients simplifies derivations but is not necessary as shown in [15]. In [9], a geometric approach is applied to more general initial data than in [14]. A renewal of the random field is used to get the appropriate mixing properties during the evolution. In [25], the authors consider random potentials that are correlated in time with finite range. They proved the semiclassical limit of Schrödinger equation by pure PDE techniques.

In the setting of the Dirac equation, the energy matrix has to be decomposed into different propagating and non-propagating modes, which behave quite differently in the high-frequency limit. This is the main difference from the scalar case of the Schrödinger equation. As a result, we will see that the limiting Wigner transform satisfies a system of coupled radiative transfer equations.

The rest of the paper is organized as follows. We describe the setting of the problem and state our main results in Section 2. Next, we sketch the outline of the proof and discuss in details the construction of the perturbed test functions in Section 3. In Section 4, we prove the convergence of the different modes. The case of slower fluctuations in time, which allow us to recover limiting kinetic models with elastic scattering, is briefly discussed in Section 5. Conclusions and some further discussions are presented in Section 6.

2 Main results

In this section, we first describe our setting, then introduce the Wigner transform as our main tool, and present our construction of the random electromagnetic field. We finally state the main results of the paper.

The Dirac equation in three dimensions of space reads

$$\varepsilon \partial_t \Psi^\varepsilon + P(x, \varepsilon D_x)\Psi^\varepsilon = 0, \quad x \in \mathbb{R}^3,\ t \in \mathbb{R}, \tag{2.1}$$

with the differential operator $P(x, D)$ defined as:

$$P(x,\xi) = i\left(\sum_{k=1}^{3} \gamma^0\gamma^k(\xi_k - eA_k(x)) + m_0 c\gamma^0 - eA_0(x)I_4\right).$$

Here $\Psi^\varepsilon = \Psi^\varepsilon(t,x) \in \mathbb{C}^4$ is the wave function, $\varepsilon = \hbar$ is our small parameter and stands for the Planck constant, e is the unit charge, m_0 is the electron's rest mass, c is the velocity of light, and $A_k(x) \in \mathbb{R}$, $k = 0, 1, 2, 3$ are the components of the prescribed electromagnetic field. In particular, A_0 is the electric potential and (A_1, A_2, A_3) is the magnetic potential vector.

The matrices $\gamma^k \in \mathbb{C}^{4\times 4}$, $k = 0, 1, 2, 3$ are the 4×4 Dirac matrices, which are closely related to the 2×2 Pauli matrices. They consist of entries $0, 1, i$, and satisfy

$$\begin{aligned}
&\gamma^{0*} = \gamma^0, \gamma^{k*} = -\gamma^k, k = 1, 2, 3, (\gamma^0\gamma^k)^* = \gamma^0\gamma^k,\\
&\gamma^m\gamma^n + \gamma^n\gamma^m = 0, m \neq n, {\gamma^0}^2 = I_4, {\gamma^k}^2 = -I_4, k = 1, 2, 3.
\end{aligned}$$

In our proofs, the explicit form of the Dirac matrices does not play any special role.

The relativistic current density J^ε is a 4 dimensional vector with elements J^ε_k given by

$$J^\varepsilon_k = \Psi^{\varepsilon*}\gamma^0\gamma^k\Psi^\varepsilon,$$

and the relativistic position density $n(t, x)$ is

$$n^\varepsilon(t,x) = J^\varepsilon_0(t,x) = \Psi^{\varepsilon*}(t,x)\Psi^\varepsilon(t,x).$$

We investigate the limiting behavior of the solution to (2.1) as $\varepsilon \to 0$. Given the conservation of $\int_{\mathbb{R}^d} n^\varepsilon(t,x)dx$ and the fact that $n^\varepsilon(t,x)$ does not admit a closed-formed equation, the interested quantity here is the Wigner transform of Ψ^ε, which we introduce next. For more results on the Wigner transform, see [16].

2.1 Wigner transform and pseudo-differential calculus

We introduce the Wigner transform and some pseudo-differential calculus that is to be used.

The matrix-valued Wigner transform of two spatially-dependent d dimensional vector fields $u(x)$ and $v(x)$ is defined as

$$W[u,v](x,\xi) = \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} e^{i\xi\cdot y} u\left(x - \frac{\varepsilon y}{2}\right) v^*\left(x + \frac{\varepsilon y}{2}\right) dy,$$

where v^* is the transposition and possible complex conjugate of v. It may be seen as the inverse Fourier transform of the two point correlation function of $u(x)$ and $v(x)$, where we define Fourier transform using the convention $\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x} f(x)dx$. We will also use the notation $\mathcal{F}$ depending on which one is most convenient. We check that

$$(W[u,u](x,\xi))^* = W[u,u](x,\xi),$$

as well as

$$\int_{\mathbb{R}^d} W[u,v](t,x,\xi)d\xi = (uv^*)(t,x).$$

This allows us to interpret the Wigner transform as an energy density in phase space, although the Wigner transform is positive only in the limit $\varepsilon \to 0$ [16].

We recall from [1] some simple results about pseudo-differential calculus that are needed in this paper.

Proposition 2.1. *Let $P(\varepsilon D)$ be a matrix-valued pseudo-differential operator defined by*

$$P(\varepsilon D)u(x) = \int_{\mathbb{R}^d} e^{ip\cdot x} P(i\varepsilon p)\hat{u}(p)\frac{dp}{(2\pi)^d}.$$

Then, we have

$$W[P(\varepsilon D)u, v](x,\xi) = P\left(i\xi + \frac{\varepsilon D}{2}\right) W[u,v](x,\xi),$$

and

$$W[u, P(\varepsilon D)v](x,\xi) = W[u,v](x,\xi)P^*\left(i\xi - \frac{\varepsilon D}{2}\right).$$

Above, $W[u,v](x,\xi)P^*(i\xi - \varepsilon D/2)$ is defined as the inverse Fourier transform $\mathcal{F}^{-1}_{p\to x}$ of the matrix $\hat{W}[u,v](p,\xi)P^*(i\xi - i\varepsilon p/2)$.

Proposition 2.2. *Let $V(x)$ be a real matrix-valued function. Then we have*

$$W\left[V\left(\frac{x}{\varepsilon}\right)u, v\right](x,\xi) = \int_{\mathbb{R}^d} \frac{e^{ix\cdot p/\varepsilon}}{(2\pi)^d}\hat{V}(p)W[u,v]\left(x, \xi - \frac{p}{2}\right)dp,$$

and

$$W\left[u, V\left(\frac{x}{\varepsilon}\right)v\right](x,\xi) = \int_{\mathbb{R}^d} \frac{e^{ix\cdot p/\varepsilon}}{(2\pi)^d} W[u,v]\left(x, \xi + \frac{p}{2}\right)\hat{V}^t(p)dp,$$

where $\hat{V}$ is the Fourier transform of V component by component.

Remark 2.3. Throughout the paper, we assume that the spatial dimension d is equal to 3 although we occasionally use "d" for expressions that hold independently of dimension.

2.2 The random field

The A_k, appearing in (2.1) are the components of a random electromagnetic field, which we assume to be time-dependent and of mean zero.

We follow the same construction of the random field as in [21, Section 2.2]. We do not give here all the details that can be found in [21] and only present the most important points.

The $A_k(t,x)$ are independent Ornstein-Uhlenbeck processes, i.e. stationary Gauss-Markov processes with correlation function

$$R_k(t,x) := \mathbb{E}\{A_k(t+s,x+y)A_k(s,y)\}, \quad k=0,1,2,3. \tag{2.2}$$

Above, R_k is assumed to have the form

$$R_k(t,x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot p}e^{-t\gamma_k(p)}\mu_k(p)dp,$$

where γ_k and μ_k are positive even functions in $\mathcal{S}(\mathbb{R}^d)$, with $\gamma_k(p) \geq \gamma_* > 0$ for all $p \in \mathbb{R}^d$. These regularity assumptions are for technical convenience and could be considerably weakened.

The process $\{A_k(t), t \geq 0\}$ is Markovian with a unique invariant measure denoted by π_k. For each $t \geq 0$ and π_k a.s., $A_k(t)$ belongs to a Hilbert space $\mathcal{E}$ that can be chosen as the weighted Sobolev space $H^s_{-\rho}$ with norm

$$\|f\|^2_{H^s_{-\rho}} = \int_{\mathbb{R}^d}\theta_s(\xi)|\mathcal{F}(f\theta_{-\rho})(\xi)|^2 d\xi, \qquad \theta_s(x) = (1+|x|^2)^{s/2},$$

for $\rho > d$ and any $s \geq 0$ (this holds since $\mu \in \mathcal{S}(\mathbb{R}^d)$). Above, we recall that $\mathcal{F}$ denotes Fourier transform. We will work with $s = d+1$ and $\rho = d+1$ for instance, and in that case, when $d = 3$, the space $H^s_{-\rho}$ is continuously embedded into the space of C^1 functions that we denote by $\mathcal{C}^1(\mathbb{R}^d)$. It can also be shown that the trajectories of A_k are continuous in time with values in $H^s_{-\rho}$. This shows in particular that, for any $T > 0$,

$$\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d}|\theta_{-\rho}(x)\nabla^\alpha_x A_k(x)| \leq C, \quad \alpha = 0,1, \qquad \pi_k\ a.s. \tag{2.3}$$

We will denote by $\tilde{A}_k(t,p)$ the Fourier transform of A_k with respect to x only. Since $A_k(t) \in \mathcal{E}$ for each $t \geq 0$, $\tilde{A}_k(t)$ is an element of $H^{-\rho}(\mathbb{R}^d)$, the usual Sobolev space. Besides, as A_k is real, we have $\tilde{A}^*_k(t,p) = \tilde{A}_k(t,-p)$.

The process A_k satisfies a stochastic differential equation of the form

$$dA_k = C_k A_k dt + \mathcal{R}_k^{1/2}dW_k, \tag{2.4}$$

where W_k is a standard cylindrical Wiener process and $\mathcal{R}_k$ an appropriate nonnegative trace class self-adjoint operator. Above, the operator C_k is the infinitesimal generator of the strongly continuous semigroup $S_k(t) : \mathcal{E} \to \mathcal{E}$,

$$S_k(t)A(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot p}e^{-t\gamma_k(p)}\tilde{A}(p)dp.$$

We assume that the initial condition for (2.4) $A_k(t=0)$ is drawn according to the invariant measure π_k. The generator of $\{A_k(t), t \geq 0\}$ is denoted by $\mathcal{Q}_k$. According to [11], section 4, its spectral gap is the same as C_k, that is γ_*. Hence, for any $g \in L^2(\mathcal{E},\pi_k)$ with $\mathbb{E}_k g = 0$ ($\mathbb{E}_k$ denotes expectation with respect to π_k), we have

$$\|e^{t\mathcal{Q}_k}g\|_{L^2(\mathcal{E},\pi_k)} \leq \|g\|_{L^2(\mathcal{E},\pi_k)}e^{-\gamma_* t/2}. \tag{2.5}$$

Given (2.5), the Fredholm alternative holds for the Poisson equation

$$\mathcal{Q}_k f = g,$$

provided that $\mathbb{E}_k g = 0$. It has a unique solution $f \in L^2(\mathcal{E}, \pi_k)$ with $\mathbb{E}_k f = 0$ that is given explicitely by

$$f(A) = -\int_0^\infty dr e^{r\mathcal{Q}_k} g(A), \qquad A \in \mathcal{E},$$

and the integral converges absolutely in $L^2(\mathcal{E}, \pi_k)$ thanks to (2.5). The latter relation will be used in the construction of the perturbed test functions. Moreover, given the definition of the generator $\mathcal{Q}_k$ for the first equality below, and (2.4) together with the Markovian property of $\{A_k(t), t \geq 0\}$ for the second, we have

$$e^{t\mathcal{Q}_k} A_k(s) = \mathbb{E}_k\{A_k(t+s)|A_k(s)\} = S_k(t)A_k(s).$$

The action of $\exp(t\mathcal{Q}_k)$ on $\tilde{A}_k$ is therefore represented by the multiplication operator (with a slight abuse of notation):

$$e^{t\mathcal{Q}_k} \tilde{A}_k(s,p) := \mathcal{F}^{-1}\left(e^{t\mathcal{Q}_k} A_k(s)\right) = e^{-t\gamma_k(p)} \tilde{A}_k(s,p). \tag{2.6}$$

In the same way, one can show that

$$\begin{aligned} e^{t\mathcal{Q}_k} \Big(\tilde{A}_k(s,p)\tilde{A}_k(s,q) - \mathbb{E}_{\pi_k}\{\tilde{A}_k(s,p)\tilde{A}_k(s,q)\}\Big) = \\ e^{-t(\gamma_k(p)+\gamma_k(q))} \tilde{A}_k(s,p)\tilde{A}_k(s,q) - (2\pi)^d \delta(p+q) e^{-2t\gamma_k(p)} \mu_k(p). \end{aligned} \tag{2.7}$$

We will use the relation below to control fourth order moments of $\tilde{A}_k$ in the estimation of the perturbed test functions. It is a consequence of the gaussianity of A_k. We have

$$\begin{aligned} (2\pi)^{-2d}\, \mathbb{E}_k\{\tilde{A}_k(t_1,p_1)\tilde{A}_k(t_2,p_2)\tilde{A}_k(t_3,p_3)\tilde{A}_k(t_4,p_4)\} = \\ \delta(p_1+p_2)\tilde{R}(t_1-t_2,p_1)\delta(p_3+p_4)\tilde{R}(t_3-t_4,p_3) \\ + \delta(p_1+p_4)\tilde{R}(t_1-t_4,p_1)\delta(p_2+p_3)\tilde{R}(t_2-t_3,p_2) \\ + \delta(p_1+p_3)\tilde{R}(t_1-t_3,p_1)\delta(p_2+p_4)\tilde{R}(t_2-t_4,p_2), \end{aligned} \tag{2.8}$$

where $\tilde{R}_k$ is the Fourier transform of R_k w.r.t. x. More generally, we have

$$\mathbb{E}_{\pi_k}\left\{\prod_{i=1}^{2n} \tilde{A}_k(t_i,p_i)\right\} = (2\pi)^{nd} \sum_{\mathcal{P}} \prod_{(\alpha,\beta)\in\mathcal{P}} \delta(p_\alpha+p_\beta)\tilde{R}(t_\alpha-t_\beta,p_\alpha) \tag{2.9}$$

where the sum runs over the pairings $\mathcal{P}$ of $\{1,\dots,2n\}$. A pairing over vertices of $\{1,\dots,2n\}$ is a partition of this set made of n pairs of couples (α,β), for which $\alpha < \beta$ and such that all the elements of $\{1,\dots,2n\}$ appear in only one of the pairs. We will use (2.9) for $n = 3$.

In the sequel, we will work with vector processes $A(t) = (A_0(t), A_1(t), A_2(t), A_3(t))$, where the components are independent and are Ornstein-Uhlenbeck processes as just described. The generator of A is denoted by $\mathcal{Q} = \sum_{k=0}^3 \mathcal{Q}_k$ and $\mathbb{E}_\pi$ will denote expectation with respect to $\pi = \otimes_{k=0}^3 \pi_k$.

2.3 Main theorem

Before stating the main results, we first derive the equation satisfied by the Wigner transform.

Recall the equation

$$\varepsilon\partial_t\Psi^\varepsilon + P(x,\varepsilon D_x)\Psi^\varepsilon = 0.$$

We replace $A_k(t,x)$ with $\sqrt{\varepsilon}A_k(\frac{t}{\varepsilon},\frac{x}{\varepsilon})$ in the weak coupling limit, so the equation for Ψ^ε becomes

$$\varepsilon\partial_t\Psi^\varepsilon + i\left(\sum_{k=1}^{3}\gamma^0\gamma^k\left[\varepsilon D_k - \sqrt{\varepsilon}eA_k\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)\right] + m_0c\gamma^0 - \sqrt{\varepsilon}eA_0\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)I_4\right)\Psi^\varepsilon = 0, \quad (2.10)$$

equipped with a deterministic initial condition Ψ_0^ε. As usual with Wigner transform, we will work with mixed states: suppose Ψ^ε depends on some $\zeta \in S$, i.e., $\Psi^\varepsilon \equiv \Psi^\varepsilon_\zeta$, and define

$$W_\varepsilon(t,x,\xi) = \int_S W[\Psi^\varepsilon_\zeta(t,.),\Psi^\varepsilon_\zeta(t,.)](x,\xi)\mathfrak{m}(\zeta)d\zeta,$$

for some positive integrable function $\mathfrak{m}$. By Propositions 2.1 and 2.2, we derive the following equation satisfied by W_ε,

$$\begin{aligned}\varepsilon\partial_t W_\varepsilon + \sum_{k=1}^{3}c\gamma^0\gamma^k P_k\left(i\xi+\frac{\varepsilon D}{2}\right)W_\varepsilon + \sum_{k=1}^{3}W_\varepsilon P_k^*\left(i\xi-\frac{\varepsilon D}{2}\right)c\gamma^0\gamma^k + im_0c^2(\gamma^0W_\varepsilon - W_\varepsilon\gamma^0)\\ -ie\sqrt{\varepsilon}\sum_{k=0}^{3}\gamma^0\gamma^k\mathcal{K}_\varepsilon^k W_\varepsilon + ie\sqrt{\varepsilon}\sum_{k=0}^{3}\tilde{\mathcal{K}}_\varepsilon^k W_\varepsilon\gamma^0\gamma^k = 0,\end{aligned} \quad (2.11)$$

where the symbol $P_k(\xi) = \xi_k$, and the operators $\mathcal{K}_\varepsilon^k, \tilde{\mathcal{K}}_\varepsilon^k$ are defined by

$$\begin{aligned}\mathcal{K}_\varepsilon^k\lambda(t,x,\xi) &= \int_{\mathbb{R}^d}\frac{dp\tilde{A}_k(\frac{t}{\varepsilon},p)}{(2\pi)^d}e^{ix\cdot p/\varepsilon}\lambda(t,x,\xi-\frac{p}{2}),\\ \tilde{\mathcal{K}}_\varepsilon^k\lambda(t,x,\xi) &= \int_{\mathbb{R}^d}\frac{dp\tilde{A}_k(\frac{t}{\varepsilon},p)}{(2\pi)^d}e^{ix\cdot p/\varepsilon}\lambda(t,x,\xi+\frac{p}{2}).\end{aligned}$$

If we divide both sides of (2.11) by ε and compare with the scalar case when the dynamics is given by the random Schrödinger equation, we see that the structures of the equations are similar except for the term

$$\frac{1}{\varepsilon}\sum_{k=1}^{3}\left(c\gamma^0\gamma^k P_k(i\xi)W_\varepsilon + W_\varepsilon P_k^*(i\xi)c\gamma^0\gamma^k\right).$$

Weak solutions to the Wigner equation. We define the L^2 inner product for matrix valued functions as follows,

$$\langle F(\cdot),G(\cdot)\rangle = \int_{\mathbb{R}^\ell}\mathrm{Tr}(F^*G)(y)dy,$$

with the same notation for the usual inner product for scalar functions. With an abuse of notation, the corresponding L^2 norm will be denoted by $\|\cdot\|_{L^2(\mathbb{R}^\ell)}$ for both matrix-valued and scalar functions. We will in particular use the same notation for inner products involving W_ε defined on $(L^2(\mathbb{R}^6))^{16}$ and $W_\varepsilon \otimes W_\varepsilon$ defined on $(L^2(\mathbb{R}^{12}))^{16^2}$ since there is no possible confusion. We consider matrix-valued test functions $\lambda(x,\xi)$ in the space $\mathcal{S}_d := (\mathcal{S}(\mathbb{R}^{2d}))^{(d+1)^2}$. We will construct weak solutions to (2.11), for an initial condition that is the Wigner transform of some mixed state wavefunction, in the following sense: given an Ornstein-Uhlenbeck process $\{A(t),\ t \geq 0\}$ and $T > 0$ arbitrary, find $W_\varepsilon \in (\mathcal{C}^0([0,T], L^2(\mathbb{R}^6)))^{16}$ such that, for any test function $\lambda \in \mathcal{S}_3$,

$$\forall t \in [0,T], \qquad \varepsilon\langle W_\varepsilon(t), \lambda\rangle - \varepsilon\langle W_\varepsilon^0, \lambda\rangle = \int_0^t ds \langle (\mathcal{I}_\varepsilon + \mathcal{K}_\varepsilon)\lambda, W_\varepsilon(s)\rangle ds, \quad \pi \quad a.s. \tag{2.12}$$

where

$$\begin{aligned}
\mathcal{I}_\varepsilon \lambda &= -\sum_{k=1}^{3} c\gamma^0\gamma^k P_k\left(i\xi + \frac{\varepsilon D}{2}\right)^* \lambda - \lambda \sum_{k=1}^{3} P_k\left(i\xi - \frac{\varepsilon D}{2}\right) c\gamma^0\gamma^k + im_0c^2(\gamma^0\lambda - \lambda\gamma^0),\\
\mathcal{K}_\varepsilon \lambda &= -ie\sqrt{\varepsilon}\sum_{k=0}^{3} \gamma^0\gamma^k \mathcal{K}_\varepsilon^k \lambda + ie\sqrt{\varepsilon}\sum_{k=0}^{3} \tilde{\mathcal{K}}_\varepsilon^k \lambda \gamma^0\gamma^k.
\end{aligned}$$

Above, W_ε^0 is the Wigner transform of the initial condition $\Psi_{0,\zeta}^\varepsilon$. We use (2.3) together with the following relation to make sense of the last term in (2.12):

$$(\mathcal{F}_2 \mathcal{K}_\varepsilon^k \lambda_{mn})(x,y) = A_k\left(\frac{x}{\varepsilon} - \frac{y}{2}\right) \mathcal{F}_2 \lambda_{mn}(x,y),$$

where $\mathcal{F}_2$ denotes the Fourier transform with respect to the second variable. We sketch a proof of the existence of weak solutions by a fairly standard argument: consider a smooth cut-off function χ_n with support in the ball of radius n centered at zero. According to (2.3), $A^n := \chi_n A$, where $A = (A_0, A_1, A_2, A_3)$, is a bounded function in $\mathcal{C}^0([0,T]), \mathcal{C}^1(\mathbb{R}^d))$. Owing to this regularity, and with $\Psi_{0,\zeta}^\varepsilon \in (H^1(\mathbb{R}^d))^4$ uniformly bounded in ζ for instance, it is direct to construct a unique solution in $\mathcal{C}^0([0,T], (H^1(\mathbb{R}^d))^4) \cap \mathcal{C}^1([0,T], (L^2(\mathbb{R}^d))^4)$ to the Dirac equation with potential A^n. This can be done e.g. by using the properties of the free Dirac semigroup stated in [12], Chapter 15, §5, and a fixed point procedure. One can then show that the corresponding Wigner transform $W_{\varepsilon,n}$ satisfies (2.11) pointwise in t, a.e. in (x,ξ), and π a.s. Direct algebra then shows that, as usual, the Wigner equation preserves the L^2 norm, that is

$$\|W_{\varepsilon,n}(t)\|_{L^2(\mathbb{R}^{2d})} = \|W_\varepsilon^0\|_{L^2(\mathbb{R}^{2d})}, \qquad \forall t \in [0,T].$$

This allows us to obtain by weak-$*$ compactness a limit in $L^\infty((0,T), L^2(\mathbb{R}^{2d}))$ denoted by W_ε. Moreover, it can be shown from (2.3) and (2.12) with A replaced by A^n, that $\langle W_{\varepsilon,n}(t), \lambda\rangle$ is equicontinuous on $[0,T]$. This is enough to pass to the limit and obtain that W_ε satisfies (2.12) with

$$\|W_\varepsilon(t)\|_{L^2(\mathbb{R}^{2d})} \leq \|W_\varepsilon^0\|_{L^2(\mathbb{R}^{2d})}, \qquad \forall t \in [0,T]. \tag{2.13}$$

We will suppose that $W^0_\varepsilon(x,\xi)$ is uniformly bounded in ε in $L^2(\mathbb{R}^{2d})$ and converges weakly as $\varepsilon \to 0$ to $W^0(x,\xi) \in L^2(\mathbb{R}^{2d})$ without restriction to a subsequence. This holds for instance if Ψ^ε_ζ has the form $\varphi^\varepsilon e^{ix\cdot\zeta/\varepsilon}$ with $\varphi^\varepsilon \in (\mathcal{S}(\mathbb{R}^d))^4$ uniformly bounded and $\zeta \in \mathbb{R}^d$. In such a case, $W^0_\varepsilon(x,\xi)$ reads

$$W^0_\varepsilon(x,\xi) = \int_{\mathbb{R}^d} e^{i\xi\cdot y}\varphi^\varepsilon(x-\frac{\varepsilon y}{2})(\varphi^\varepsilon)^*(x+\frac{\varepsilon y}{2})\hat{\mathfrak{m}}(y)dy.$$

When $\hat{\mathfrak{m}} \in L^2(\mathbb{R}^d)$, it follows that

$$\|W^0_\varepsilon\|_{L^2(\mathbb{R}^{2d})} \leq \|\varphi^\varepsilon\|^2_{L^4(\mathbb{R}^d)}\|\hat{\mathfrak{m}}\|_{L^2(\mathbb{R}^d)} \leq C.$$

The entire sequence converges provided that, for some $f_{ij} \in L^2(\mathbb{R}^{2d})$,

$$\varphi^\varepsilon_i(x-\frac{\varepsilon y}{2})(\varphi^\varepsilon_j)^*(x+\frac{\varepsilon y}{2})\hat{\mathfrak{m}}(y) \to f_{ij}(x,y) \qquad \text{weakly in} \qquad L^2(\mathbb{R}^{2d}).$$

Mode decomposition. We need to decompose the Wigner function into various propagating modes in order to state the main theorem. For this, we write first the formal expansion $W_\varepsilon = W_0 + \sqrt{\varepsilon}W_{1,\varepsilon} + \varepsilon W_{2,\varepsilon} + \ldots$. Dividing then (2.11) by ε and expanding in ε, we obtain, by setting the $1/\varepsilon$ order term to be zero,

$$\sum_{k=1}^3 c\gamma^0\gamma^k P_k(i\xi)W_0 + W_0\sum_{k=1}^3 P^*_k(i\xi)c\gamma^0\gamma^k + im_0c^2(\gamma^0W_0 - W_0\gamma^0) = 0,$$

where we recall that $P_k(\xi) = \xi_k$. This enables us to define the dispersion matrix

$$Q(\xi) = \sum_{k=1}^3 \gamma^0\gamma^k\xi_k + m_0c\gamma^0, \tag{2.14}$$

and we expect the limit to satisfy that

$$Q(\xi)W_0(t,x,\xi) = W_0(t,x,\xi)Q(\xi). \tag{2.15}$$

By the properties of $\gamma^0\gamma^k$, we find that Q is Hermitian, as well as $Q^2 = (m_0^2c^2 + |\xi|^2)I_4$. Let

$$\lambda_\pm(\xi) = \pm\sqrt{m_0^2c^2 + |\xi|^2}$$

be the eigenvalues of Q corresponding to the energy levels of electrons and positrons, respectively. The orthonormal eigenvectors are

$$x_1(\xi) = \begin{pmatrix} 0 \\ \frac{m_0c}{\sqrt{2\lambda_+(\lambda_+-\xi_3)}} \\ \frac{\xi_1-i\xi_2}{\sqrt{2\lambda_+(\lambda_+-\xi_3)}} \\ \sqrt{\frac{\lambda_+-\xi_3}{2\lambda_+}} \end{pmatrix}, \qquad x_2(\xi) = \begin{pmatrix} \sqrt{\frac{\lambda_+-\xi_3}{2\lambda_+}} \\ -\frac{\xi_1+i\xi_2}{\sqrt{2\lambda_+(\lambda_+-\xi_3)}} \\ \frac{m_0c}{\sqrt{2\lambda_+(\lambda_+-\xi_3)}} \\ 0 \end{pmatrix},$$

$$y_1(\xi) = \begin{pmatrix} \frac{m_0c}{\sqrt{2\lambda_+(\lambda_+-\xi_3)}} \\ 0 \\ -\sqrt{\frac{\lambda_+-\xi_3}{2\lambda_+}} \\ \frac{\xi_1+i\xi_2}{\sqrt{2\lambda_+(\lambda_+-\xi_3)}} \end{pmatrix}, \qquad y_2(\xi) = \begin{pmatrix} \frac{\xi_1-i\xi_2}{\sqrt{2\lambda_+(\lambda_+-\xi_3)}} \\ \sqrt{\frac{\lambda_+-\xi_3}{2\lambda_+}} \\ 0 \\ -\frac{m_0c}{\sqrt{2\lambda_+(\lambda_+-\xi_3)}} \end{pmatrix},$$

where x_1, x_2 correspond to λ_+, and y_1, y_2 correspond to λ_-. Because $\{x_1, x_2, y_1, y_2\}$ is a basis in $\mathbb{C}^4$, we can decompose W_ε as

$$W_\varepsilon = \sum_{i,j=1,2} a_{ij}^\varepsilon x_i x_j^* + \sum_{i,j=1,2} b_{ij}^\varepsilon y_i y_j^* + \sum_{i,j=1,2} c_{ij}^\varepsilon x_i y_j^* + \sum_{i,j=1,2} d_{ij}^\varepsilon y_i x_j^*.$$

On the one hand, we have $W_\varepsilon^* = W_\varepsilon$, so

$$\begin{aligned} a_{ii}^\varepsilon \in \mathbb{R}, \quad i = 1, 2, \quad a_{12}^\varepsilon = \overline{a_{21}^\varepsilon}, \\ b_{ii}^\varepsilon \in \mathbb{R}, \quad i = 1, 2, \quad b_{12}^\varepsilon = \overline{b_{21}^\varepsilon}. \end{aligned}$$

On the other hand, since $W_\varepsilon \in (\mathcal{C}([0,T], L^2(\mathbb{R}^{2d})))^{(d+1)^2}$, we have that all the coefficients $a_{ij}^\varepsilon, b_{ij}^\varepsilon, c_{ij}^\varepsilon, d_{ij}^\varepsilon$ belong to $\mathcal{C}([0,T], L^2(\mathbb{R}^{2d}))$. We will see later that the dispersion relation (2.15) implies that $c_{ij}^\varepsilon, d_{ij}^\varepsilon$ converge to zero as $\varepsilon \to 0$. Thus, we write the limit in the form of

$$W_0 = \sum_{i,j=1,2} a_{ij} x_i x_j^* + \sum_{i,j=1,2} b_{ij} y_i y_j^*.$$

We denote by $\Pi_\pm$ the orthogonal projection of $\mathbb{C}^4$ on the eigenspace associated to $\lambda_\pm$, and they admit the expressions

$$\begin{aligned} \Pi_+ &= x_1 x_1^* + x_2 x_2^* = \frac{1}{2}\left(I_4 + \frac{Q}{\lambda_+}\right), \\ \Pi_- &= y_1 y_1^* + y_2 y_2^* = \frac{1}{2}\left(I_4 - \frac{Q}{\lambda_+}\right). \end{aligned}$$

Define $\alpha_+^\varepsilon = \mathrm{Tr}(\Pi_+ W_\varepsilon) = a_{11}^\varepsilon + a_{22}^\varepsilon$ and $\alpha_-^\varepsilon = \mathrm{Tr}(\Pi_- W_\varepsilon) = b_{11}^\varepsilon + b_{22}^\varepsilon$. We check that

$$\int_{\mathbb{R}^d} (\alpha_+^\varepsilon + \alpha_-^\varepsilon)(t, x, \xi) d\xi = n^\varepsilon(t, x).$$

Therefore, we can interpret α_+^ε and α_-^ε as energy densities in the phase space corresponding to electrons and positrons, respectively. They are the physical quantities we are interested in.

Define $L_M^2(\mathbb{R}^{2d}) = \{f : \|f\|_{L^2(\mathbb{R}^{2d})} \leq M\}$. The functional space we use for $(\alpha_+^\varepsilon(t), \alpha_-^\varepsilon(t))$ is

$$H_M = L_M^2(\mathbb{R}^{2d}) \bigoplus L_M^2(\mathbb{R}^{2d}) = \{X = (X_1, X_2), \quad X_1, X_2 \in L_M^2(\mathbb{R}^{2d})\},$$

endowed with the metric

$$d_H(X, Y) = \sum_{n=1}^\infty \left(\frac{|\langle X_1 - Y_1, e_n\rangle|}{2^n} + \frac{|\langle X_2 - Y_2, e_n\rangle|}{2^n} \right),$$

where $\{e_n\}$ is an orthonormal basis of $L_M^2(\mathbb{R}^{2d})$. It is straightforward to check that d_H induces the weak topology on $L_M^2(\mathbb{R}^{2d}) \bigoplus L_M^2(\mathbb{R}^{2d})$, and that (H_M, d_H) is a complete separable metric space.

Now we can state our main theorem.

Theorem 2.4. *Suppose W_ε is a weak solution to (2.11) with a deterministic initial condition $W_\varepsilon(0,x,\xi)$ that is the Wigner transform of some mixed states wave function. Suppose that $W_\varepsilon(0,x,\xi)$ is uniformly bounded in $L^2(\mathbb{R}^{2d})$ and converges weakly in $L^2(\mathbb{R}^{2d})$ as $\varepsilon \to 0$ to $W_0(0,x,\xi) \in L^2(\mathbb{R}^{2d})$. We decompose W_ε as*

$$W_\varepsilon = \sum_{i,j=1,2} a_{ij}^\varepsilon x_i x_j^* + \sum_{i,j=1,2} b_{ij}^\varepsilon y_i y_j^* + \sum_{i,j=1,2} c_{ij}^\varepsilon x_i y_j^* + \sum_{i,j=1,2} d_{ij}^\varepsilon y_i x_j^*.$$

Then, for the cross modes c_{ij}^ε, d_{ij}^ε, we have, with $u = c_{ij}^\varepsilon, d_{ij}^\varepsilon$, $i,j = 1,2$,

$$\mathbb{E}_\pi \left| \int_0^T \int_{\mathbb{R}^{2d}} u(t,x,\xi) f(t,x,\xi) dx d\xi dt \right| \le C_{f,T} \sqrt{\varepsilon}$$

for any function $f \in \mathcal{C}^1([0,T], \mathcal{S}(\mathbb{R}^{2d}))$. For the propagating modes, define $\alpha_+^\varepsilon = a_{11}^\varepsilon + a_{22}^\varepsilon$, $\alpha_-^\varepsilon = b_{11}^\varepsilon + b_{22}^\varepsilon$, and suppose $\mathbb{P}_\varepsilon$ is the family of probability measures induced by $(\alpha_+^\varepsilon, \alpha_-^\varepsilon)$ on $\mathcal{C}([0,T], H_M)$. Then, as $\varepsilon \to 0$, $\mathbb{P}_\varepsilon$ converges weakly to the probability measure $\mathbb{P} = \delta_{(\alpha_+,\alpha_-)}$, where (α_+,α_-) is the unique deterministic weak solution in $\mathcal{C}^0([0,T], L^2(\mathbb{R}^{2d}))^2$ to the following system of transport equations:

$$\begin{aligned} \partial_t \alpha_+ + \frac{c\xi \cdot \nabla_x \alpha_+}{\lambda_+(\xi)} &= \mathcal{T}(\alpha_+, \alpha_-), \\ \partial_t \alpha_- + \frac{c\xi \cdot \nabla_x \alpha_-}{\lambda_-(\xi)} &= \mathcal{T}(\alpha_-, \alpha_+). \end{aligned} \tag{2.16}$$

The initial conditions are given by $\alpha_\pm(0,x,\xi) = \mathrm{Tr}(\Pi_\pm W_0(0,x,\xi))$, and the scattering operator $\mathcal{T}$ is defined as

$$\begin{aligned} \mathcal{T}(\alpha_+,\alpha_-) =& \frac{e^2}{(2\pi)^d} \int_{\mathbb{R}^d} (\alpha_+(q) - \alpha_+(\xi)) \sum_{k=0}^3 \omega_k(\xi,q) \hat{R}_k(c\lambda_+(q) - c\lambda_+(\xi), q-\xi) dq \\ &+ \frac{e^2}{(2\pi)^d} \int_{\mathbb{R}^d} (\alpha_-(q) - \alpha_+(\xi)) \sum_{k=0}^3 \tilde{\omega}_k(\xi,q) \hat{R}_k(c\lambda_+(q) + c\lambda_+(\xi), q-\xi) dq, \end{aligned}$$

where $\omega_k(\xi,q) + \tilde{\omega}_k(\xi,q) = 1$ and

$$\begin{aligned} \omega_0(\xi,q) &= \frac{\lambda_+(q)\lambda_+(\xi) + \xi_1 q_1 + \xi_2 q_2 + \xi_3 q_3 + m_0^2 c^2}{2\lambda_+(q)\lambda_+(\xi)}, \\ \omega_1(\xi,q) &= \frac{\lambda_+(q)\lambda_+(\xi) + \xi_1 q_1 - \xi_2 q_2 - \xi_3 q_3 - m_0^2 c^2}{2\lambda_+(q)\lambda_+(\xi)}, \\ \omega_2(\xi,q) &= \frac{\lambda_+(q)\lambda_+(\xi) - \xi_1 q_1 + \xi_2 q_2 - \xi_3 q_3 - m_0^2 c^2}{2\lambda_+(q)\lambda_+(\xi)}, \\ \omega_3(\xi,q) &= \frac{\lambda_+(q)\lambda_+(\xi) - \xi_1 q_1 - \xi_2 q_2 + \xi_3 q_3 - m_0^2 c^2}{2\lambda_+(q)\lambda_+(\xi)}. \end{aligned}$$

Above, $\hat{R}_k$ denotes the Fourier transform of the correlation function R_k in both t and x.

Remark 2.5. The existence and uniqueness of a weak solution to (2.16) can be established in a straightforward manner since the scattering operator $\mathcal{T}$ is a bounded operator on $(L^2(\mathbb{R}^d))^2$, see for instance the results and methods of [13], Chapter 21, §2.

Remark 2.6. We assumed here that the components or the random electromagnetic field A were independent. Our results could be generalized to coupled components with a given, smooth, correlation function.

Remark 2.7. We see from the structure of the scattering operator that due to the temporal regularization, the energy $\sqrt{m_0^2c^2 + |\xi|^2}$ is no longer conserved. We also observe a coupling between the propagating modes α_+ and α_-.

3 Outline of the proof of Theorem 2.4

The proof of the convergence of the cross modes to zero is adressed in section 4.1 and is a direct consequence of the Wigner equation itself and the a priori L^2 bound of W_ε. For the propagating modes $\alpha^\varepsilon_\pm$, we use martingale techniques and the classical perturbed test functions method, see e.g. [6, 7, 8, 15]. The basic idea is to use the Markovian property of $A(t)$ and to construct appropriate test functions on which expansions will be performed. In order to illustrate the method, let us define first the operator $\mathcal{L}$ as

$$\mathcal{L}\lambda_0(x,\xi) = \frac{e^2}{(2\pi)^d}\sum_{k=0}^{3}\int_0^\infty\int_{\mathbb{R}^d} dqdr\,(L_{1,k} - L_{2,k})(r,q,x,\xi) \tag{3.1}$$

where

$$L_{1,k}(r,q,x,\xi) = \gamma^0\gamma^k\tilde{R}_k(r,q)e^{-rA_1(\xi-\frac{q}{2},-q)}\left[\lambda_0(\xi-q)\gamma^0\gamma^k - \gamma^0\gamma^k\lambda_0(\xi)\right]e^{-rA_2(\xi-\frac{q}{2},-q)}$$
$$L_{2,k}(r,q,x,\xi) = e^{-rA_1(\xi-\frac{q}{2},q)}\left[\lambda_0(\xi)\gamma^0\gamma^k - \gamma^0\gamma^k\lambda_0(\xi-q)\right]e^{-rA_2(\xi-\frac{q}{2},q)}\gamma^0\gamma^k\tilde{R}_k(r,q).$$

We introduce as well

$$\mathcal{A}\lambda_0 = \frac{1}{2}\sum_{k=1}^{3}\left(c\gamma^0\gamma^kD_k\lambda_0 + \frac{1}{2}D_k\lambda_0c\gamma^0\gamma^k\right) + \mathcal{L}\lambda_0, \tag{3.2}$$

with

$$\begin{aligned} A_1(\xi,p) &= \sum_{k=1}^{3}c\gamma^0\gamma^kP_k^*\left(i\xi+\frac{ip}{2}\right) - im_0c^2\gamma^0,\\ A_2(\xi,p) &= \sum_{k=1}^{3}P_k\left(i\xi-\frac{ip}{2}\right)c\gamma^0\gamma^k + im_0c^2\gamma^0. \end{aligned} \tag{3.3}$$

Above, e^{-rA_1} for instance, is the exponential of the matrix rA_1, and we recall that D_k is the derivative w.r.t. the variable x_k. The operator $\mathcal{L}$ is related to the scattering operator $\mathcal{T}$ after appropriate projection onto the propagating modes. We claim that for W_0 the limiting Wigner function and for any test function λ_0 such that $Q\lambda_0 = \lambda_0 Q$, we have that W_0 satisfies the equation

$$\langle \lambda_0, W_0\rangle(t) - \langle \lambda_0, W_0\rangle(0) = \int_0^t ds \langle (\partial_t + \mathcal{A})\lambda_0, W_0\rangle(s).$$

This motivates us to define the first approximate functional $\mathcal{G}^1_{\lambda_0} : \mathcal{C}^0([0,T], L^2(\mathbb{R}^{2d})) \to \mathcal{C}^0([0,T])$ as

$$\mathcal{G}^1_{\lambda_0}[W](t) = \langle \lambda_0, W\rangle(t) - \int_0^t ds \langle (\partial_t + \mathcal{A})\lambda_0, W\rangle(s).$$

Assume $\tilde{\mathbb{P}}_\varepsilon$ is the probability measure induced by W_ε on $(\mathcal{C}^0([0,T], L^2(\mathbb{R}^{2d})))^{(d+1)^2}$. We will show that, for all $t \in [0,T]$,

$$\left| \mathbb{E}^{\tilde{\mathbb{P}}_\varepsilon}\{\mathcal{G}^1_{\lambda_0}[W](t)\} - \langle \lambda_0, W^0_\varepsilon\rangle \right| \leq C\sqrt{\varepsilon}, \tag{3.4}$$

where we recall that W^0_ε is the Wigner transform of the initial data. Supposing then that $(\alpha^\varepsilon_+, \alpha^\varepsilon_-)$ converges in law to (α_+, α_-) in $\mathcal{C}^0([0,T], H_M)$ and choosing appropriate test functions λ_0 in (3.4) (that are defined further in section 4.2 and are related to projectors on α^ε_+ and α^ε_-; for these test functions, the measure $\tilde{\mathbb{P}}_\varepsilon$ reduces to the measure $\mathbb{P}_\varepsilon$ generated by $(\alpha^\varepsilon_+, \alpha^\varepsilon_-)$), we obtain from the above relation that $(\mathbb{E}\{\alpha_+\}, \mathbb{E}\{\alpha_-\})$ satisfies the transport equation system (2.16) by sending ε to zero. The proof of tightness of $(\alpha^\varepsilon_+, \alpha^\varepsilon_-)$ is addressed in section 4.4.

In order to show that the limit of $(\alpha^\varepsilon_+, \alpha^\varepsilon_-)$ is deterministic, we introduce a second functional. For any test function F_0 and G_0 satisfying the dispersion relation, i.e., $QF_0 = F_0 Q$, and $QG_0 = G_0 Q$, let $\mu_0 = F_0 \otimes G_0$. The second approximate functional $\mathcal{G}^2_{\mu_0}$: $\mathcal{C}^0([0,T], L^2(\mathbb{R}^{4d}))^{(d+1)^4} \to \mathcal{C}^0([0,T])$ is defined by

$$\mathcal{G}^2_{\mu_0}[W](t) = \langle F_0 \otimes G_0, W \otimes W\rangle(t) - \int_0^t ds \langle F_0 \otimes [(\partial_t + \mathcal{A})G_0] + [(\partial_t + \mathcal{A})F_0] \otimes G_0, W \otimes W\rangle(s).$$

Above, we have used the notation $(W \otimes W)(t, x_1, x_2, \xi_1, \xi_2) = W(t, x_1, \xi_1) \otimes W(t, x_2, \xi_2)$. We will show that, for all $t \in [0,T]$,

$$\left| \mathbb{E}^{\tilde{\mathbb{P}}_\varepsilon}\{\mathcal{G}^2_{\mu_0}[W](t)\} - \langle F_0 \otimes G_0, W^0_\varepsilon \otimes W^0_\varepsilon\rangle \right| \leq C\sqrt{\varepsilon}. \tag{3.5}$$

Letting $\varepsilon \to 0$, we will obtain that $\mathbb{E}\{\alpha_+ \otimes \alpha_+\} = \mathbb{E}\{\alpha_+\} \otimes \mathbb{E}\{\alpha_+\}$ and $\mathbb{E}\{\alpha_- \otimes \alpha_-\} = \mathbb{E}\{\alpha_-\} \otimes \mathbb{E}\{\alpha_-\}$, which implies that (α_+, α_-) is deterministic and the solution to the transport equation system (2.16).

We next go through the construction of test functions.

3.1 Construction of $\lambda_{1,\varepsilon}, \lambda_{2,\varepsilon}$

Denote by $\bar{\mathbb{P}}_\varepsilon$ the probability measure induced by the jointly Markov process $(W_\varepsilon(t), A(t/\varepsilon))$ on the space $\mathcal{C}^0([0,T], (L^2(\mathbb{R}^{2d}))^{(d+1)^2} \times \mathcal{E}^4)$. Here, $A = (A_0, A_1, A_2, A_3)$ is the random electromagnetic field constructed in section 2.2. Given $V \in \mathcal{E}^4$, $W \in (L^2(\mathbb{R}^{2d}))^{(d+1)^2}$and a smooth function F, the conditional expectation is defined as

$$\mathbb{E}^{\bar{\mathbb{P}}_\varepsilon}_{W,V,t}\{F(A(\tau), W(\tau))\} := \mathbb{E}^{\bar{\mathbb{P}}_\varepsilon}\{F(A(\tau), W(\tau))|A(t) = V, W(t) = W\}, \quad \tau \geq t.$$

The only functions we will be interested in are those of the form $F(V, W) = \langle \lambda(V), W\rangle$. The weak form of the infinitesimal generator of the Markov process induced by $(W_\varepsilon(t), A(t/\varepsilon))$ is then

$$\frac{d}{dh}\mathbb{E}^{\bar{\mathbb{P}}_\varepsilon}_{W,V,t}\left\{\langle \lambda(A(t+h)), W(t+h)\rangle\right\}|_{h=0} = \left\langle (\frac{1}{\varepsilon}\mathcal{Q} + \partial_t + \mathcal{A}_\varepsilon)\lambda(t), W\right\rangle. \tag{3.6}$$

Above, $\mathcal{Q}$ is the generator of A introduced in section 2.2 and $\varepsilon\mathcal{A}_\varepsilon := \mathcal{I}_\varepsilon + \mathcal{K}_\varepsilon$, the latter operators being defined below (2.12). Let $\lambda_\varepsilon = \lambda_0 + \sqrt{\varepsilon}\lambda_{1,\varepsilon} + \varepsilon\lambda_{2,\varepsilon}$ with $\lambda_0 \in \mathcal{C}^1([0,T], \mathcal{S}_d)$ satisfying $Q\lambda_0 = \lambda_0 Q$. Plugging this expressions into $(\frac{1}{\varepsilon}Q + \partial_t + \mathcal{A}_\varepsilon)\lambda_\varepsilon$ and equating like powers of ε, we find that the term of order $1/\varepsilon$ is equal to zero.

Considering the term of order $1/\sqrt{\varepsilon}$, we introduce the fast variable $z = x/\varepsilon$ and define $\lambda_{1,\varepsilon} = \lambda_1(A, t, x, x/\varepsilon, \xi)$, where $\lambda_1(A, t, x, z, \xi)$ solves

$$\mathcal{Q}\hat{\lambda}_1 - A_1\hat{\lambda}_1 - \hat{\lambda}_1 A_2 = F_1. \tag{3.7}$$

Above, $\hat{\lambda}_1 = \mathcal{F}_{z\to p}\lambda_1$, A_1, A_2 are defined in (3.3), and

$$F_1 = ieG_t(p)\lambda_0\left(\xi - \frac{p}{2}\right) - ie\lambda_0\left(\xi + \frac{p}{2}\right)G_t(p),$$
$$G_t(p) = \sum_{k=0}^{3}\gamma^0\gamma^k\tilde{A}_k\left(\frac{t}{\varepsilon}, p\right).$$

Since $\mathbb{E}_\pi\{F_1\} = 0$, (3.7) can be solved as explained in section 2.2, and its solution is given by

$$\lambda_1 = -\int_0^\infty e^{r\mathcal{Q}}\int_{\mathbb{R}^d}\frac{e^{iz\cdot p}}{(2\pi)^d}e^{-rA_1}F_1 e^{-rA_2}dpdr. \tag{3.8}$$

Above, and in the sequel, $e^{r\mathcal{Q}}$ is a scalar operator acting on each entries of the matrix that follows it, here the Fourier integral of $e^{-rA_1}F_1e^{-rA_2}$. We will show in section 4.2 that the entries of λ_1 are well defined as elements of $L^2(\mathbb{R}^{2d} \times \mathcal{E}^4)$.

Similarly, we can define $\lambda_{2,\varepsilon} = \lambda_2(A, t, x, x/\varepsilon, \xi)$ with $\lambda_2(A, t, x, z, \xi)$ solving

$$\mathcal{Q}\hat{\lambda}_2 - A_1\hat{\lambda}_2 - \hat{\lambda}_2 A_2 = F_2 + \mathcal{L}\lambda_0\,(2\pi)^d\delta(p), \tag{3.9}$$

where $\delta(p)$ is the Dirac measure and

$$F_2 = \frac{ie}{(2\pi)^d}\int_{\mathbb{R}^d}G_t(q)\hat{\lambda}_1(p-q, \xi - \frac{q}{2})dq - \frac{ie}{(2\pi)^d}\int_{\mathbb{R}^d}\hat{\lambda}_1(p-q, \xi + \frac{q}{2})G_t(q)dq.$$

The solvability comes from the fact that $\mathbb{E}_\pi\{F_2\} + \mathcal{L}\lambda_0\,(2\pi)^d\delta(p) = 0$, and the solution is given by

$$\lambda_2 = -\int_0^\infty e^{r\mathcal{Q}}\int_{\mathbb{R}^d}\frac{e^{iz\cdot p}}{(2\pi)^d}e^{-rA_1}(F_2 + \mathcal{L}\lambda_0(2\pi)^d\,\delta(p))e^{-rA_2}dpdr. \tag{3.10}$$

Again, we will show in section 4.2 that λ_2 is well defined.

Now that we have the test functions $\lambda_{1,\varepsilon}$ and $\lambda_{2,\varepsilon}$, we know by (3.6) that

$$\mathcal{G}^{1,\varepsilon}_{\lambda_\varepsilon}[W](t) := \langle \lambda_\varepsilon, W\rangle(t) - \int_0^t ds \left\langle (\frac{1}{\varepsilon}\mathcal{Q} + \partial_t + \mathcal{A}_\varepsilon)\lambda_\varepsilon, W \right\rangle(s)$$

is a $\bar{\mathbb{P}}_\varepsilon$-martingale, so that

$$\mathbb{E}^{\bar{\mathbb{P}}_\varepsilon}\{\mathcal{G}^{1,\varepsilon}_{\lambda_\varepsilon}[W](t)\} = \mathbb{E}^{\bar{\mathbb{P}}_\varepsilon}\{\mathcal{G}^{1,\varepsilon}_{\lambda_\varepsilon}[W](0)\} = \mathbb{E}^{\bar{\mathbb{P}}_\varepsilon}\{\langle \lambda_\varepsilon, W^0_\varepsilon\rangle\}. \tag{3.11}$$

With $\lambda_\varepsilon = \lambda_0 + \sqrt{\varepsilon}\lambda_{1,\varepsilon} + \varepsilon\lambda_{2,\varepsilon}$, it follows that

$$\mathcal{G}^{1,\varepsilon}_{\lambda_\varepsilon}[W](t) = \mathcal{G}^{1}_{\lambda_0}[W](t) + \langle C_1, W\rangle(t) + \int_0^t ds \langle C_2 + C_3 + C_4, W\rangle(s), \tag{3.12}$$

where the correctors C_i are given by

$$\begin{aligned}
C_1 &= \sqrt{\varepsilon}\lambda_{1,\varepsilon} + \varepsilon\lambda_{2,\varepsilon},\\
C_2 &= -\partial_t(\sqrt{\varepsilon}\lambda_{1,\varepsilon} + \varepsilon\lambda_{2,\varepsilon}),\\
C_3 &= \sqrt{\varepsilon} ie \sum_{k=0}^{3} \gamma^0\gamma^k \mathcal{K}^k_\varepsilon \lambda_{2,\varepsilon} - \sqrt{\varepsilon} ie \sum_{k=0}^{3} \tilde{\mathcal{K}}^k_\varepsilon \lambda_{2,\varepsilon}\gamma^0\gamma^k,\\
C_4 &= \frac{1}{\varepsilon}\sum_{k=1}^{3} c\gamma^0\gamma^k P^*_k\left(\frac{\varepsilon D_x}{2}\right)(\sqrt{\varepsilon}\lambda_{1,\varepsilon} + \varepsilon\lambda_{2,\varepsilon}) + \frac{1}{\varepsilon}\sum_{k=1}^{3}(\sqrt{\varepsilon}\lambda_{1,\varepsilon} + \varepsilon\lambda_{2,\varepsilon}) P_k\left(-\frac{\varepsilon D_x}{2}\right) c\gamma^0\gamma^k.
\end{aligned}$$

Remark 3.1. In C_4, the derivative D_x is with respect to the slow variable x.

In order to prove (3.4), we will show in section 4.2 that the correctors are small:

$$\|C_i(t)\|_{L^2_{x,\xi,\pi}} \le C_{\lambda_0,T}\sqrt{\varepsilon}, \qquad i = 1, \cdots, 4,$$

uniformly in $t \in [0,T]$. Above, we have used the notation, for both matrix-valued and scalar functions f defined on $\mathbb{R}^d \times \mathbb{R}^d \times \mathcal{E}^4$:

$$\|f\|^2_{L^2_{x,\xi,\pi}} = \mathbb{E}_\pi \|f\|^2_{L^2(\mathbb{R}^{2d})}.$$

3.2 Construction of $\mu_{1,\varepsilon}, \mu_{2,\varepsilon}$

First of all, we derive the equation satisfied by $W_\varepsilon \otimes W_\varepsilon(t, x_1, x_2, \xi_1, \xi_2)$. Define the 16×16 matrices

$$\Gamma^k = \begin{pmatrix} \gamma^k_{11} I_4 & \gamma^k_{12} I_4 & \gamma^k_{13} I_4 & \gamma^k_{14} I_4 \\ \gamma^k_{21} I_4 & \gamma^k_{22} I_4 & \gamma^k_{23} I_4 & \gamma^k_{24} I_4 \\ \gamma^k_{31} I_4 & \gamma^k_{32} I_4 & \gamma^k_{33} I_4 & \gamma^k_{34} I_4 \\ \gamma^k_{41} I_4 & \gamma^k_{42} I_4 & \gamma^k_{43} I_4 & \gamma^k_{44} I_4 \end{pmatrix}, \qquad \tilde{\Gamma}^k = \begin{pmatrix} \gamma^k & 0 & 0 & 0 \\ 0 & \gamma^k & 0 & 0 \\ 0 & 0 & \gamma^k & 0 \\ 0 & 0 & 0 & \gamma^k \end{pmatrix}.$$

We have

$$\begin{aligned}\varepsilon\partial_t W_\varepsilon\otimes W_\varepsilon &= -im_0c^2\big(\Gamma^0+\tilde\Gamma^0\big)W_\varepsilon\otimes W_\varepsilon + im_0c^2 W_\varepsilon\otimes W_\varepsilon\big(\Gamma^0+\tilde\Gamma^0\big)\\ &\quad -\sum_{k=1}^3\left[c\Gamma^0\Gamma^k P_{k1}\left(i\xi+\frac{\varepsilon D}{2}\right)+c\tilde\Gamma^0\tilde\Gamma^k P_{k2}\left(i\xi+\frac{\varepsilon D}{2}\right)\right]W_\varepsilon\otimes W_\varepsilon\\ &\quad -W_\varepsilon\otimes W_\varepsilon\sum_{k=1}^3\left[c\Gamma^0\Gamma^k P_{k1}^*\left(i\xi-\frac{\varepsilon D}{2}\right)+c\tilde\Gamma_0\tilde\Gamma^k P_{k2}^*\left(i\xi-\frac{\varepsilon D}{2}\right)\right]\\ &\quad +ie\sqrt{\varepsilon}\sum_{k=0}^3\left(\Gamma^0\Gamma^k\mathcal{K}_{\varepsilon1}^k+\tilde\Gamma^0\tilde\Gamma^k\mathcal{K}_{\varepsilon2}^k\right)W_\varepsilon\otimes W_\varepsilon\\ &\quad -ie\sqrt{\varepsilon}W_\varepsilon\otimes W_\varepsilon\sum_{k=0}^3\left(\Gamma^0\Gamma^k\tilde{\mathcal{K}}_{\varepsilon1}^k+\tilde\Gamma^0\tilde\Gamma^k\tilde{\mathcal{K}}_{\varepsilon2}^k\right)=0, \end{aligned}\tag{3.13}$$

where $P_{ki}, P_{ki}^*, \mathcal{K}_{\varepsilon i}^k, \tilde{\mathcal{K}}_{\varepsilon i}^k$ are the same operators as before but acting now on the variables $(x_i,\xi_i), i=1,2$.

Writing (3.13) as $\partial_t(W_\varepsilon\otimes W_\varepsilon)=\mathcal{B}_\varepsilon^*(W_\varepsilon\otimes W_\varepsilon)$, the weak form of the infinitesimal generator of the Markov process $\langle\mu, W_\varepsilon\otimes W_\varepsilon\rangle$ is

$$\frac{d}{dh}\mathbb{E}^{\bar{\mathbb{P}}_\varepsilon}_{W,V,t}\{\langle\mu(A(t+h)),W\otimes W(t+h)\rangle\}|_{h=0}=\left\langle(\frac{1}{\varepsilon}\mathcal{Q}+\partial_t+\mathcal{B}_\varepsilon)\mu, W\otimes W\right\rangle$$

Define $\mu_\varepsilon=\mu_0+\sqrt{\varepsilon}\mu_{1,\varepsilon}+\varepsilon\mu_{2,\varepsilon}$. Then

$$\mathcal{G}^{2,\varepsilon}_{\mu_0}[W](t):=\langle\mu_\varepsilon,W\otimes W\rangle(t)-\int_0^t ds\left\langle(\frac{1}{\varepsilon}\mathcal{Q}+\partial_t+\mathcal{B}_\varepsilon)\mu_\varepsilon, W\otimes W\right\rangle(s)$$

is a $\bar{\mathbb{P}}_\varepsilon$-martingale. Consider $(\frac{1}{\varepsilon}\mathcal{Q}+\partial_t+\mathcal{B}_\varepsilon)\mu_\varepsilon$, and after expanding in ε, the term of order $1/\varepsilon$ is

$$\begin{aligned}(I)=&-\frac{1}{\varepsilon}\sum_{k=1}^3 c\left[\Gamma^0\Gamma^kP_{k1}^*(i\xi)+\tilde\Gamma^0\tilde\Gamma^kP_{k2}^*(i\xi)\right]\mu_0-\frac{1}{\varepsilon}\mu_0\sum_{k=1}^3 c\left[\Gamma^0\Gamma^kP_{k1}(i\xi)+\tilde\Gamma^0\tilde\Gamma^kP_{k2}(i\xi)\right]\\ &+\frac{1}{\varepsilon}im_0c^2(\Gamma^0+\tilde\Gamma^0)\mu_0-\frac{1}{\varepsilon}im_0c^2\mu_0(\Gamma^0+\tilde\Gamma^0).\end{aligned}$$

If we choose $\mu_0=F_0\otimes G_0$ with F_0, G_0 both satisfying the dispersion relation, i.e.,

$$QF_0=F_0Q,\quad QG_0=G_0Q,$$

we can check that $(I)=0$. For cancelling the term of order $1/\sqrt{\varepsilon}$, we introduce similarly fast variables $z_1=x_1/\varepsilon, z_2=x_2/\varepsilon$, and define

$$\mu_{1,\varepsilon}(A,t,x_1,x_2,\xi_1,\xi_2)=\mu_1\left(A,t,x_1,\frac{x_1}{\varepsilon},x_2,\frac{x_2}{\varepsilon},\xi_1,\xi_2\right),$$

with μ_1 solving

$$\begin{aligned}\mathcal{Q}\mu_1-&\sum_{k=1}^{3}c\left[\Gamma^0\Gamma^kP_{k1}^*\left(i\xi+\frac{D_z}{2}\right)+\tilde{\Gamma}^0\tilde{\Gamma}^kP_{k2}^*\left(i\xi+\frac{D_z}{2}\right)\right]\mu_1+im_0c^2\left(\Gamma^0+\tilde{\Gamma}^0\right)\mu_1\\ &-\mu_1\sum_{k=1}^{3}c\left[\Gamma^0\Gamma^kP_{k1}\left(i\xi-\frac{D_z}{2}\right)+\tilde{\Gamma}^0\tilde{\Gamma}^kP_{k2}\left(i\xi-\frac{D_z}{2}\right)\right]-im_0c^2\mu_1\left(\Gamma^0+\tilde{\Gamma}^0\right)\\ &=ie\sum_{k=0}^{3}\left(\Gamma^0\Gamma^k\mathcal{K}_{\varepsilon1}^k+\tilde{\Gamma}^0\tilde{\Gamma}^k\mathcal{K}_{\varepsilon2}^k\right)\mu_0-ie\mu_0\sum_{k=0}^{3}\left(\Gamma^0\Gamma^k\tilde{\mathcal{K}}_{\varepsilon1}^k+\tilde{\Gamma}^0\tilde{\Gamma}^k\tilde{\mathcal{K}}_{\varepsilon2}^k\right).\end{aligned} \tag{3.14}$$

We can check that the solution $\mu_1=F_1\otimes G_0+F_0\otimes G_1$, where F_1,G_1 solve (3.7) with λ_0 replaced by F_0,G_0 respectively.

Remark 3.2. In (3.14), we replace x_i/ε by z_i in the operators $\mathcal{K}_{\varepsilon i}^k,\tilde{\mathcal{K}}_{\varepsilon i}^k$.

In the same way, we define

$$\mu_{2,\varepsilon}(\tilde{A},t,x_1,x_2,\xi_1,\xi_2)=\mu_2\left(\tilde{A},t,x_1,\frac{x_1}{\varepsilon},x_2,\frac{x_2}{\varepsilon},\xi_1,\xi_2\right),$$

and μ_2 solves

$$\begin{aligned}\mathcal{Q}\mu_2-&\sum_{k=1}^{3}c\left[\Gamma^0\Gamma^kP_{k1}^*\left(i\xi+\frac{D_z}{2}\right)+\tilde{\Gamma}^0\tilde{\Gamma}^kP_{k2}^*\left(i\xi+\frac{D_z}{2}\right)\right]\mu_2+im_0c^2\left(\Gamma^0+\tilde{\Gamma}^0\right)\mu_2\\ &-\mu_2\sum_{k=1}^{3}c\left[\Gamma^0\Gamma^kP_{k1}\left(i\xi-\frac{D_z}{2}\right)+\tilde{\Gamma}^0\tilde{\Gamma}^kP_{k2}\left(i\xi-\frac{D_z}{2}\right)\right]-im_0c^2\mu_2\left(\Gamma^0+\tilde{\Gamma}^0\right)\\ &=ie\sum_{k=0}^{3}\left(\Gamma^0\Gamma^k\mathcal{K}_{\varepsilon1}^k+\tilde{\Gamma}^0\tilde{\Gamma}^k\mathcal{K}_{\varepsilon2}^k\right)\mu_1-ie\mu_1\sum_{k=0}^{3}\left(\Gamma^0\Gamma^k\tilde{\mathcal{K}}_{\varepsilon1}^k+\tilde{\Gamma}^0\tilde{\Gamma}^k\tilde{\mathcal{K}}_{\varepsilon2}^k\right)+\mathbf{L}\mu_0.\end{aligned} \tag{3.15}$$

We define $\mathbf{L}\mu_0$ such that the r.h.s. of (3.15) has mean zero so that the equation is solvable, i.e.,

$$\mathbf{L}\mu_0=\mathbb{E}_\pi\left\{-ie\sum_{k=0}^{3}\left(\Gamma^0\Gamma^k\mathcal{K}_{\varepsilon1}^k+\tilde{\Gamma}^0\tilde{\Gamma}^k\mathcal{K}_{\varepsilon2}^k\right)\mu_1+ie\mu_1\sum_{k=0}^{3}\left(\Gamma^0\Gamma^k\tilde{\mathcal{K}}_{\varepsilon1}^k+\tilde{\Gamma}^0\tilde{\Gamma}^k\tilde{\mathcal{K}}_{\varepsilon2}^k\right)\right\}.$$

Injecting $\mu_1=F_1\otimes G_0+F_0\otimes G_1$, we decompose $\mathbf{L}\mu_0$ into two parts, i.e., $\mathbf{L}\mu_0=\mathbf{L}_1\mu_0+\mathbf{L}_2\mu_0$, where

$$\begin{aligned}\mathbf{L}_1\mu_0=&ie\mathbb{E}_\pi\left\{-\sum_{k=0}^{3}\Gamma^0\Gamma^k\mathcal{K}_{\varepsilon1}^kF_1\otimes G_0+F_1\otimes G_0\sum_{k=0}^{3}\Gamma^0\Gamma^k\tilde{\mathcal{K}}_{\varepsilon1}^k\right\}\\ &-ie\mathbb{E}_\pi\left\{\sum_{k=0}^{3}\tilde{\Gamma}^0\tilde{\Gamma}^k\mathcal{K}_{\varepsilon2}^kF_0\otimes G_1+F_0\otimes G_1\sum_{k=0}^{3}\tilde{\Gamma}^0\tilde{\Gamma}^k\tilde{\mathcal{K}}_{\varepsilon2}^k\right\}\\ =&(\mathcal{L}F_0)\otimes G_0+F_0\otimes(\mathcal{L}G_0),\end{aligned}$$

where $\mathcal{L}$ is defined in (3.1) and

$$\begin{aligned}\mathbf{L}_2\mu_0 =& ie\mathbb{E}_\pi\left\{-\sum_{k=0}^3\Gamma^0\Gamma^k\mathcal{K}_{\varepsilon 1}^k F_0\otimes G_1 + F_0\otimes G_1\sum_{k=0}^3\Gamma^0\Gamma^k\tilde{\mathcal{K}}_{\varepsilon 1}^k\right\}\\ &- ie\mathbb{E}_\pi\left\{\sum_{k=0}^3\tilde{\Gamma}^0\tilde{\Gamma}^k\mathcal{K}_{\varepsilon 2}^k F_1\otimes G_0 + F_1\otimes G_0\sum_{k=0}^3\tilde{\Gamma}^0\tilde{\Gamma}^k\tilde{\mathcal{K}}_{\varepsilon 2}^k\right\}.\end{aligned} \tag{3.16}$$

By this decomposition, we can write $\mu_2 = F_0\otimes G_2 + F_2\otimes G_0 + \tilde{\mu}_2$, where F_2, G_2 solve (3.9) with λ_0 replaced by F_0, G_0 and $\tilde{\mu}_2$ solves the following equation

$$\begin{aligned}\mathcal{Q}\tilde{\mu}_2 -& \sum_{k=1}^3 c\left[\Gamma^0\Gamma^k P_{k1}^*\left(i\xi+\frac{D_z}{2}\right)+\tilde{\Gamma}^0\tilde{\Gamma}^k P_{k2}^*\left(i\xi+\frac{D_z}{2}\right)\right]\tilde{\mu}_2 + im_0c^2\left(\Gamma^0+\tilde{\Gamma}^0\right)\tilde{\mu}_2\\ &-\tilde{\mu}_2\sum_{k=1}^3 c\left[\Gamma^0\Gamma^k P_{k1}\left(i\xi-\frac{D_z}{2}\right)+\tilde{\Gamma}^0\tilde{\Gamma}^k P_{k2}\left(i\xi-\frac{D_z}{2}\right)\right] - im_0c^2\tilde{\mu}_2\left(\Gamma^0+\tilde{\Gamma}^0\right)\\ &= ie\left(\sum_{k=0}^3\Gamma^0\Gamma^k\mathcal{K}_{\varepsilon 1}^k F_0\otimes G_1 - F_0\otimes G_1\sum_{k=0}^3\Gamma^0\Gamma^k\tilde{\mathcal{K}}_{\varepsilon 1}^k\right)\\ &\quad + ie\left(\sum_{k=0}^3\tilde{\Gamma}^0\tilde{\Gamma}^k\mathcal{K}_{\varepsilon 2}^k F_1\otimes G_0 - F_1\otimes G_0\sum_{k=0}^3\tilde{\Gamma}^0\tilde{\Gamma}^k\tilde{\mathcal{K}}_{\varepsilon 2}^k\right) + \mathbf{L}_2\mu_0.\end{aligned}$$

With $\mu_\varepsilon = \mu_0 + \sqrt{\varepsilon}\mu_{1,\varepsilon} + \varepsilon\mu_{2,\varepsilon}$, we have

$$\mathcal{G}_{\mu_0}^{2,\varepsilon}[W](t) = \mathcal{G}_{\mu_0}^2[W](t) + \langle\bar{C}_1, W\otimes W\rangle(t) + \int_0^t ds\left\langle\sum_{i=2}^5\bar{C}_i, W\otimes W\right\rangle(s),$$

and the correctors $\bar{C}_i$ are

$$\begin{aligned}\bar{C}_1 &= \sqrt{\varepsilon}\mu_{1,\varepsilon}+\varepsilon\mu_{2,\varepsilon},\\ \bar{C}_2 &= -\partial_t(\sqrt{\varepsilon}\mu_{1,\varepsilon}+\varepsilon\mu_{2,\varepsilon}),\\ \bar{C}_3 &= ie\sqrt{\varepsilon}\sum_{k=0}^3\left(\Gamma^0\Gamma^k\mathcal{K}_{\varepsilon 1}^k+\tilde{\Gamma}^0\tilde{\Gamma}^k\mathcal{K}_{\varepsilon 2}^k\right)\mu_{2,\varepsilon} - ie\sqrt{\varepsilon}\mu_{2,\varepsilon}\sum_{k=0}^3\left(\Gamma^0\Gamma^k\tilde{\mathcal{K}}_{\varepsilon 1}^k+\tilde{\Gamma}^0\tilde{\Gamma}^k\tilde{\mathcal{K}}_{\varepsilon 2}^k\right),\\ \bar{C}_4 &= -\mathbf{L}_2\mu_0,\\ \bar{C}_5 &= \frac{1}{\varepsilon}\sum_{k=1}^3 c\left[\Gamma^0\Gamma^k P_{k1}^*\left(\frac{\varepsilon D_x}{2}\right)+\tilde{\Gamma}^0\tilde{\Gamma}^k P_{k2}^*\left(\frac{\varepsilon D_x}{2}\right)\right](\sqrt{\varepsilon}\mu_{1,\varepsilon}+\varepsilon\mu_{2,\varepsilon})\\ &\quad+\frac{1}{\varepsilon}(\sqrt{\varepsilon}\mu_{1,\varepsilon}+\varepsilon\mu_{2,\varepsilon})\sum_{k=1}^3 c\left[\Gamma^0\Gamma^k P_{k1}\left(-\frac{\varepsilon D_x}{2}\right)+\tilde{\Gamma}^0\tilde{\Gamma}^k P_{k2}\left(-\frac{\varepsilon D_x}{2}\right)\right].\end{aligned}$$

Remark 3.3. In $\bar{C}_5$, D_x is with respect to the slow variable.

We will prove in section 4.3 that the correctors are small in L^2.

4 Proof of the main theorem

We first prove the convergence of the cross modes in section 4.1, and address the convergence of the propagating modes in section 4.2 for the expectation and in section 4.3 for the second moment. In section 4.4, we prove the tightness and finish the proof of the main theorem.

From now on, we use the notation $a \lesssim b$ when there exists a constant C such that $a \leq Cb$.

4.1 Cross modes

We show here that the cross modes converge weakly to zero. Rewriting the weak formulation (2.12) for a time-dependent test function $F \in \mathcal{C}^1([0,T], \mathcal{S}_d)$, we have, for all $t \in [0,T]$,

$$\varepsilon\langle F, W_\varepsilon\rangle(t) - \varepsilon\langle F, W_\varepsilon\rangle(0) = \int_0^t ds \langle(\varepsilon\partial_t + \mathcal{I}_\varepsilon + \mathcal{K}_\varepsilon)F, W_\varepsilon\rangle(s),$$

where $\mathcal{I}_\varepsilon$ and $\mathcal{K}_\varepsilon$ are defined below (2.12). We rewrite first $\mathcal{I}_\varepsilon$ as

$$\begin{aligned}\mathcal{I}_\varepsilon F &= \left(\sum_{k=1}^3 c\gamma^0\gamma^k i\xi_k + im_0c^2\gamma^0\right)F - F\left(\sum_{k=1}^3 i\xi_k c\gamma^0\gamma^k + im_0c^2\gamma^0\right)\\ &\quad - \varepsilon\sum_{k=1}^3 c\gamma^0\gamma^k P_k^*\left(\frac{D}{2}\right)F - \varepsilon F\sum_{k=1}^3 P_k\left(-\frac{D}{2}\right)c\gamma^0\gamma^k \\ &:= QF - FQ + \varepsilon R_0 F,\end{aligned} \tag{4.1}$$

where Q is the dispersion matrix defined in (2.14). Then, using (2.2) and the regularity of R_k, we find

$$\begin{aligned}\|\mathcal{K}_\varepsilon^k F_{mn}(s,\cdot)\|^2_{L^2_{x,\xi,\pi}} &= \int_{\mathbb{R}^{3d}} dxdydy' R_k(0,(y-y')/2)\mathcal{F}_2F_{mn}(x,y)(\mathcal{F}_2F_{mn})^*(s,x,y')\\ &\lesssim \|F_{mn}(s,\cdot)\|^2_{L^2(\mathbb{R}^{2d})},\end{aligned}$$

with a similar estimate for $\tilde{\mathcal{K}}_\varepsilon^k$, and this implies that $\mathcal{K}_\varepsilon/\sqrt{\varepsilon}$ is a bounded operator on L^2. Thanks to the estimate (2.13), we therefore obtain from the calculations above

$$\mathbb{E}_\pi\left|\int_0^T ds\langle QF - FQ, W_\varepsilon\rangle(s)\right| \lesssim \sqrt{\varepsilon}.$$

Recalling that

$$W_\varepsilon = \sum_{i,j=1,2} a_{ij}^\varepsilon x_i x_j^* + \sum_{i,j=1,2} b_{ij}^\varepsilon y_i y_j^* + \sum_{i,j=1,2} c_{ij}^\varepsilon x_i y_j^* + \sum_{i,j=1,2} d_{ij}^\varepsilon y_i x_j^*,$$

with a similar decomposition for the matrix F, we have

$$QF - FQ = 2\lambda_+ \sum_{i,j=1,2} c_{ij}^{F,\varepsilon} x_i y_j^* + 2\lambda_- \sum_{i,j=1,2} d_{ij}^{F,\varepsilon} y_i x_j^*.$$

If we choose $c_{ij}^{F,\varepsilon} = f/\lambda_+$ for some function $f \in \mathcal{S}(\mathbb{R}^{2d})$, and $a_{ij}^{F,\varepsilon} = b_{ij}^{F,\varepsilon} = d_{ij}^{F,\varepsilon} = 0$, we have

$$\mathbb{E}_\pi \left| \int_0^T ds \langle f, c_{ij}^\varepsilon \rangle(s) \right| \lesssim \sqrt{\varepsilon}.$$

The same is true for d_{ij}^ε if we choose other test functions. We can therefore conclude that the cross modes c_{ij}^ε and d_{ij}^ε converge to zero weakly.

Remark 4.1. We do not necessarily have the weak convergence of $c_{ij}^\varepsilon, d_{ij}^\varepsilon$ as processes in $\mathcal{C}^0([0,T], L^2(\mathbb{R}^{2d}))$ here. The following is a heuristic argument: rewrite the equation (2.11) satisfied by W_ε as

$$\begin{aligned}
&\partial_t W_\varepsilon + \frac{1}{\varepsilon}\left[\sum_{k=1}^3 c\gamma^0\gamma^k P_k(i\xi) W_\varepsilon + \sum_{k=1}^3 W_\varepsilon P_k^*(i\xi) c\gamma^0\gamma^k + i m_0 c^2(\gamma^0 W_\varepsilon - W_\varepsilon \gamma^0)\right] + \dots \\
&= \partial_t W_\varepsilon + \frac{ic}{\varepsilon}(Q W_\varepsilon - W_\varepsilon Q) + \dots = 0.
\end{aligned}$$

All terms that are not reported here are those of order 1 or $1/\sqrt{\varepsilon}$. So we have

$$x_i^* \partial_t W_\varepsilon y_j + \frac{ic}{\varepsilon} x_i^*(Q W_\varepsilon - W_\varepsilon Q) y_j + \dots = 0,$$

which leads to

$$\partial_t c_{ij}^\varepsilon + \frac{2ic\lambda_+}{\varepsilon} c_{ij}^\varepsilon + \dots = 0.$$

We see that as $\varepsilon \to 0$, the cross modes c_{ij}^ε are highly oscillatory in the time variable, and go to zero weakly in spacetime. The same happens to d_{ij}^ε. From this perspective, we can only expect the weak convergence in time as a L^2 function rather than as a process in $\mathcal{C}^0([0,T], L^2(\mathbb{R}^{2d}))$.

4.2 Convergence of the expectation

We have first the following lemmas concerning $\lambda_{1,\varepsilon}$ and $\lambda_{2,\varepsilon}$.

Lemma 4.2. *$\|\lambda_{1,\varepsilon}(t)\|_{L^2_{x,\xi,\pi}} \le C_{\lambda_0,T}$ uniformly for $t \in [0,T]$.*

Proof. Recall from (3.8), we have $\lambda_{1,\varepsilon}(t,x,\xi) = \lambda_1(t,x,x/\varepsilon,\xi)$, and

$$\begin{aligned}
&\lambda_1(t,x,z,\xi) \\
&= -\int_0^\infty dr e^{r\mathcal{Q}} \int_{\mathbb{R}^d} \frac{dp e^{iz\cdot p}}{(2\pi)^d} e^{-rA_1(\xi,p)} ie \left[G_t(p)\lambda_0(\xi - \frac{p}{2}) - \lambda_0(\xi + \frac{p}{2}) G_t(p)\right] e^{-rA_2(\xi,p)}.
\end{aligned}$$

where A_1, A_2 are defined in (3.3) and $G_t(p)$ is defined below (3.7).

Take one of the two terms above, for example

$$\begin{aligned}
(I) &= -\int_0^\infty dr e^{r\mathcal{Q}} \int_{\mathbb{R}^d} \frac{dp \tilde{A}_k(\frac{t}{\varepsilon}, p)}{(2\pi)^d} e^{iz\cdot p} ie e^{-rA_1(\xi,p)} \gamma^0\gamma^k \lambda_0(\xi - \frac{p}{2}) e^{-rA_2(\xi,p)} \\
&:= -\int_0^\infty dr e^{r\mathcal{Q}} f(r,t,x,\xi).
\end{aligned}$$

The Minkowski integral inequality together with (2.5) yields

$$\|(I)_{mn}\|_{L^2_{x,\xi,\pi}} \leq \int_0^\infty dr \|e^{r\mathcal{Q}}(f(t,r,\cdot,\cdot))_{mn}\|_{L^2_{x,\xi,\pi}} \leq \int_0^\infty dr e^{-\gamma_* r/2}\|(f(t,r,\cdot,\cdot))_{mn}\|_{L^2_{x,\xi,\pi}}.$$

Above, we used the fact that $e^{r\mathcal{Q}}f = e^{r\mathcal{Q}_k}f$ since f only depends on A_k. According to (2.2), we have

$$\begin{aligned}\|(f(t,r,\cdot,\cdot))_{mn}\|^2_{L^2_{x,\xi,\pi}} \quad &= \quad e^2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} dxd\xi\left[\int_{\mathbb{R}^d}\frac{\tilde{R}_k(0,p)}{(2\pi)^d}\left(e^{-rA_1(\xi,p)}\gamma^0\gamma^k\lambda_0(x,\xi-\frac{p}{2})e^{-rA_2(\xi,p)}\right)_{mn}\right.\\ &\qquad\left.\times\left(e^{-rA_1(\xi,p)}\gamma^0\gamma^k\lambda_0(x,\xi-\frac{p}{2})e^{-rA_2(\xi,p)}\right)^*_{mn}\right].\end{aligned}$$

Since the dispersion matrix Q is hermitian and $A_1(\xi,p) = -icQ(\xi+\frac{p}{2})$, $A_2(\xi,p) = icQ(\xi-\frac{p}{2})$, it follows that any element in $e^{-rA_1(\xi,p)}$ and $e^{-rA_2(\xi,p)}$ is uniformly bounded, and thus

$$\left|\left[e^{-rA_1(\xi,p)}\gamma^0\gamma^k\lambda_0(x,\xi-\frac{p}{2})e^{-rA_2(\xi,p)}\right]_{mn}\right| \lesssim \sum_{i,j}\left|\left[\lambda_0(x,\xi-\frac{p}{2})\right]_{ij}\right|. \tag{4.2}$$

Since $\tilde{R}\in\mathcal{S}(\mathbb{R}\times\mathbb{R}^d)$ and $\lambda_0\in\mathcal{S}_d$, it follows that $(I)_{mn}$ is bounded in $L^2_{x,\xi,\pi}$ uniformly in time. The proof is similar for the other terms in the definition of λ_1. □

Lemma 4.3. $\|\lambda_{2,\varepsilon}(t)\|_{L^2_{x,\xi,\pi}} \leq C_{\lambda_0,T}$ *for* $t\in[0,T]$.

Proof. The proof is a little more involved than the one for $\lambda_{1,\varepsilon}$. Recall that

$$\mathcal{Q}\hat{\lambda}_2 - A_1\hat{\lambda}_2 - \hat{\lambda}_2A_2 = F_2 - \mathbb{E}_\pi\{F_2\},$$

where

$$F_2 = \frac{ie}{(2\pi)^d}\int_{\mathbb{R}^d}G_t(q)\hat{\lambda}_1(p-q,\xi-\frac{q}{2})dq - \frac{ie}{(2\pi)^d}\int_{\mathbb{R}^d}\hat{\lambda}_1(p-q,\xi+\frac{q}{2})G_t(q)dq,$$

and

$$\hat{\lambda}_1(p,\xi) = -\int_0^\infty dre^{r\mathcal{Q}}e^{-rA_1(\xi,p)}\left[ieG_t(p)\lambda_0(\xi-\frac{p}{2}) - ie\lambda_0(\xi+\frac{p}{2})G_t(p)\right]e^{-rA_2(\xi,p)}.$$

According to (2.6), this can be recast as

$$\hat{\lambda}_1(p,\xi) = -ie\sum_{k=0}^3\int_0^\infty dre^{-r\gamma_k(p)}\tilde{A}_k(t,p)e^{-rA_1(\xi,p)}\left[\gamma^0\gamma^k\lambda_0(\xi-\frac{p}{2}) - \lambda_0(\xi+\frac{p}{2})\gamma^0\gamma^k\right]e^{-rA_2(\xi,p)}.$$

We can then write $F_2 = \frac{-e^2}{(2\pi)^d}(B_1 - B_2)$, where

$$\begin{aligned}B_1 \quad &= \quad -\sum_{k=0}^3\int_{\mathbb{R}^d}dqG_t(q)\int_0^\infty dre^{-r\gamma_k(p-q)}\tilde{A}_k(t,p-q)e^{-rA_1(\xi-\frac{q}{2},p-q)}\\ &\qquad\times\left[\gamma^0\gamma^k\lambda_0(\xi-\frac{p}{2}) - \lambda_0(\xi+\frac{p}{2}-q)\gamma^0\gamma^k\right]e^{-rA_2(\xi-\frac{q}{2},p-q)}\\ &:= \quad B_{1,1} - B_{1,2},\end{aligned} \tag{4.3}$$

and

$$
\begin{aligned}
B_2 &= -\sum_{k=0}^{3}\int_{\mathbb{R}^d} dq \int_0^\infty dr e^{-r\gamma_k(p-q)}\tilde{A}_k(t,p-q)e^{-rA_1(\xi+\frac{q}{2},p-q)} \\
&\qquad \times\left[\gamma^0\gamma^k\lambda_0(\xi+q-\frac{p}{2})-\lambda_0(\xi+\frac{p}{2})\gamma^0\gamma^k\right]e^{-rA_2(\xi+\frac{q}{2},p-q)}G_t(q) \\
&:= B_{2,1}-B_{2,2}.
\end{aligned}
$$

Hence, λ_2 admits the expression

$$
\lambda_2(t,x,z,\xi) = -\int_0^\infty dr e^{r\mathcal{Q}}\int_{\mathbb{R}^d}\frac{dp e^{iz\cdot p}}{(2\pi)^d}e^{-rA_1}\frac{-e^2}{(2\pi)^d}(B_1-B_2-\mathbb{E}_\pi B_1+\mathbb{E}_\pi B_2)e^{-rA_2}. \quad (4.4)
$$

Consider the term in $B_1-\mathbb{E}_\pi B_1$, that we denote by $\lambda_{2,1}$. Then, as in the proof of Lemma 4.2 , it follows that

$$
\|(\lambda_{2,1})_{mn}\|_{L^2_{x,\xi,\pi}} \leq C\int_0^\infty dr e^{-\gamma_* r/2}\left\|\int_{\mathbb{R}^d} dp e^{iz\cdot p}\left(e^{-rA_1}(B_1-\mathbb{E}_\pi B_1)e^{-rA_2}\right)_{mn}\right\|_{L^2_{x,\xi,\pi}}.
$$

We split the last term into four terms involving B_{ij}, and consider for example the one with $B_{1,1}$. Recalling that $G_t(q)=\sum_{k=0}^3\gamma^0\gamma^k\tilde{A}_k(\frac{t}{\varepsilon},q)$, we need to estimate terms of the form

$$
D_{kk'} := \left\|\int_{\mathbb{R}^d} dq\tilde{A}_k(t,q)\int_{\mathbb{R}^d} dp\tilde{A}_{k'}(t,p-q)\big(f_{kk'}(r,x,\xi,p,q)\big)_{mn}\right\|_{L^2_{x,\xi,\pi}}
$$

where

$$
\begin{aligned}
f_{kk'}(r,x,\xi,p,q) &:= e^{iz\cdot p}\int_0^\infty dr' e^{-\gamma_{k'}(p-q)r'}e^{-rA_1(\xi,p)}\gamma^0\gamma^k \\
&\qquad \times e^{-r'A_1(\xi-\frac{q}{2},p-q)}\gamma^0\gamma^{k'}\lambda_0(x,\xi-\frac{p}{2})e^{-r'A_2(\xi-\frac{q}{2},p-q)}e^{-rA_2(\xi,p)}. (4.5)
\end{aligned}
$$

We only treat the most difficult cases corresponding to $k=k'$ as they involve fourth order moments of $\tilde{A}_k$. The case $k\neq k'$ only involves second order moments since $\tilde{A}_k$ and $\tilde{A}_{k'}$ are independent. Following (2.8), there are three terms to estimate in $(D_{kk'})^2$, and they are all bounded in the same way. Picking say the last one in (2.8), we find

$$
(I) := (2\pi)^{2d}\int_{\mathbb{R}^{2d}} dx d\xi\int_{\mathbb{R}^{2d}} dq_1 dq_2 (f_{kk}(r,x,\xi,q_1+q_2,q_1))_{mn}\tilde{R}_k(0,2q_2)(f^*_{kk}(r,x,\xi,q_1+q_2,q_2))_{mn}\tilde{R}_k(0,-2q_1).
$$

Proceeding as in the proof of Lemma 4.2, we find that

$$
|\,(f_{kk'}(r,x,\xi,p,q))_{mn}\,| \leq C\sum_{i,j}\left|\left(\lambda_0(x,\xi-\frac{p}{2})\right)_{ij}\right|. \quad (4.6)
$$

Since $\tilde{R}\in\mathcal{S}(\mathbb{R}\times\mathbb{R}^d)$ and $\lambda_0\in\mathcal{S}_d$, it follows that (I) is bounded uniformly in time. As was already claimed, all other terms in λ_2 are treated in the same way and are therefore uniformly bounded. This ends the proof. $\square$

The terms C_2 and C_4 are treated in the same way as C_1 since it suffices to replace λ_0 by its derivatives w.r.t. t and the slow variable x. Regarding C_3, we have the following lemma.

Lemma 4.4. $\|C_3(t)\|_{L^2_{x,\xi,\pi}} \leq C_{\lambda_0,T}\sqrt{\varepsilon}$ *for all* $t \in [0,T]$.

Proof. The proof is similar to that of Lemma 4.3, with the only difference that we now need to estimate sixth order moments of A. There are several terms that have to be estimated, and since they are all treated in the same way, we only detail one representative calculation. We will control $\mathcal{K}_\varepsilon^k \lambda_{2,\varepsilon}$ in $L^2_{x,\xi,\pi}$ norm. We start with expression (4.4) for $\lambda_{2,\varepsilon}$, and treat for instance the term in $B_{1,1} - \mathbb{E}_\pi B_{1,1}$, see (4.3). As in the proof of Lemma 4.3, we select the combination of A_k that leads to the highest order moment. This contribution to $\lambda_{2,\varepsilon}$ has the form (we omit the time-dependence for simplicity)

$$\int_0^\infty dr e^{r\mathcal{Q}} \int_{\mathbb{R}^{2d}} dpdq \left(\tilde{A}_k(p-q)\tilde{A}_k(q) - \mathbb{E}_\pi\{\tilde{A}_k(p-q)\tilde{A}_k(q)\}\right) f_{kk}(x,\xi,r,p,q)$$

where f_{kk} is defined in (4.5). Using (2.7) and keeping only the first term resulting from it, the above expression becomes

$$I_k(x,\xi) := \int_{\mathbb{R}^{2d}} dpdq \tilde{A}_k(p-q)\tilde{A}_k(q) g(x,\xi,p,q),$$

where

$$g(x,\xi,p,q) = \int_0^\infty dr e^{-r(\gamma_k(p-q)+\gamma_k(q))} f_{kk}(x,\xi,r,p,q).$$

We then estimate

$$\|\mathcal{K}_\varepsilon^k I_k\|^2_{L^2_{x,\xi,\pi}} = \mathbb{E}_\pi \int_{\mathbb{R}^{8d}} dX \left(\tilde{A}_k(v)\tilde{A}_k(p'-q')\tilde{A}_k(q')\right)^* \tilde{A}_k(u)\tilde{A}_k(p-q)\tilde{A}_k(q) h(X), \qquad (4.7)$$

where $dX = dxd\xi dudvdpdqdp'dq'$ and

$$h(x,\xi,u,v,p,q,p',q') = (2\pi)^{-2d} e^{ix\cdot(u-v)/\varepsilon} g(x,\xi-u/2,p,q) g^*(x,\xi-v/2,p',q').$$

Using $\tilde{A}^*(p) = \tilde{A}(-p)$ and (2.9), there are $5!! = 15$ pairings and therefore 15 terms in the expectation in (4.7). They are all treated in the same way, and we pick as an example the one that is proportional to

$$\delta(u-v)\tilde{R}(0,u)\delta(p-q-p'+q')\tilde{R}(0,p-q)\delta(q-q')\tilde{R}(0,q).$$

This yields a contribution of the form

$$\begin{aligned}\int_{\mathbb{R}^{5d}} dxd\xi dudpdq \tilde{R}(0,u)\tilde{R}(0,p-q)\tilde{R}(0,q)|g(x,\xi-u/2,p,q)|^2 \\ \lesssim \|\tilde{R}(0,\cdot)\|^3_{L^1(\mathbb{R}^d)} \sup_{p,q} \int_{\mathbb{R}^{2d}} dxd\xi |g(x,\xi,p,q)|^2 \\ \lesssim \|\lambda_0\|^2_{L^2(\mathbb{R}^{2d})},\end{aligned}$$

where we used (4.6) in the last line. All calculations done, we find that

$$\|\mathcal{K}_\varepsilon^k \lambda_{2,\varepsilon}\|^2_{L^2_{x,\xi,\pi}} \lesssim \|\lambda_0\|^2_{L^2(\mathbb{R}^{2d})},$$

which ends the proof. □

We are now in position to prove (3.4). We start from (3.11) and have first, since W_ε^0 is deterministic,

$$\mathbb{E}^{\bar{\mathbb{P}}_\varepsilon}\{\langle \lambda_\varepsilon, W_\varepsilon^0\rangle\} = \langle \lambda_0, W_\varepsilon^0\rangle + \sqrt{\varepsilon}\mathbb{E}_\pi\{\langle \lambda_{1,\varepsilon}, W_\varepsilon^0\rangle\} + \varepsilon\mathbb{E}_\pi\{\langle \lambda_{2,\varepsilon}, W_\varepsilon^0\rangle\}.$$

According to Lemmas 4.2 and 4.3, the last two terms are of order $\sqrt{\varepsilon}$. Above, we have used the fact that the expectation over $\bar{\mathbb{P}}_\varepsilon$ reduces to the expectation over π for functions of the potential A only. Then, using (3.12) together with Lemmas 4.2, 4.3, 4.4 and (2.13), it follows that, for all $t \in [0, T]$,

$$\left|\mathbb{E}^{\bar{\mathbb{P}}_\varepsilon}\{\mathcal{G}^1_{\lambda_\varepsilon}[W](t)\} - \mathbb{E}^{\bar{\mathbb{P}}_\varepsilon}\{\mathcal{G}^1_{\lambda_0}[W](t)\}\right| \leq C\sqrt{\varepsilon},$$

which then yields (3.4). Now we can prove the following proposition about convergence of expectation.

Proposition 4.5. *If $(\alpha_+^\varepsilon, \alpha_-^\varepsilon)$ converges weakly to some (α_+, α_-) in $\mathcal{C}^0([0, T], H_M)$, then $(\mathbb{E}\alpha_+, \mathbb{E}\alpha_-)$ is the weak solution to (2.10).*

Proof. In order to simplify notation, we will denote by $\mathbb{E}$ both $\mathbb{E}^{\tilde{\mathbb{P}}_\varepsilon}$ and $\mathbb{E}^{\mathbb{P}_\varepsilon}$, knowing implicitly that $\mathbb{E}^{\tilde{\mathbb{P}}_\varepsilon}$ applies to functions of W_ε and $\mathbb{E}^{\mathbb{P}_\varepsilon}$ to functions of $(\alpha_+^\varepsilon, \alpha_-^\varepsilon)$. First of all, we examine the initial condition for $(\mathbb{E}\alpha_+, \mathbb{E}\alpha_-)$. For $f_0 \in \mathcal{S}(\mathbb{R}^{2d})$, choosing the test function to be $\lambda_0 = f_0(x_1x_1^* + x_2x_2^*)$ (this is allowed since the vectors x_1 and x_2 are smooth as can be seen from their definition), we have

$$\langle \lambda_0, \mathbb{E}W_\varepsilon\rangle(0) = \langle f_0, \mathbb{E}\alpha_+^\varepsilon\rangle(0).$$

Then, since $\langle \lambda_0, \mathbb{E}W_\varepsilon\rangle(0) \to \langle \lambda_0, W_0\rangle(0) = \langle f_0, \mathrm{Tr}(\Pi_+W_0)\rangle(0)$ and $\langle f_0, \mathbb{E}\alpha_+^\varepsilon\rangle(0) \to \langle f_0, \mathbb{E}\alpha_+\rangle(0)$, we obtain $\mathbb{E}\alpha_+(0) = \mathrm{Tr}(\Pi_+W_0(0))$. The same discussion holds for $\mathbb{E}\alpha_-(0)$.

Besides, according to (3.4), we have

$$\left|\langle \lambda_0, \mathbb{E}W_\varepsilon\rangle(t) - \langle \lambda_0, \mathbb{E}W_\varepsilon\rangle(0) - \int_0^t ds\langle(\partial_t + \mathcal{A})\lambda_0, \mathbb{E}W_\varepsilon\rangle(s)\right| \leq C_{\lambda_0,T}\sqrt{\varepsilon}.$$

By the choice of λ_0, we further obtain

$$\begin{aligned}
&\langle \lambda_0, \mathbb{E}W_\varepsilon\rangle(t) - \langle \lambda_0, \mathbb{E}W_\varepsilon\rangle(0) - \int_0^t ds\langle \partial_t\lambda_0, \mathbb{E}W_\varepsilon\rangle(s)\\
&= \langle f_0, \mathbb{E}\alpha_+^\varepsilon\rangle(t) - \langle f_0, \mathbb{E}\alpha_+^\varepsilon\rangle(0) - \int_0^t ds\langle \partial_t f_0, \mathbb{E}\alpha_+^\varepsilon\rangle(s)\\
&\to \langle f_0, \mathbb{E}\alpha_+\rangle(t) - \langle f_0, \mathbb{E}\alpha_+\rangle(0) - \int_0^t ds\langle \partial_t f_0, \mathbb{E}\alpha_+\rangle(s)
\end{aligned}$$

as $\varepsilon \to 0$.

For the remaining term $\int_0^t ds\langle \mathcal{A}\lambda_0, \mathbb{E}W_\varepsilon\rangle(s)$, we need to calculate $x_1^*\mathcal{A}^*(\mathbb{E}W_\varepsilon)x_1 + x_2^*\mathcal{A}^*(\mathbb{E}W_\varepsilon)x_2$. After some lengthy algebra, we get

$$x_1^*\mathcal{A}^*(\mathbb{E}W_\varepsilon)x_1 + x_2^*\mathcal{A}^*(\mathbb{E}W_\varepsilon)x_2 = -\frac{c\xi \cdot \nabla_x\mathbb{E}\alpha_+^\varepsilon}{\lambda_+(\xi)} + \mathcal{T}(\mathbb{E}\alpha_+^\varepsilon, \mathbb{E}\alpha_-^\varepsilon) + (I),$$

where (I) includes terms containing $c^\varepsilon_{ij}, d^\varepsilon_{ij}$. Using the convergence results on the cross modes obtained in section 4.1, we check that $\mathbb{E}_\pi|\int_0^t ds\langle f_0,(I)\rangle(s)| \to 0$ as $\varepsilon\to 0$. Thus, we have shown that $(\mathbb{E}\alpha_+,\mathbb{E}\alpha_-)$ is the weak solution to

$$\partial_t\mathbb{E}\alpha_+ + \frac{c\xi\cdot\nabla_x\mathbb{E}\alpha_+}{\lambda_+(\xi)} = \mathcal{T}(\mathbb{E}\alpha_+,\mathbb{E}\alpha_-)$$

with the initial condition given by $\mathbb{E}\alpha_\pm(0) = \mathrm{Tr}(\Pi_\pm W_0(0))$.

In the same way, if we choose $\lambda_0 = f_0(y_1y_1^* + y_2y_2^*)$, we obtain

$$\partial_t\mathbb{E}\alpha_- + \frac{c\xi\cdot\nabla_x\mathbb{E}\alpha_-}{\lambda_-(\xi)} = \mathcal{T}(\mathbb{E}\alpha_-,\mathbb{E}\alpha_+).$$

By uniqueness of the solutions to the above transport equations, the proof is complete. $\square$

4.3 Convergence of the second moment

We now prove (3.5). Recalling the construction of $\mu_\varepsilon = \mu_0 + \sqrt{\varepsilon}\mu_{1,\varepsilon} + \varepsilon\mu_{2,\varepsilon}$, we need to prove that the correctors $\bar{C}_i, i = 1,\ldots,5$ are small. The only term that requires a different treatment than the correctors C_i of the previous section is $\bar{C}_4$, for which we have the following lemma.

Lemma 4.6. $\|\bar{C}_4(t)\|_{L^2(\mathbb{R}^{4d})} \le C_{\mu_0,T}\varepsilon^{d/2}$ *uniformly in* $t\in[0,T]$.

Proof. There are four terms in the definition of $\mathbf{L}_2\mu_0$ in (3.16), all of which are treated in the same way. We only consider one term, for example

$$\sum_{k=0}^3 \Gamma^0\Gamma^k\mathcal{K}^k_{\varepsilon 1}F_0\otimes G_1 = \sum_{k=0}^3\left(\gamma^0\gamma^k\int_{\mathbb{R}^d}\frac{dp\tilde{A}_k(\frac{t}{\varepsilon},p)}{(2\pi)^d}e^{ix\cdot p/\varepsilon}F_0(x,\xi-\frac{p}{2})\right)\otimes G_1.$$

Since

$$\begin{aligned}&G_1(t,x,\frac{x}{\varepsilon},\xi)\\ =&-\int_0^\infty dr e^{r\mathcal{Q}}\int_{\mathbb{R}^d}\frac{dpe^{ix\cdot p/\varepsilon}}{(2\pi)^d}e^{-rA_1(\xi,p)}ie\left[G_t(p)G_0(\xi-\frac{p}{2})-G_0(\xi+\frac{p}{2})G_t(p)\right]e^{-rA_2(\xi,p)},\end{aligned}$$

and $G_t(p) = \sum_{k=0}^3\gamma^0\gamma^k\tilde{A}_k(\frac{t}{\varepsilon},p)$, taking one term in the sum and ignoring constant, we have

$$\begin{aligned}(I) =& \left(\gamma^0\gamma^m\int_{\mathbb{R}^d}\frac{dp\tilde{A}_m(\frac{t}{\varepsilon},p)}{(2\pi)^d}e^{ix\cdot p/\varepsilon}F_0(x,\xi-\frac{p}{2})\right)\otimes\\ &\left(\int_0^\infty dre^{r\mathcal{Q}}\int_{\mathbb{R}^d}\frac{dp\tilde{A}_n(\frac{t}{\varepsilon},p)}{(2\pi)^d}e^{ix\cdot p/\varepsilon}e^{-rA_1(\xi,p)}\gamma^0\gamma^nG_0(x,\xi-\frac{p}{2})e^{-rA_2(\xi,p)}\right).\end{aligned}$$

For any element in matrix $\mathbb{E}_\pi\{I\}$, we know that it is a linear combination of terms of the following form

$$(i) = \int_0^\infty dr\int_{\mathbb{R}^d}dpe^{\frac{i(x_1-x_2)\cdot p}{\varepsilon}}\tilde{R}_k(r,p)f(x_1,\xi_1-\frac{p}{2})g(x_2,\xi_2+\frac{p}{2})T_1(r,\xi_2,p)T_2(r,\xi_2,p),$$

and f, g, T_1, T_2 are from $F_0, G_0, e^{-rA_1(\xi_2,p)}, e^{-rA_2(\xi_2,-p)}$ respectively and we can show they are all real. So

$$\|(i)\|^2_{L^2(\mathbb{R}^{2d})} = \int_{\mathbb{R}^{6d}} \int_0^\infty \int_0^\infty H\tilde{R}_k(s_1,p_1)\tilde{R}_k(s_2,p_2)e^{\frac{i(x_1-x_2)\cdot(p_1-p_2)}{\varepsilon}} ds_1 ds_2 dp_1 dp_2 dx_1 dx_2 d\xi_1 d\xi_2,$$

where

$$H = \prod_{i=1,2} f(x_1, \xi_1 - \frac{p_i}{2}) \prod_{i=1,2} g(x_2, \xi_2 + \frac{p_i}{2}) \prod_{i=1,2} T_i(s_1, \xi_2, p_1) \prod_{i=1,2} T_i(s_2, \xi_2, p_2).$$

By density argument, we can assume $f(x_1,\xi_1)g(x_2,\xi_2) = h_1(x_1+x_2)h_2(x_1-x_2)h_3(\xi_1)h_4(\xi_2)$ for some smooth function h_i, then we have

$$H = |h_1(x_1+x_2)|^2|h_2(x_1-x_2)|^2 \prod_{i=1,2} h_3(\xi_1-\frac{p_i}{2}) \prod_{i=1,2} h_4(\xi_2+\frac{p_i}{2}) \prod_{i=1,2} T_i(s_1,\xi_2,p_1) \prod_{i=1,2} T_i(s_2,\xi_2,p_2).$$

Changing variables $y_1 = x_1 + x_2, y_2 = x_1 - x_2$, and integrating in y_i, we have

$$\|(i)\|^2_{L^2(\mathbb{R}^{4d})} = C\int_{\mathbb{R}^{4d}} \int_0^\infty \int_0^\infty \prod_{i=1,2} h_3(\xi_1 - \frac{p_i}{2}) \prod_{i=1,2} h_4(\xi_2 + \frac{p_i}{2}) \prod_{i=1,2} T_i(s_1,\xi_2,p_1) \prod_{i=1,2} T_i(s_2,\xi_2,p_2)\tilde{R}_k(s_1,p_1)\tilde{R}_k(s_2,p_2)\hat{\nu}\left(\frac{p_2-p_1}{\varepsilon}\right) ds_1 ds_2 dp_1 dp_2 d\xi_1 d\xi_2$$

for some constant C and $\hat{\nu}$ is the Fourier transform of of $|h_2|^2$. Since T_i is bounded, integrating in ξ_1 and ξ_2 yields:

$$\|(i)\|^2_{L^2(\mathbb{R}^{4d})} \lesssim \int_{\mathbb{R}^{2d}} \int_0^\infty \int_0^\infty |\tilde{R}_k(s_1,p_1)\tilde{R}_k(s_2,p_2)\hat{\nu}\left(\frac{p_2-p_1}{\varepsilon}\right)| ds_1 ds_2 dp_1 dp_2.$$

Changing variable $p_2 = p_1 + u$, then integrating in p_1, s_1, s_2, since $\tilde{R}_k \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^d)$, we have

$$\|(i)\|^2_{L^2(\mathbb{R}^{4d})} \lesssim \int_{\mathbb{R}^d} \left|\hat{\nu}\left(\frac{u}{\varepsilon}\right)\right| du \lesssim \varepsilon^d,$$

which ends the proof. □

Since $\mu_1 = F_0\otimes G_1 + F_1 \otimes G_0$ and $\mu_2 = F_0 \otimes G_2 + F_2 \otimes G_0 + \tilde{\mu}_2$, and we already have the control for F_1, G_1, F_2, G_2, we only need to show that $\mathbb{E}_\pi\|\tilde{\mu}_2\|^2_{L^2(\mathbb{R}^{4d})} \le C$. For this, we note that the matrices

$$\begin{aligned}
\mathbf{A}_1 &:= \sum_{k=1}^3 c\left[\Gamma^0\Gamma^k P^*_{k1}\left(i\xi + \frac{ip}{2}\right) + \tilde{\Gamma}^0\tilde{\Gamma}^k P^*_{k2}(i\xi + \frac{ip}{2})\right] - im_0c^2(\Gamma^0 + \tilde{\Gamma}^0)\\
\mathbf{A}_2 &:= \sum_{k=1}^3 c\left[\Gamma^0\Gamma^k P_{k1}\left(i\xi - \frac{ip}{2}\right) + \tilde{\Gamma}^0\tilde{\Gamma}^k P_{k2}\left(i\xi - \frac{ip}{2}\right)\right] + im_0c^2(\Gamma^0 + \tilde{\Gamma}^0),
\end{aligned}$$

are both of the form iM for some hermitian matrix M, thus any element in the matrices $e^{-r\mathbf{A}_1}, e^{-r\mathbf{A}_2}$ is bounded. The rest of the proof is similar to the one for $\lambda_{2,\varepsilon}$, the details are omitted: the equation satisfied by $\tilde{\mu}_2$ is inverted and the r.h.s. involves quadratic quantities the random field A that are treated in the same way as with $\lambda_{2,\varepsilon}$ by using fourth order moments as in the proof of Lemma 4.3. At this point, we have therefore, for all $t \in [0,T]$,

$$\mathbb{E}_\pi \|\tilde{\mu}_1(t)\|^2_{L^2(\mathbb{R}^{4d})} + \mathbb{E}_\pi \|\tilde{\mu}_2(t)\|^2_{L^2(\mathbb{R}^{4d})} \leq C.$$

Similar bounds can be obtained for the derivatives w.r.t. t and the slow variable x, which lead to, for all $t \in [0,T]$,

$$\mathbb{E}_\pi \|\bar{C}_i(t)\|^2_{L^2(\mathbb{R}^{4d})} \leq C\varepsilon, \qquad i = 1,2,5.$$

The control of the corrector $\bar{C}_4$ is a consequence of Lemma 4.6. The bound on $\bar{C}_3$ is obtained by following the lines of Lemma 4.4 on the corrector C_3, and using the above remark about $\mathbf{A}_1$ and $\mathbf{A}_2$: $\bar{C}_3$ involves the product of three random fields with different arguments, which leads to the estimation of sixth order moments of A. We do not provide the details since the calculations are the same as in Lemma 4.4. Proceeding as for (3.4), we finally obtain the estimate 3.5. We are then able to prove the following proposition.

Proposition 4.7. *If $(\alpha^\varepsilon_+, \alpha^\varepsilon_-)$ converges weakly to (α_+, α_-) in $\mathcal{C}^0([0,T], H_M)$, then (α_+, α_-) is the unique weak solution to (2.10).*

Proof. We use the notation of Proposition 4.5 and recall that $\alpha_\pm(0,x,\xi) = \mathrm{Tr}(\Pi_\pm W_0(0,x,\xi))$ since W^0_ε is deterministic. We set $F_0 = f_0(x_1 x_1^* + x_2 x_2^*)$ and $G_0 = g_0(x_1 x_1^* + x_2 x_2^*)$ for smooth functions f_0 and g_0 in $\mathcal{S}(\mathbb{R}^{2d})$. From (3.5), we have

$$\begin{aligned}
&|\langle F_0 \otimes G_0, \mathbb{E}\{W_\varepsilon \otimes W_\varepsilon\}\rangle(t) - \langle F_0 \otimes G_0, \mathbb{E}\{W_\varepsilon \otimes W_\varepsilon\}\rangle(0) \\
&\quad - \int_0^t ds \, \langle F_0 \otimes [(\partial_t + \mathcal{A})G_0] + [(\partial_t + \mathcal{A})F_0] \otimes G_0, \mathbb{E}\{W_\varepsilon \otimes W_\varepsilon\}\rangle \, (s)| \lesssim \sqrt{\varepsilon}.
\end{aligned}$$

By the choice of the aforementioned F_0, G_0, we have

$$\langle F_0 \otimes G_0, \mathbb{E}\{W_\varepsilon \otimes W_\varepsilon\}\rangle = \langle f_0 \otimes g_0, \mathbb{E}\{\alpha^\varepsilon_+ \otimes \alpha^\varepsilon_+\}\rangle$$

as well as

$$\begin{aligned}
&\langle F_0 \otimes [(\partial_t + \mathcal{A})G_0] + [(\partial_t + \mathcal{A})F_0] \otimes G_0, \mathbb{E}\{W_\varepsilon \otimes W_\varepsilon\}\rangle \\
&\quad = \langle f_0 \otimes g_0, \mathbb{E}\{\alpha^\varepsilon_+ \otimes (x_1^* \mathcal{A}^* W_\varepsilon x_1 + x_2^* \mathcal{A}^* W_\varepsilon x_2) + (x_1^* \mathcal{A}^* W_\varepsilon x_1 + x_2^* \mathcal{A}^* W_\varepsilon x_2) \otimes \alpha^\varepsilon_+\}\rangle \\
&\quad + \langle f_0 \otimes \partial_t g_0 + \partial_t f_0 \otimes g_0, \mathbb{E}\{\alpha^\varepsilon_+ \otimes \alpha^\varepsilon_+\}\rangle.
\end{aligned}$$

By following similar arguments as in the proof of Proposition 4.5, we find as $\varepsilon \to 0$,

$$\begin{aligned}
&\langle f_0 \otimes g_0, \mathbb{E}\{\alpha_+ \otimes \alpha_+\}\rangle(t) - \langle f_0 \otimes g_0, \mathbb{E}\{\alpha_+ \otimes \alpha_+\}\rangle(0) \\
&\quad = \int_0^t ds \left\langle \left(\partial_t + \frac{c\xi_1 \cdot \nabla_{x_1}}{\lambda_+(\xi_1)}\right) f_0 \otimes g_0 + f_0 \otimes \left(\partial_t + \frac{c\xi_2 \cdot \nabla_{x_2}}{\lambda_+(\xi_2)}\right) g_0, \mathbb{E}\{\alpha_+ \otimes \alpha_+\} \right\rangle (s) \\
&\qquad + \int_0^t ds \langle f_0 \otimes g_0, \mathbb{E}\{\alpha_+ \otimes \mathcal{T}(\alpha_+, \alpha_-) + \mathcal{T}(\alpha_+, \alpha_-) \otimes \alpha_+\}\rangle(s).
\end{aligned}$$

Note that the term $\mathbb{E}\{\alpha_+ \otimes \mathcal{T}(\alpha_+, \alpha_-) + \mathcal{T}(\alpha_+, \alpha_-) \otimes \alpha_+\}$ can be written as a linear functional of $(\mathbb{E}\{\alpha_+ \otimes \alpha_+\}, \mathbb{E}\{\alpha_+ \otimes \alpha_-\}, \mathbb{E}\{\alpha_- \otimes \alpha_+\})$. By choosing other forms of F_0, G_0, we can derive other equations satisfied by $(\mathbb{E}\{\alpha_+ \otimes \alpha_-\}, \mathbb{E}\{\alpha_- \otimes \alpha_+\}, \mathbb{E}\{\alpha_- \otimes \alpha_-\})$. Introducing $U = (\mathbb{E}\{\alpha_+ \otimes \alpha_+\}, \mathbb{E}\{\alpha_+ \otimes \alpha_-\}, \mathbb{E}\{\alpha_- \otimes \alpha_+\}, \mathbb{E}\{\alpha_- \otimes \alpha_-\})$, we then obtain a system of transport equations of the form

$$\partial_t U + \left(\frac{c\xi_1 \cdot \nabla_{x_1}}{\lambda_+(\xi_1)} + \frac{c\xi_2 \cdot \nabla_{x_2}}{\lambda_+(\xi_2)} \right) U = \mathcal{H} U, \tag{4.8}$$

where the linear operator $\mathcal{H}$ acts on the variables (ξ_1, ξ_2) and is bounded on $(L^2(\mathbb{R}^{4d}))^2$. As for (2.16), the existence and uniqueness of a weak solution to (4.8) can be established without difficulty using the methods of [13] Chapter 21, §2. Then, it is directly checked that $(\mathbb{E}\alpha_+ \otimes \mathbb{E}\alpha_+, \mathbb{E}\alpha_+ \otimes \mathbb{E}\alpha_-, \mathbb{E}\alpha_- \otimes \mathbb{E}\alpha_+, \mathbb{E}\alpha_- \otimes \mathbb{E}\alpha_-)$ satisfies (4.8) with the same initial condition as U. By uniqueness, this shows that (α_+, α_-) is deterministic and satisfies (2.16). The proof is complete. □

4.4 Tightness

In this section, we prove that $(\alpha_+^\varepsilon, \alpha_-^\varepsilon)$ is tight in $\mathcal{C}^0([0,T], H_M)$. As a consequence, $(\alpha_+^\varepsilon, \alpha_-^\varepsilon)$ converges in law, up to an extraction of a subsequence, to some random variable (α_+, α_-). Together with Propositions 4.5 and 4.7, it follows that (α_+, α_-) is deterministic and the solution to the transport equation (2.16). Since the latter admits a unique solution, it follows that the whole sequence $(\alpha_+^\varepsilon, \alpha_-^\varepsilon)$ converges and concludes the proof of the main theorem.

Proposition 4.8. *$(\alpha_+^\varepsilon, \alpha_-^\varepsilon)$ is tight in $\mathcal{C}^0([0,T], H_M)$.*

Proof. Since the set H_M is compact when endowed with the metric d_H, we remark first that we only have to show that the process $\langle f, \alpha_\pm^\varepsilon \rangle \in \mathcal{C}^0([0,T], \mathbb{R})$ is tight for any function $f \in L^2(\mathbb{R}^{2d})$, and thus by density for any smooth function $f \in \mathcal{S}(\mathbb{R}^{2d})$. We choose f to be real-valued. Take α_+^ε for example and note that

$$\langle f, \alpha_+^\varepsilon \rangle = \langle f(x_1 x_1^* + x_2 x_2^*), W_\varepsilon \rangle.$$

Defining $\lambda_0 = f(x_1 x_1^* + x_2 x_2^*)$, we have $\langle f, \alpha_+^\varepsilon \rangle = \langle \lambda_0, W_\varepsilon \rangle$. Choosing then the test function $\lambda_\varepsilon = \lambda_0 + \sqrt{\varepsilon}\lambda_{1,\varepsilon}$, where $\lambda_{1,\varepsilon}$ is defined in section 3.1, we have

$$\begin{aligned}
\langle \lambda_0, W_\varepsilon \rangle(t) &= \int_0^t ds \left\langle R_0(\lambda_0 + \sqrt{\varepsilon}\lambda_{1,\varepsilon}) + \frac{1}{\sqrt{\varepsilon}} \mathcal{K}_\varepsilon \lambda_{1,\varepsilon}, W_\varepsilon \right\rangle (s) + \mathcal{G}^{1,\varepsilon}_{\lambda_\varepsilon}[W_\varepsilon](t) \\
&\quad - \sqrt{\varepsilon} \langle \lambda_{1,\varepsilon}, W_\varepsilon \rangle(t), \\
&:= x_\varepsilon(t) + \mathcal{G}^{1,\varepsilon}_{\lambda_\varepsilon}[W_\varepsilon](t) + R_\varepsilon(t),
\end{aligned}$$

where the operator R_0 is defined in (4.1), $\mathcal{K}_\varepsilon$ in (2.12), and

$$R_\varepsilon(t) = \sqrt{\varepsilon} \int_0^t ds \, \langle R_0 \lambda_{1,\varepsilon}, W_\varepsilon \rangle (s) - \sqrt{\varepsilon} \langle \lambda_{1,\varepsilon}, W_\varepsilon \rangle(t).$$

We will show below that $\sup_{t \in [0,T]} |R_\varepsilon(t)|$ converges to zero in probability, which means we only need to address the tightness of $x_\varepsilon(t) + \mathcal{G}^{1,\varepsilon}_{\lambda_\varepsilon}[W_\varepsilon](t)$. Thanks to the uniform bound on

W_ε and adapting Lemma 4.2 to $R_0\lambda_{1,\varepsilon}$, the first term in R_ε, denoted by R^1_ε, can easily be shown to verify

$$\mathbb{E}^{\bar{\mathbb{P}}_\varepsilon}\left\{\sup_{t\in[0,T]}|R^1_\varepsilon(t)|\right\}\lesssim\sqrt{\varepsilon}.$$

Regarding the second term, recall that $\lambda_{1,\varepsilon}$ reads, using (2.6),

$$\begin{aligned}\lambda_{1,\varepsilon}(t,x,\xi)=-ie\sum_{k=0}^{3}\int_0^\infty dr\int_{\mathbb{R}^d}\frac{dpe^{ix\cdot p/\varepsilon}}{(2\pi)^d}\tilde{A}_k\left(\frac{t}{\varepsilon},p\right)e^{-r\gamma_k(p)}&\\ \times e^{-rA_1(\xi,p)}\left[\gamma_0\gamma_k\lambda_0(\xi-\frac{p}{2})-\lambda_0(\xi+\frac{p}{2})\gamma_0\gamma_k\right]e^{-rA_2(\xi,p)}&\\ :=\sum_{k=0}^{3}\lambda_{1,\varepsilon,k}.&\end{aligned}$$

Note that with our choice of λ_0, the matrix $\lambda_{1,\varepsilon}$ is hermitian. For some smooth function φ, let

$$V_1(t):=\int_{\mathbb{R}^d}dp\tilde{A}_k\,(t,p)\,\varphi(p).$$

Using the gaussianity of A_k, it is shown in [17], Lemma 7.1, that

$$\mathbb{E}_\pi\left\{\sup_{t\in[0,T]}\left|V_1\left(\frac{t}{\varepsilon}\right)\right|^2\right\}\lesssim\left(\int_0^\theta\left(\log\left(\frac{C}{r^2\varepsilon}\right)\right)^{1/2}dr\right)^2,$$

where

$$\theta^2=2\int_{\mathbb{R}^d}|\varphi(p)|^2\mu_k(p)dp.$$

It then follows that, $\forall\alpha\in(0,1)$,

$$\mathbb{E}_\pi\left\{\sup_{t\in[0,T]}\left|V_1\left(\frac{t}{\varepsilon}\right)\right|^2\right\}\lesssim(\sqrt{\varepsilon}\theta)^{2(1-\alpha)}/\varepsilon \tag{4.9}$$

Applying (4.9) with

$$\varphi_{x,\xi}(p)=\int_0^\infty dre^{ix\cdot p/\varepsilon}e^{-r\gamma_k(p)}\left(e^{-rA_1(\xi,p)}\left[\gamma_0\gamma_k\lambda_0(\xi-\frac{p}{2})-\lambda_0(\xi+\frac{p}{2})\gamma_0\gamma_k\right]e^{-rA_2(\xi,p)}\right)_{mn},$$

together with (4.2), we can conclude that

$$\mathbb{E}_\pi\int_{\mathbb{R}^{2d}}\left(\sup_{t\in[0,T]}|\lambda_{1,\varepsilon}|^2\right)dxd\xi\lesssim\varepsilon^{-\alpha},\qquad\forall\alpha\in(0,1).$$

Using the uniform bound on W_ε and the Chebychev inequality, we can then conclude that

$$\sqrt{\varepsilon}\,\bar{\mathbb{P}}_\varepsilon\left\{\sup_{t\in[0,T]}|\langle\lambda_{1,\varepsilon},W_\varepsilon\rangle(t)|>\delta\right\}\lesssim\varepsilon^{\frac{1}{2}-\alpha},\qquad\forall\alpha\in(0,1/2),$$

which yields the convergence of R_ε to zero as announced.

We now show the tightness of $x_\varepsilon(t)$ that we recall reads

$$x_\varepsilon(t) = \int_0^t ds \left\langle (R_0\lambda_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{K}_\varepsilon\lambda_{1,\varepsilon}, W_\varepsilon \right\rangle (s) := x_\varepsilon^1(t) + x_\varepsilon^2(t).$$

Following e.g. the discussion in section 2.3 of [4], we prove a Kolmogorov moment condition in the form

$$\mathbb{E}^{\bar{\mathbb{P}}_\varepsilon}\left\{|x_\varepsilon(t)-x_\varepsilon(t_1)|^\gamma |x_\varepsilon(t_1)-x_\varepsilon(s)|^\gamma\right\} \le C_{f,T}|t-s|^{1+\beta}, \tag{4.10}$$

with $0\le s\le t_1\le t\le T$ and $\gamma>0$, $\beta>0$. Using the L^2 bound on W_ε, it is direct to show that

$$\left|x_\varepsilon^1(t)-x_\varepsilon^1(s)\right|^2 \le C(t-s)^2, \tag{4.11}$$

so that we focus on x_ε^2. We have

$$\begin{aligned}\mathbb{E}^{\bar{\mathbb{P}}_\varepsilon}\left\{\left|x_\varepsilon^2(t)-x_\varepsilon^2(s)\right|^2\right\} &\le C\sum_{k=0}^3 \mathbb{E}^{\bar{\mathbb{P}}_\varepsilon}\left\{\left|\int_s^t d\tau \left(\|\mathcal{K}_\varepsilon^k\lambda_{1,\varepsilon}\|_{L^2(\mathbb{R}^{2d})} + \|\tilde{\mathcal{K}}_\varepsilon^k\lambda_{1,\varepsilon}\|_{L^2(\mathbb{R}^{2d})}\right)(\tau)\right|^2\right\} \\ &\le C(t-s)\sum_{k=0}^3\int_s^t d\tau \mathbb{E}^{\bar{\mathbb{P}}_\varepsilon}\left\{\left(\|\mathcal{K}_\varepsilon^k\lambda_{1,\varepsilon}\|^2_{L^2(\mathbb{R}^{2d})} + \|\tilde{\mathcal{K}}_\varepsilon^k\lambda_{1,\varepsilon}\|^2_{L^2(\mathbb{R}^{2d})}\right)(\tau)\right\}.\end{aligned}$$

Following the lines of Lemma 4.4 (the proof here is actually a little simpler since it only involves fourth-order moments of the random field), we can show that

$$\mathbb{E}^{\bar{\mathbb{P}}_\varepsilon}\|\mathcal{K}_\varepsilon^k\lambda_{1,\varepsilon}\|^2_{L^2(\mathbb{R}^{2d})} \le C,$$

with a similar bound for the term in $\tilde{\mathcal{K}}_\varepsilon^k$. As a consequence,

$$\mathbb{E}^{\bar{\mathbb{P}}_\varepsilon}\left|x_\varepsilon^2(t)-x_\varepsilon^2(s)\right|^2 \le C(t-s)^2.$$

This together with (4.11) and a Cauchy-Schwarz inequality, leads to (4.10) with $\gamma=\beta=1$. Since $x_\varepsilon(0)=0$, we obtain the tightness of $x_\varepsilon(t)$.

In remains to show that $\mathcal{G}^{1,\varepsilon}_{\lambda_\varepsilon}[W_\varepsilon](t)$ is tight. Note that it is real valued since all matrices in the definition of $\mathcal{G}^{1,\varepsilon}_{\lambda_\varepsilon}[W_\varepsilon](t)$ are hermitian. Since it is a martingale, we calculate the increasing process, that is usually denoted by $\langle\mathcal{G}^{1,\varepsilon}_{\lambda_\varepsilon}[W_\varepsilon]\rangle_t$. A classical calculation, see e.g. [19], Proposition 4.7.11, shows that

$$\langle\mathcal{G}^{1,\varepsilon}_{\lambda_0}[W_\varepsilon](t)\rangle_t = \frac{1}{\varepsilon}\int_0^t ds\left(\langle\mathcal{Q}(\lambda_\varepsilon\otimes\lambda_\varepsilon), W_\varepsilon\otimes W_\varepsilon\rangle - 2\langle\mathcal{Q}\lambda_\varepsilon, W_\varepsilon\rangle\langle\lambda_\varepsilon, W_\varepsilon\rangle\right)(s).$$

With our choice of λ_ε, we find that

$$\langle\mathcal{G}^{1,\varepsilon}_{\lambda_0}[W_\varepsilon](t)\rangle_t = \int_0^t ds\left(\langle\mathcal{Q}(\lambda_{1,\varepsilon}\otimes\lambda_{1,\varepsilon}), W_\varepsilon\otimes W_\varepsilon\rangle - 2\langle\mathcal{Q}\lambda_{1,\varepsilon}, W_\varepsilon\rangle\langle\lambda_{1,\varepsilon}, W_\varepsilon\rangle\right)(s).$$

According to (2.7), we have

$$\mathcal{Q}_k\left(\tilde{A}_k(t/\varepsilon,p)\tilde{A}_k(t/\varepsilon,q)\right) = -[\gamma_k(p)+\gamma_k(q)]\tilde{A}_k(t/\varepsilon,p)\tilde{A}_k(t/\varepsilon,q)+2(2\pi)^d\gamma_k(p)\delta(p+q)\tilde{R}_k(0,p).$$

This, together with the uniform bound on W_ε and a similar proof as the one of Lemma 4.3 involving fourth-order moments of A, allows us to obtain

$$\mathbb{E}^{\bar{\mathbb{P}}_\varepsilon}|\langle \mathcal{Q}(\lambda_{1,\varepsilon}\otimes\lambda_{1,\varepsilon}), W_\varepsilon\otimes W_\varepsilon\rangle| \le C\mathbb{E}^{\bar{\mathbb{P}}_\varepsilon}\|\mathcal{Q}(\lambda_{1,\varepsilon}\otimes\lambda_{1,\varepsilon})\|_{L^2(\mathbb{R}^{4d})} \le C.$$

It can be shown similarly that the other term in $\langle \mathcal{G}^{1,\varepsilon}_{\lambda_0}[W_\varepsilon](t)\rangle_t$ is uniformly bounded. This implies finally that, for $t \ge s$,

$$\mathbb{E}^{\bar{\mathbb{P}}_\varepsilon}\{\langle \mathcal{G}_{\lambda_0}\rangle_t - \langle \mathcal{G}_{\lambda_0}\rangle_s\} \le C(t-s).$$

Following the discussion at the end of section 5.5 of [4], this shows that $\mathcal{G}^{1,\varepsilon}_{\lambda_\varepsilon}[W_\varepsilon](t)$ is tight and ends the proof of the proposition. $\square$

5 Slow time fluctuations

In this section, we discuss briefly the case when the temporal random fluctuations are slower than the spatial fluctuations of the potential. This leads to a kinetic regime with elastic collisions as the random fluctuations are now too slow to affect such a scattering; see [4] for a similar derivation in the setting of Schrödinger equations.

More precisely, we consider the Dirac equation (2.10) with $A_i(\frac{t}{\varepsilon},\frac{x}{\varepsilon})$ replaced by $A_i(\frac{t}{\varepsilon^\alpha},\frac{x}{\varepsilon})$ for some $\alpha \in (0,1)$. The proof of convergence for α sufficiently large remains almost the same as before except that the infinitesimal generator of $(W_\varepsilon(t), \tilde{A}(t/\varepsilon^\alpha))$ becomes

$$\frac{d}{dh}\mathbb{E}^{\bar{\mathbb{P}}_\varepsilon}_{W,V,t}\{\langle \lambda(V(t+h)), W(t+h)\rangle\}|_{h=0} = \left\langle \frac{1}{\varepsilon^\alpha}\mathcal{Q}\lambda + \partial_t\lambda + \mathcal{A}_\varepsilon\lambda, W\right\rangle.$$

So our test functions λ_1, λ_2 in (3.8) and (3.10) become

$$\begin{aligned}
\lambda_1 &= -\int_0^\infty e^{r\varepsilon^{1-\alpha}\mathcal{Q}}\int_{\mathbb{R}^d}\frac{e^{iz\cdot p}}{(2\pi)^d}e^{-rA_1}F_1e^{-rA_2}drdp,\\
\lambda_2 &= -\int_0^\infty e^{r\varepsilon^{1-\alpha}\mathcal{Q}}\int_{\mathbb{R}^d}\frac{e^{iz\cdot p}}{(2\pi)^d}e^{-rA_1}(F_2+\mathcal{L}_\varepsilon\lambda_0(2\pi)^d\delta(p))e^{-rA_2}drdp,
\end{aligned}$$

where $\mathcal{L}_\varepsilon\lambda_0$ is defined by replacing $\tilde{R}_k(r,q)$ in $\mathcal{L}\lambda_0$ by $\tilde{R}_k(r\varepsilon^{1-\alpha},q)$. Thus we see that $\lambda_{1,\varepsilon}$ is of order $\varepsilon^{\alpha-1}$ and $\lambda_{2,\varepsilon}$ is of order $\varepsilon^{2\alpha-2}$. To make those correctors C_i small, we have to choose $\alpha \in (\frac{3}{4},1)$. Note that for C_1, C_2, C_4, $\alpha > \frac{1}{2}$ is enough, but for C_3, it has to be greater than $\frac{3}{4}$.

In the inequalities (3.4) and (3.5), we replace $\mathcal{A}\lambda_0$ by the $\varepsilon-$dependent operator

$$\frac{1}{2}\sum_{k=1}^3(c\gamma^0\gamma^kD_k\lambda_0 + \frac{1}{2}D_k\lambda_0c\gamma^0\gamma^k) + \mathcal{L}_\varepsilon\lambda_0.$$

The rest of the proof is the same except that when we pass to the limit, we need to calculate

$x_1^*\mathcal{L}_\varepsilon^* x_1 + x_2^*\mathcal{L}_\varepsilon^* x_2$, and this leads to some $\varepsilon-$dependent scattering operator of the form

$$\begin{aligned}\mathcal{T}_\varepsilon(\alpha_+,\alpha_-) =& \frac{e^2}{(2\pi)^d}\int_{\mathbb{R}^d}(\alpha_+(q)-\alpha_+(\xi))\sum_{k=0}^{3}\omega_k(\xi,q)\varepsilon^{\alpha-1}\hat{R}_k\left(\frac{c\lambda_+(q)-c\lambda_+(\xi)}{\varepsilon^{1-\alpha}},q-\xi\right)dq\\ &+\frac{e^2}{(2\pi)^d}\int_{\mathbb{R}^d}(\alpha_-(q)-\alpha_+(\xi))\sum_{k=0}^{3}\tilde{\omega}_k(\xi,q)\varepsilon^{\alpha-1}\hat{R}_k\left(\frac{c\lambda_+(q)+c\lambda_+(\xi)}{\varepsilon^{1-\alpha}},q-\xi\right)dq.\end{aligned}$$

Following the same type of proof as in [4] and letting $\varepsilon\to 0$, we arrive at the elastic scattering operator

$$\mathcal{T}(\alpha_+) = e^2\int_{\mathbb{R}^d}(\alpha_+(q)-\alpha_+(\xi))\sum_{k=0}^{3}\omega_k(\xi,q)\delta(c\lambda_+(q)-c\lambda_+(\xi))\tilde{R}_k(0,q-\xi)dq. \tag{5.1}$$

Therefore, in the slow temporal fluctuation case, when $\alpha\in(\frac{3}{4},1)$, we have the following transport equation system of the limit (α_+,α_-):

$$\partial_t\alpha_\pm + \frac{c\xi\cdot\nabla_x\alpha_\pm}{\lambda_\pm(\xi)} = \mathcal{T}(\alpha_\pm), \tag{5.2}$$

where $\mathcal{T}$ is the elastic scattering operator defined in (5.1).

We see that the coupling between α_+ and α_- appeared in (2.16) is inactive, and we expect (5.2) to hold in the limit of no time-dependent regularization, i.e., formally for $\alpha=0$.

Remark 5.1. The condition that $\alpha>\frac{3}{4}$ could be relaxed somewhat. As a matter of fact, if we construct the test function as

$$\lambda_\varepsilon = \lambda_0 + \sum_{n=1}^{N}\varepsilon^{\frac{n}{2}}\lambda_{n,\varepsilon},$$

and follow the same procedure, we can show that $\lambda_{n,\varepsilon}$ is of order $\varepsilon^{n\alpha-n}$, and $\frac{n}{2}+n\alpha-n>0$ gives us $\alpha>\frac{1}{2}$. For the corrector

$$C_3 = \frac{1}{\sqrt{\varepsilon}}ie\sum_{k=0}^{3}\gamma^0\gamma^k\mathcal{K}_\varepsilon^k\varepsilon^{\frac{N}{2}}\lambda_{N,\varepsilon} - \frac{1}{\sqrt{\varepsilon}}ie\sum_{k=0}^{3}\tilde{\mathcal{K}}_\varepsilon^k\varepsilon^{\frac{N}{2}}\lambda_{N,\varepsilon}\gamma^0\gamma^k,$$

it is of the order $\varepsilon^{N\alpha-\frac{N+1}{2}}$, and $N\alpha-\frac{N+1}{2}>0$ leads to

$$\alpha > \frac{1}{2}+\frac{1}{2N}.$$

Therefore, by expanding in higher order, we can relax the assumption to be $\alpha\in(\frac{1}{2},1)$. Note that $\alpha=\frac{1}{2}$ corresponds to the regime considered in [9].

For slower time fluctuations of the media, i.e., when $\alpha<\frac{1}{2}$ or even $\alpha\to 0$, other techniques than the Markovian regularization considered here presumably need to be developed, and the use of the diagrammatic techniques as in [14, 28] is currently unavoidable.

6 Conclusion and further discussions

In this work, we have derived the kinetic limit for the Dirac equation with a time-dependent random electromagnetic field. We have shown that the cross modes $c^{\varepsilon}_{ij}, d^{\varepsilon}_{ij}$ converged to zero weakly in space while the limiting propagating modes $\alpha_+ = a_{11} + a_{22}$ and $\alpha_- = b_{11}+b_{22}$ satisfy a system of transport equations. In addition, we have seen that the temporal regularization brings in some new features (inelastic scattering) to the collision structure, which disappear when the random fluctuations are slower in time.

The method we use relies on the fact that the random field $\tilde{A}(t,p)$ is Markovian in time. By constructing appropriate test functions, we proved some approximating inequalities, and together with the tightness result, we were able to pass to the limit. We should mention that our approach is restricted to the L^2 case, in the sense that the initial Wigner transform should be bounded in L^2.

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